Wild Solutions and Turbulence

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Revisit the basic notions relating turbulence and properties of the Navier-Stokes and Euler equations in the light of the recent contributions of Camillo DeLellis and Laszlo Szekelyhidi.

I will not touch the “Clay problem" assuming that with smooth initial data and convenient boundary conditions solutions of these equations are smooth.

On the other hand I will relate "turbulence " to two effects: non regular initial datas and boundary effects when viscosity goes to 0.

Equations that we all know.

Compressible Navier-Stokes equation for perfect gazes with $\epsilon =$ Knudsen number.

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0\,,\\
&\rho\Big(\partial_t + u \cdot \nabla\Big) u + \nabla p = \epsilon \nabla \cdot [\nu \sigma(u)]\,,\\
&\frac{3}{2}\rho\Big(\partial_t + u \cdot \nabla\Big)\theta + \rho\theta \nabla \cdot u = \frac{1}{2}\epsilon\nu(\sigma(u) : \sigma(u)) + \epsilon \nabla[\kappa \nabla \theta]\,,
\end{aligned}
$$

$$
p = \rho\theta\,, \quad \sigma(u) = \Big(\nabla_x u + (\nabla_x u)^T\Big) - \frac{2}{3}(\nabla_x \cdot u) I\,.
$$

Small Mach number (of the order of the Knudsen number) approximation.

For the fluctuations of order ϵ of density, velocity and temperature about a constant background state, $(\overline{\rho}, 0, \overline{\theta})$, namely,

$$\rho = \overline{\rho} + \epsilon\tilde{\rho}, \quad \theta = \overline{\theta} + \epsilon\tilde{\theta}, \quad u = \epsilon\tilde{u}\,.$$

In order to observe the cumulative effect of the small velocity, one has to consider the system for a time scale long enough (of the order of ϵ^{-1}), and hence by rescaling the time, by factor of ϵ^{-1}

$$\begin{aligned}
&\nabla \cdot \tilde{u} = 0\,, \quad \nabla(\overline{\theta}\tilde{\rho} + \overline{\rho}\tilde{\theta}) = 0\,,\\
&\overline{\rho}(\partial_t \tilde{u} + (\tilde{u} \cdot \nabla)\tilde{u}) + \nabla \pi = \nu \Delta \tilde{u}\,,\\
&\frac{3}{2}\overline{\rho}(\partial_t \tilde{\theta} + (\tilde{u} \cdot \nabla)\tilde{\theta}) = \kappa \Delta \tilde{\theta}\,,
\end{aligned}$$

where π is the unknown pressure.

This system is often referred to as the incompressible *Navier-Stokes-Fourier system of equations.*

Observe that the equations for the velocity are independent of the temperature, while the advection term, $(\tilde{u} \cdot \nabla)\tilde{\theta}$, in the temperature equation depends on the velocity. The term incompressible comes from the $\nabla \cdot \tilde{u} = 0$ condition, which implies that the Lagrangian flow map defined by:

$$x \mapsto \phi(x,t)\,, \quad \phi(x,0) = x\,, \quad \dot{\phi}(x,t) = \tilde{u}(\phi(x,t),t)\,,$$

is volume preserving. The equation $\nabla(\bar{\theta}\tilde{\rho} + \overline{\rho}\tilde{\theta}) = 0$ is known as the Bernoulli relation.

Together with the equation for the temperature they describe how the flow, at this approximation level, propagate the initial fluctuations of density and temperature.

In particular, if $\tilde{\rho} = \tilde{\theta} = 0$ the system reduces (hereafter the tilde will be omitted, $\overline{\rho}$ will be taken equal to 1, and following the tradition in this field we will replace the notation for the pressure π by p) to the standard incompressible Navier-Stokes equations:

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0\,, \quad \nabla \cdot u = 0\,. \tag{1}$$

In a typical engineering or physical situation characteristic length, velocity and time scales, L , U and T, respectively, are introduced.

After rescaling the relevant quantities in (**??**), retaining the same notation for the rescaled dimensionless velocity, spatial and temporal variables, one obtains the undimensionlized Navier-Stokes equations.

$$\partial_t u + (u \cdot \nabla) u - \frac{1}{Re} \Delta u + \nabla p = 0\,, \quad \nabla \cdot u = 0\,, \tag{2}$$

where

$$Re = \frac{UL}{\nu}$$

is the dimensionless Reynolds number, which measures the ratio between a typical strength of the nonlinear advection term and a typical linear viscous dissipation in the Navier-Stokes equations. Since equations (**??**) and (**??**) look the same (symbolically), we will abuse, hereafter, the notation and let ν represent the inverse of the Reynolds number.

In most practical applications the Reynolds number is very large (for instance, the Reynolds number is about 100 *for the air around a moving bicycle, about* 10^8 *for a moving car, and is of the order of* 10^{12} *in climate and meteorological applications). This naturally suggests to investigate and compare the behavior of the Navier-Stokes equations (**??**), for large values of the Reynolds numbers, to that of Euler equations, which are obtained formally by substituting* $Re = \infty$ *in (**??**):*

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0\,, \ \nabla \cdot u = 0\,. \tag{3}$$

Compare the behavior of the solutions of Navier-Stokes to those of Euler, as the Reynolds number $Re \to \infty$*.*

However, as it will be developed below, the relationship between these two systems is far from being simple.

Boundary Conditions in a domain Ω with boundary

(not the whole space not a periodic box).
For the Euler equation:

Impermeability condition $u_\nu \cdot \vec{n} = 0$

For Navier-Stokes equations.

$$u_\nu \cdot \vec{n} = 0\,, \quad \nu(\partial_{\vec{n}} u_\nu + C(x) u_\nu)_\tau + \lambda u_\nu = 0 \text{ on } \partial\Omega\,, \tag{4}$$

$$\text{with} \quad \lambda(\nu, x) \geq 0\,,\ C(x) \in C(\mathbb{R}^n; \mathbb{R}^n)\,, \tag{5}$$

With $\lambda(\nu) = \infty$, the second statement corresponds to the Dirichlet boundary condition: *the component of the velocity on the plane tangent to $\partial\Omega$ being also zero and, therefore, it is called the “no-slip" boundary condition.*

In the more general case $C(x)$ is a linear bounded operator acting on the space of tangent vectors to $\partial\Omega$ with the $L^2(\partial\Omega)-$norm. With a convenient choice of $C(x)$ this statement includes the following conditions

$$(S(u_\nu)\cdot\vec{n})_\tau = (\partial_{\vec{n}} u_\nu)_\tau - (\nabla^t\vec{n}\cdot u_\nu)_\tau \Rightarrow \nu(S(u_\nu)\cdot\vec{n})_\tau + \lambda u_\nu = 0\,, \quad (6)$$

and

$$(\nabla\wedge u_\nu)\wedge\vec{n} = (\partial_{\vec{n}} u_\nu)_\tau + (\nabla^t\vec{n}\cdot u_\nu)_\tau \Rightarrow \nu(\nabla\wedge u_\nu)\wedge\vec{n} + \lambda u_\nu = 0\,. \quad (7)$$

Condition (**??**) is the standard Fourier law, which can be either derived by a phenomenological arguments, or deduced from a boundary condition, for the Boltzmann equations, due to the atomic structure of the wall. Condition (**??**) emphasize the role of the vorticity.

Energy balance

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(x)|u_\nu(x,t)|^2 d\sigma = \nu \int_{\partial\Omega} C(u_\nu)_\tau u_\nu d\sigma\,; \tag{8}$$

and the $\nu-$uniform estimate:

$$\frac{d}{dt}\int_\Omega \frac{|u_\nu(x,t)|^2}{2} dx + \nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(x)|u_\nu(x,t)|^2 d\sigma = o(\nu)\,. \tag{9}$$

With $\nu \to 0$ and $u_\nu(x,0) \to u_0$ in $\mathrm{strong} - L^2(\Omega)$

The only uniform estimate is:

$$\int_\Omega \frac{|u_\nu(x,t)|^2}{2} dx + \nu \int_0^t \int_\Omega |\nabla u_\nu(x,s)|^2 dx ds = \int_\Omega \frac{|u_\nu(x,0)|^2}{2} dx + o(\nu)\,. \tag{10}$$

Hence the problem of the weak limit in $L^\infty_{\mathrm{loc}}(\mathbb{R}_t; L^2(\Omega))$.

Oscillation and concentration: $\overline{u \otimes u} \neq \overline{u} \otimes \overline{u}$ and boundary effects.

Dissipative solution

Comparison with a smooth solution, or solenidal vector field

$$w(x,t)\,, S(w) = \frac{\nabla w + \nabla^{\perp} w}{2} \in L^1(0,T;L^\infty(\Omega)).$$

u *such that for every smooth divergence-free test vector field,* w*, tangent to the boundary* $\partial\Omega$ *(i.e.,* $w\cdot\vec{n}=0$ *on* $\partial\Omega$*), and with bounded support in* Ω*, one has the estimate:*

$$\frac{1}{2}\int_\Omega |u(x,t)-w(x,t)|^2 \le \int_0^t\int_\Omega |(E(w(x,s)),u(x,s)-w(x,s))|dxds$$
$$+\int_0^t\int_\Omega |(u(x,s)-w(x,s)S(w)u(x,s)-w(x,s))|dxds$$
$$+\frac{1}{2}\int_\Omega |u(x,0)-w(x,0)|^2 dx\,,$$

with $E(w) = \partial_t w + P_\sigma(w\cdot\nabla w)\,, P_\sigma$ *Leray projection.*

Theorem

1 For w regular solution of Euler equation and u dissipative solution one has stability:

$$\int_\Omega |u(x,t)-w(x,t)|^2 \leq \int_\Omega |u(x,0)-w(x,0)|^2 dx e^{2\int_0^t |S(w)|_{L^\infty} ds} \quad (11)$$

In particular any dissipative solution is admissible (i.e) dissipate energy:

$$\text{with} \quad w \equiv 0\,, \quad \int_\Omega |u(x,t)|^2 \leq \int_\Omega |u(x,0|^2 dx \quad (12)$$

2 When $\partial\Omega = \emptyset$ Any admissible weak solution of Euler equation is a dissipative solution.

3 When $\partial\Omega = \emptyset$ Any weak viscosity limit of solution of Navier-Stokes equation is a dissipative solution.

4 When $\partial\Omega = \emptyset$ any weak limit of

$$u_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{R}_v} vF_\epsilon(x,v,t)dv$$

with F_ϵ solution of the Boltzmann equation

$$\epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon^{1+q}}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad (x,v)\in\Omega\times\mathbf{R}^N \qquad (13)$$

in the incompressible Euler scaling is a dissipative solution (Laure Saint Raymond).

With oscillating initial data and or boundary effects things are more complicated.

Basic example from DiPerna-Majda Shear flow Pressure-less gas:

$$u(x,t) = (u_1(x_2), 0, u_3(x_1 - tu_1(x_2), x_2)) = 0$$

$$u_\epsilon(x,t) = (\sin(\frac{x_2}{\epsilon}), 0, w(x_1 - t\sin(\frac{x_2}{\epsilon}))$$

which is uniformly bounded and which weakly converges to

$$\overline{u} = (0, 0, \int_0^1 w(x_1 - t\sin s)ds)\,.$$

Obviously this is not a solution of Euler equations, by direct inspection and also because the term $u_\epsilon^1 \partial_{x_1} w_\epsilon$ *does not go to zero! Also not a dissipative solution.*

With Boundary.

In dimensions 2 *or* 3*, for any Leray-Hopf weak solution, on the time interval* $[0,T]$*, of Navier-Stokes equations*

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = 0\,,$$

in a domain Ω*, with boundary conditions*

$$u_\nu \cdot \vec{n} = 0\,, \nu(\partial_{\vec{n}} u_\nu + C(x) u_\nu)_\tau + \lambda(\nu, x) u_\nu = 0 \;\; \textit{on}\; \partial\Omega\,, \quad (14)$$

$$\textit{and}\; \lambda(\nu, x) \geq 0\,,\;\; C(x) \in C(\mathbb{R}^n; \mathbb{R}^n)\,; \quad (15)$$

and with fixed initial data u_0*,*

one has the following THEOREM:

1. If u_ν converges weak$-*$, in $L^\infty((0,T);L^2(\Omega))$, to a function $u(x,t)$, which is, for all $t\in(0,T)$, a smooth solution of Euler equations, then u_ν converges strongly to u, and it does not dissipate energy in the limit as $\nu\to 0$; namely,

$$\epsilon(\nu,T)=\lim_{\nu\to 0}\int_0^T(\nu\int_\Omega|\nabla u_\nu|^2dx+\int_{\partial\Omega}\lambda(\nu)|u_\nu(x,t)|^2d\sigma)dt=0\,. \tag{16}$$

2. If $\nu(\frac{\partial u_\nu}{\partial\vec{n}})_\tau\to 0$, as $\nu\to 0$, even in a very weak sense (for instance, in the sense of distribution in $\mathcal{D}'(\partial\Omega\times(0,T))$), then any weak limit, $\overline{u}$, of the sequence u_ν is a dissipative solution up to the boundary of Euler equations.

3. If $\lambda(\nu)$ goes to 0 then the hypothesis of the point 2 is satisfied.

4. (Kato criteria) If the sequence u_ν satisfies the estimate:

$$\lim_{\nu\to 0}\int_0^T \nu \int_{\{d(x,\partial\Omega)<c\nu\}\bigcap\Omega} |\nabla u_\nu(x,t)|^2 dx dt = 0\,, \tag{17}$$

then the hypothesis of the point 2 above is satisfied, and therefore any weak limit of the sequence u_ν is a dissipative solution up to the boundary of Euler equations.

Proof

*Equation (**??**) represents the dissipation of energy over the time* $[0, T]$*, the term* $\int_0^T (\nu \int_\Omega |\nabla u_\nu|^2 dx) dt$ *represents the dissipation of energy due to the fluid viscosity, and* $\int_0^T (\int_{\partial\Omega} \lambda(\nu) |u_\nu(x,t)|^2 d\sigma) dt$ *represents the dissipation of energy due to the friction of the fluid flow with the boundary. In the special case of the Dirichlet boundary condition for the Navier-Stokes equations this second term does not appear because the velocity field,* u_ν*, is zero on the boundary.*

Point 1, follows from the energy estimate

$$\frac{1}{2}\int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_0^T \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(x) |u_\nu(x,t)|^2 d\sigma \\ \leq \int_\Omega |u_0|^2 dx + o(\nu)\,.$$

Observe, thanks to the weak$-*$ convergence in $L^\infty((0,T);L^2(\Omega))$ of u_ν to a smooth solution u of Euler equations, that one has:

$$\begin{aligned}\frac{1}{2}\int_\Omega |u_0(x))|^2dx = \frac{1}{2}\int_\Omega |u(x,t)|^2dx = \frac{1}{2}\int_\Omega |u(x,t)|^2dx \\ \leq \lim_{\nu\to 0}\frac{1}{2}\int_\Omega |u_\nu(x,t)|^2dx\,.\end{aligned} \tag{18}$$

For point 2, the main difference between the problem without physical boundary and the current problem with boundary comes from the, only uncontrolled, extra term in the computation of

$$-\nu \int_0^T \int_\Omega (\Delta u_\nu \cdot (u_\nu - w)) dx dt \,,$$

*which, thanks to the boundary conditions $w \cdot \vec{n} = 0$ and (**??**)-(**??**), gives a boundary term of the form*

$$\nu \int_0^T \int_{\partial\Omega} ((\frac{\partial u_\nu}{\partial \vec{n}})_\tau \cdot w_\tau) d\sigma dt \,. \tag{19}$$

2 and 3 follow.

For any $w(x,t)$, which is tangent to $\partial\Omega$ for all $t \in (0,T]$, introduce a sequence $w_\nu(s,\tau,t)$ (where τ and s are the local tangential and normal coordinates at $\partial\Omega$, respectively) of divergence-free, tangent to the boundary, test functions with the following properties: They coincide with w on $\partial\Omega \times (0,T)$, their support is contained in

$$\Omega_\nu \times (0,T) = \{x \in \Omega : \, d(x,\partial\Omega) < \nu\} \times (0,T)\,,$$

and they satisfy the estimate

$$|\nabla_{\tau,t} w_\nu|_{L^\infty} \le C\,, \quad |\partial_s w_\nu|_{L^\infty} \le \frac{C}{\nu}\,. \tag{20}$$

Then the expression $((\frac{\partial u_\nu}{\partial \vec{n}})_\tau, w_\nu)_{L^2(\partial\Omega\times(0,T))}$ *is deduced, from Navier-Stokes equations subject to the boundary conditions (**??**)-(**??**), according to the formula:*

$$0 = \int_0^T (0, w_\nu) dt = \int_0^T ((\partial_t u_\nu + \nabla\cdot(u_\nu \otimes u_\nu) - \nu\Delta u_\nu + \nabla p_\nu), w_\nu) dt =$$
$$-\int_0^T (u_\nu, \partial_t w_\nu) dt - \int_0^T ((u_\nu\cdot\nabla) w_\nu, u_\nu) dt$$
$$+\nu\int_0^T (\nabla u_\nu, \nabla w_\nu) dt - (\nu\partial_{\vec{n}} u_\nu, w_\nu)_{L^2(\partial\Omega\times(0,T))}, \tag{21}$$

which implies the estimate:

$$|(\nu\partial_{\vec{n}}u_\nu, w_\nu)_{L^2(\partial\Omega\times(0,T))}| = |\int_0^T ((u_\nu\cdot\nabla)w_\nu, u_\nu)dt| + o(\nu)\,. \quad (22)$$

*Eventually, using (**??**), the control of energy and the Poincaré inequality, one obtains*

$$|\int_0^T ((u_\nu\cdot\nabla)w_\nu, u_\nu)dt| \le C\int_0^T\int_{\Omega_\nu}\nu|\nabla u_\nu|^2 dxdt \quad (23)$$

to complete the proof of point 3.

When $\lambda(\nu)$ does not go to zero, in particular, in the most extreme case (the Dirichlet boundary conditions) when $\lambda(\nu) = \infty$, there is no information on the nature of the limit, at this level of generality

The validity of the Prandtl approximation is in agreement with the Kato's criteria because then one has:

$$\nu \int_0^T \int_{\Omega \cap \{d(x,\partial\Omega) \leq c\nu\}} |\nabla \wedge u_\nu(x,s)|^2 dx ds \leq C\nu \,. \tag{24}$$

More on weak and admissible solutions

Results of De Lellis and Szekelyhidi after a long history!

1) For any smooth initial data (in $C^{1,\alpha}$ there is locally in $\Omega \subset \mathbb{R}^3$ and globally in $\Omega \subset \mathbb{R}^2$ a unique (local in time in $3d$ and global in time in $2d$) smooth solution hence a unique dissipative solution

2) For any initial data $u_0 \in L^2$ there exists a weak solution (in the proof not admissible.)

3) In $3d$ (in spite of the absence of global existence result of smooth solutions) there exists a dense (in L^2) set of initial data for which there exists a global admissible solution (the proof is indirect: uses measure value solutions).

4) In $2d$ there exists a residual set (in L^2) of initial data with global unique solution (uses the Youdovitch approach plus a Baire argument).

5) There exists an infinite set of initial data (non regular) with a infinite set of admissible solutions of the Cauchy problem.

The proof involves the introduction of the Reynolds stress tensor $\mathcal{R}$ (a standard coarse graining object) ; the formulation of the problem as a linear system plus a non linear constraint and an iteration process of the Nash Moser style.

Luc Tartar decomposition and Reynolds stress tensor

With $\quad \overline{v} = \lim_{\nu \to 0} \quad$ in $\quad C(0,T; L^2_w(\mathbb{R}^d))$

$\partial_t v_\nu + \nabla(v_\nu \otimes v_\nu) + \nabla p_\nu - \nu \Delta v_\nu = 0\,,$

$\partial_t v_\nu + \nabla(v_\nu \otimes v_\nu - \frac{|v_\nu|^2}{d}\mathbf{I_d}) + \nabla(\mathbf{q}_\nu + \frac{|\mathbf{v}_\nu|^2}{\mathbf{d}}\mathbf{I_d}) = \mathbf{p}_\nu - \nu\boldsymbol{\Delta}\mathbf{v}_\nu = \mathbf{0}\,,$

$v_\nu(x,0) \to v_0(x) \text{in} L^2(\Omega)$

one has:

$$\partial_t \overline{v} + \nabla(\overline{v \otimes v}) + \nabla \overline{p} = 0\,,$$

$$\overline{u} = (\overline{v} \otimes \overline{v} + \mathcal{R}) - \text{Trace}((\overline{v} \otimes \overline{v} + \mathcal{R}))\frac{1}{d}\mathbf{I}\,,$$

$$\partial_t \overline{v} + \nabla(\overline{u}) + \nabla \overline{q} = 0\,,$$

$$\mathcal{R} \geq 0 \Rightarrow \overline{v} \otimes \overline{v} - \overline{u} \leq \frac{1}{d}\overline{|v|^2}\mathbf{I} = \frac{2}{d}\overline{e}\mathbf{I}$$

$$\lambda_{max}(\overline{v} \otimes \overline{v} - \overline{u}) \leq \frac{1}{d}\overline{|v|^2} = \frac{2}{d}\overline{e}\,.$$

And the equality

$$\overline{v} \otimes \overline{v} - \overline{u} = \frac{1}{d}\overline{|v|^2}\mathbf{I_d}$$

is equivalent to the strong convergence and the fact that

$$\overline{u} = \overline{v} \otimes \overline{v} - \frac{|\overline{v}|^2}{d}\mathbf{I_d}$$

Hence the introduction for $(v, u) \in \mathbb{R}^d \times S_0^{d\times d}$ of

$$v \circ v = (v \otimes v - \frac{|v|^2}{d}\mathbf{I}_d) \text{ and } e(v, u) = \lambda_{\text{max}}(v \otimes v - u)$$

Basic theorem

Subsolution criterion. $\overline{e} \in C(\mathbb{R}^d \times (0,T)) \cap C([0,T); L^1)$. If there exists a sub solution

$(\overline{v}, \overline{u}, \overline{q}) \in C^0(R^d \times (0,T); \mathbb{R}^d \times S_0^{d\times d} \times \mathbb{R})$

$\partial_t \overline{v} + \nabla : \overline{u} + \nabla \overline{q} = 0$

with $t \to 0_+ \overline{v}(.,t) \rightharpoonup v_0(.)$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$

$(x,t) \in \mathcal{U} \subset \mathbb{R}^d \times (0,T) = \{(x,t), \frac{1}{2}|v(x,t)|^2 < \overline{e}(x,t)\} \Rightarrow \overline{v} \otimes \overline{v} - \overline{u} < \frac{2}{d}\overline{e}\mathbf{I},,$

Moreover $\mathcal{U}$ has non empty interior and the boundary of each time slice has 0 Lebesgue measure.

Then there exists an infinite set of solutions

$$v \in C((0,T); L^2(\mathbb{R}^d)), \quad \text{with for } t \to 0_+ \; v(.,t) \rightharpoonup v_0(.),$$

$$\partial_t v + \nabla : (v \otimes v) + \nabla p = 0, \frac{1}{2}|v(x,t)|^2 = \overline{e}(x,t) \text{ a.e.}.$$

Example

Construction of Emile Wiedemann, in a periodic box . Does not provide admissible solutions but with $\lim_{t\to\infty}\int |v(x,t)|^2 dx = 0$.

Proof

$$v_0(x) \quad \nabla\cdot v_0 = 0 \int_{\mathbf{T}} v_0(x)dx = 0\,, \overline{v} = e^{t(-\Delta)^{\frac{1}{2}}} v_0$$

In Fourier Space

$$\widehat{\overline{v}}_i(k,t) = \left(e^{-|k|t} v_{i,0}(k)\,, \quad \overline{u}_{i,j}(\widehat{k},t) = -i(\frac{k_j}{|k|}\overline{v}_i(\widehat{k},t) + \frac{k_i}{|k|}\overline{v}_j(\widehat{k},t))\,, \overline{q} = -\frac{|\overline{v}|^2}{2} \right)$$

is for $\overline{e}\in C([0,\infty);L^1), \overline{e}(x,t) > e(\overline{v},\overline{u})(x,t)$ for t > 0 . a subsolution.
(With zero pressure because $\nabla\cdot\overline{v} = 0$ and $\nabla\cdot(\nabla : u) = 0$).

For any $t > 0$ it is smooth hence existence in $\mathbf{T}^2 \times (0,\infty)$ of smooth bounded $\overline{e}(x,t) > e(\overline{v},\overline{u})$ For the hypothesis of the basic theorem one needs

$$\overline{e} \in C([0,\infty); L^1(\mathbf{T})^2)$$

This results from the strong L^2 continuity of the semi group $e^{-t(-\Delta)^{\frac{1}{2}}}$

Observe that in general one has:

$$e(\overline{v},\overline{u})(x,0) > \frac{1}{2}|v_0(x)|^2$$

with at least near $t = 0$ $\quad \int_{\mathbf{T}^2} \frac{1}{2}|v(x,t)|^2 dx > \int_{\mathbf{T}^2} \frac{1}{2}|v_0(x)|^2 dx\,.$

Exemple 2 The Shear Flow in $\mathbb{T}^2$.

$$v_0(x) = v_0(x_1, x_2) = \begin{cases} (1,0) \text{ if } 0 < x_2 < \frac{1}{2}, \\ (-1,0) \text{ if } -\frac{1}{2} < x_2 < 0. \end{cases}$$

Proof Construct a convenient sub solution $(\overline{v}, \overline{u}, \overline{q})$ an open set $\mathcal{U}$

$$(\overline{v}, \overline{u}, \overline{q}, \overline{e}),$$

$$\partial_t \overline{v} + \nabla \overline{v} + \nabla \overline{q} = 0,$$

$$\overline{v} \otimes \overline{v} - \overline{u} \leq \frac{2}{d} \overline{e} I,$$

$$\overline{e} \in C([0,T); L^1(\mathbf{T}^2)); \overline{v} \in C([0,T]; L^2_{\text{weak}}(\mathbf{T}^2)),$$

$$\forall t > 0, \quad \int \overline{e}(x,t)dx \leq \int \frac{1}{2} |v_0(x)|^2 dx.$$

$$\overline{v} = (\alpha, 0)\,, \quad \overline{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix},$$

$$\begin{aligned}
&\alpha = \alpha(x_2, t)\,, \quad \beta = \beta(x_2, t)\,, \\
&\beta = \frac{1}{2}\alpha^2\,, \quad \lambda \in (0,1)\,, \quad \gamma = -\frac{\lambda}{2}(1-\alpha^2)\,, \\
&\partial_t \alpha + \frac{\lambda}{2}\partial_{x_2}\alpha^2 = 0\,.
\end{aligned}$$

Hence $\alpha(x_2)$ given as the unique viscosity solution of Burger equation:

$$\alpha(x_2, t) = \frac{x_2}{\lambda t} \quad \text{for} \quad t \geq \frac{1}{2\lambda} \quad \text{and} \quad x_2 \in (-\frac{1}{2}, \frac{1}{2})$$

$$\text{for} \quad \mathrm{t} < \frac{1}{2\lambda} \quad \alpha(\mathrm{x}_2, \mathrm{t}) = \begin{cases} -1 & -\frac{1}{2} < x_2 < -\lambda t \\ \frac{x_2}{\lambda t} & -\lambda t < x_2 < \lambda t \\ 1 & \lambda t \quad < x_2 < \frac{1}{2} \end{cases}$$

$$\mathcal{U} = \{|\alpha| < 1 = \{(x,t) = \{x_1 \in \mathbf{T}^2, |x_2| < \lambda t\}$$

in $\mathcal{U}$ $$\mathrm{e}(\overline{\mathrm{v}}, \overline{\mathrm{u}}) = \frac{\alpha^2}{2} + \frac{\lambda}{2}(1-\alpha^2)$$

$$< \overline{e} = \frac{1}{2} - \epsilon \frac{1-\lambda}{2}(1-\alpha^2)$$

$$\overline{e}(x,0) = \frac{1}{2}.$$

Consequence.

For $3d$ Shear flow initial data the viscosity limit is a selection criteria.

$$v(x,t) = (v_1(x_2), 0, v_3(x_1 - tv_1(x_2), x_2))$$
$$v_0(x) = (v_1(x_2), 0, v_3(x_1, x_2))\,, v_1(x_2) = \begin{cases} 1 & \text{if } \; 0 < x_2 < 1/2 \\ -1 & \text{if } -1/2 < x_2 < 0, \end{cases}$$

$u(x,t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ admissible solution to the 2D vortex sheet

Then, the triple

$$(u_1(x_1, x_2, t), u_2(x_1, x_2, t), w(x_1, x_2, t))$$
$$\partial_t w + u \cdot \nabla w = 0\,, \quad w(t=0) = v_3.$$

is an admissible solution of the 3D Euler equations.

Navier-Stokes equations

$$\partial_t u_1^\nu(x_2,t) - \nu\partial_{x_2}^2 u_1^\nu(x_2,t) = 0\,,$$
$$\partial_t u_3^\nu(x_1,x_2,t) + u_1^\nu(x_2,t)\partial_{x_1} u_3^\nu(x_1,x_2,t) - \nu\Delta_{x_1,x_2} u_3^\nu(x_1,x_2,t) = 0$$

obviously will converge to

$$v(x,t) = (v_1(x_2), 0, v_3(x_1 - tv_1(x_2), x_2))$$

Same construction on a disk: Consequence for the boundary condition. With boundary a weak admissible solution may not be a dissipative one (at variance with what is true without boundary.)

Restricted to the interior of the disk one finds solutions which are weak and admissible in the sense of De Lellis and Szekelyhidi but which are not dissipative

In the presence of boundary when the Kato criteria is not satisfied the general idea would be that the Reynolds Stress tensor is not zero in the domain.

However in the present state of the art one cannot exclude situations where the pathology will be only concentrated near the boundary.

Eventually the shear flow with boundary condition provides an example of a regular limit.

Critical exponents are everywhere.....
Moreover this issue is a “model" for the construction of wild solutions

First in 2 and $3d$ the Euler equation is well posed in any space $C^{1,\alpha}$ this is the classical result of Lichtenstein.

Can be extended to $\nabla \wedge u \in L^\infty$ in $2d$ (Youdovitch) space just less regular than C^1 The shear flow shows that in $3d$ C^1 is critical.

$$(u_1(x_2), 0, u_3(x_1 - tu_1(x_2), x_2)$$

with derivative

$$\frac{\partial u_3}{\partial x_2} = D_2 u_3 - t D_1 u_3 D_2 u_1$$

which is in $C^{1,\alpha}$ for initial data in this space . But which will be only in C^{0,α^2} for initial data in $C^{0,\alpha}$.

Threshold is the regularity for energy conservation.

$$(\partial_t u u) + \nabla(u \otimes u)u = 0$$
$$(\partial_t u_\epsilon, u_\epsilon) + (\nabla(u \otimes u)_\epsilon u_\epsilon) = 0$$
$$\lim_{\epsilon \to 0} (\nabla(u \otimes u)_\epsilon, u_\epsilon) = 0$$
$$|\int \nabla(u \otimes u)u|dx| \sim \int |\nabla^{\frac{1}{3}} u|^3 dx$$

Onsager: For weak solution : $u \in C^{0,\alpha}$ with $\alpha > \frac{1}{3}$ Conservation of energy. Proof : Constantin , E and Titi; Chesdikov...Constantin, Friedlander..in Besov space:

$$\|u\|_{\mathcal{B}_3^{\alpha,\infty}} = \sup_{0<r<1} (\int |\frac{u(x+r) - u(x)}{r^{\frac{1}{3}}}|^3 dx)^{\frac{1}{3}}$$

Singular solutions conserving the energy, De Lellis and Szekelyhidi and the shear flow: $(u_1(x_2), 0, u_3(x_1 - tu_1(x_2)))$. Solutions $u \in C^{0,\frac{1}{10}}$ with energy decay (De Lellis and Szekelyhidi). **Is $\frac{1}{3}$ a threshold.**

Critical exponents in isometric imbedding.

Models for the proofs of De Lellis and Szekelyhidi (and the simplest model is the first Nash Theorem)

$C = \{x, |x| = 1 \subset \mathbb{R}^2\}$. Find a map $\phi : \mathbb{R}^2 \to \mathbb{R}^m$ such that $\phi(C) \subset \{y, \|y\|\} < r$ and the arc- length on $\phi(C) =$ the arc-lenght of C. What is the smallest possible r small and best regularity for ϕ

Since an isometric C^2 maps preserves the curvature (non trivial Gauss theorem) . If $\phi \in C^2$ then one needs $r > 1$

Nash theorem is that with $\phi \in C^1$ the radius $r > 0$ can be arbitrarily small.

Proof based on sub-solutions $C = \{\cos s, \sin s\,, s \in (0, 2\pi)\}$

Then the condition is $|x| = 1 \Rightarrow |\phi(x)| < r\,,$

$$|\partial_1 \phi(\cos s, \sin s)|^2 \cos^2 s + \cos s \sin s \partial_1 \phi \cdot \partial_2 \phi + |\partial_2 \phi(\cos s, \sin s)|^2 \sin^2 s = 1\,.$$

The sub-solutions are the set of ϕ such that

$$|\partial_1 \phi(\cos s, \sin s)|^2 \cos^2 s + \cos s \sin s \partial_1 \phi \cdot \partial_2 \phi + |\partial_2 \phi(\cos s, \sin s)|^2 \sin^2 s < 1\,.$$

The ideas is to start from a sub-solution (short imbedding) and then increase it up to the saturation of the inequality by adding more and more perturbations of higher and higher frequencies and smaller and smaller amplitudes. This last ansatz already used in the proofs of Scheffer and Shnirelman.

To the best of my knowledge the existence in this case and in more general cases of a genuine regularity threshold between C^1 and C^2 is still an open problem.

Comments on the De Lellis-Szekelyhidi construction.

- 1 The formal setting with linearization and constraint (à la Tartar).

- 2 Relaxation of the constraints.

- 3 The convex hull of the set defined by 1 and 2

- 4 An amelioration lemma or introduction of a convenient improving sequences.

- 5 Conclusion with a Baire type or a Nash-Moser type argument.

Tartar 's Linearization

$$C := \{(v, u, q) \in \mathbb{R}^d, u \in S_0^{d\times d}, \mathbb{R})\}$$
$$\partial_t v + \nabla u + \nabla q = 0$$
$$\nabla \cdot v = 0\,, v(x, 0) = v_0$$

The non-linear constraint

$$u = v \otimes v - \frac{1}{d}|v|^2$$

The subsolutions X_0

$$\partial_t \overline{v} + \nabla \overline{u} + \nabla \overline{q} = 0 \quad \nabla \cdot \overline{v} = 0$$

$$z = (\overline{v}, \overline{u}, \overline{q}, \overline{e}) \in C(\mathbb{R}^n \times (0,T))\, \overline{e} \in C([0,T); L^1(\mathbb{R}^d))$$

$$\frac{|\overline{v}|^2}{2} < \overline{e}\,, \quad \overline{v} \otimes \overline{v} - \overline{u} < \frac{2}{d}\overline{e} I \text{ in } \quad \mathbb{R}^n \times (0, T)$$

$$\overline{v}(.,t) \rightharpoonup v_0 \text{ in } L^2_w$$

$$\partial_t \overline{v} + \nabla(\overline{v} \otimes \overline{v}) + \nabla(\overline{q} + \frac{2\overline{e}}{d}) = -\nabla R$$

$$R = \overline{u} - (\overline{v} \otimes \overline{v} - \frac{2}{d}\overline{e} I) \geq 0$$

First prove $X_0 \neq \emptyset$ that was done in the previous examples.

Remark

$$0 \leq \overline{u} - (\overline{v} \otimes \overline{v} - \frac{2}{d}\overline{e}I) \Rightarrow \{\frac{|\overline{v}|^2}{2} = \overline{e} \Rightarrow \overline{u} - (\overline{v} \otimes \overline{v} - \frac{|\overline{v}|^2}{d}I) = 0!\}$$

The "Baire approach"

X the closure of X_0 in $C(]0,T[; L^2_{w-\text{loc}})$ It is a compact metric space -.

Introduce the functionals

$$I_{\tau,\Omega'}(z) = \sup_{\tau \le t \le T-\tau} \int_{\Omega'} (\overline{e}(x,t) - \frac{|\overline{v}(x,t)|^2}{2}) dx \quad \tau > 0\,, \Omega' \text{ bounded}$$

$I_{\tau,\Omega'}$ is upper continuous therefore the set of is points of continuity is a residual set.

Amelioration lemma: For all (τ, Ω') there exists a strictly positive increasing function $\alpha \mapsto \beta(\alpha)$ such that:

$I_{\tau,\Omega'}(z) > \alpha > 0$ implies for all $\eta > 0$ the existence of $z' \in X$ with:

$$\|z' - z\|_{C([0,T];H^{-1})} \leq \eta$$
$$I_{\tau,\Omega'}(z') \leq I_{\tau,\Omega'}(z) - \beta(\alpha)$$

Hence for the points of continuity one has

$$I_{\tau,\Omega'}(z) = 0$$

End of Baire argument:

The set of points of continuity of all such functional intersection of residual set is also a residual set . For any such point one has:

$$\frac{|\overline{v}(x,t)|^2}{2} - \overline{e}(x,t) = 0 \Rightarrow \overline{u} - (\frac{|\overline{v}|^2}{d} I - \overline{v} \otimes \overline{v}) \geq 0 \Rightarrow \overline{u} - (\frac{|\overline{v}|^2}{d} I - \overline{v} \otimes \overline{v}) = 0!$$

The amelioration lemma

To improve the inequality: In an open domain $\mathcal{O} \subset \{(x,t)\backslash \overline{e}(x,t) > \frac{|\overline{v}(x,t)|^2}{2}$ construct a function $\tilde{z}(x,t) = (\tilde{v}(x,t), \tilde{u}(x,t))$ with support in $\mathcal{O}' \subset\subset \mathcal{O}$ such that

$$(\overline{v}, \overline{u}) + \tilde{z} \quad \text{is still a subsolution} \tag{25}$$

$$0 < \int (e(x,t) - \frac{|(\overline{v}(x,t) + \tilde{v}(x,t))|^2}{2})dx$$

$$\int \frac{|(\overline{v}(x,t) + \tilde{v}(x,t))|^2}{2} dx = \int \frac{|\overline{v}(x,t)|^2}{2} dx + \int \frac{|\tilde{v}(x,t))|^2}{2} dx$$

$$- \int (\overline{v}(x,t) \cdot \tilde{v}(x,t))dx \tag{26}$$

In (**??**) the term $\int (\overline{v}(x,t)\tilde{v}(x,t)dx$ will be made small by oscillations (i.e. convergence in H^{-1}) of $\tilde{v}$ and the term $\int \frac{|\tilde{v}(x,t))|^2}{2} dx$ big enough by a geometric construction.

Geometric lemma.

The set

$$K_r = \{(v,u) \in \mathbb{R}^d \times S_0^d ; |v| \leq r \, , v \otimes v - u \leq \frac{r^2}{d} I_d\}$$

is closed convex and its boundary is

$$\partial K_r = \{(v,u) \in \mathbb{R}^d \times S_0^d ; |v| = r \, , v \otimes v - u = \frac{r^2}{d} I_d\}$$

For any $z = (v,u)$ in the interior of K there exist a and b with the following properties:

$$|a| = |b| \, , \lambda = (a - b, a \otimes a - b \otimes b) \in \mathbb{R}^d \times S_0^d$$

$$[z - \lambda, z + \lambda] \subset \mathrm{int} K_r \, , |a - b| \geq \frac{C}{r}(r^2 - |v|^2)$$

Plane waves solutions.

Lemma For any $\lambda = (a - b, a \otimes a - b \otimes b)\,, |a| = |b| = r$ There is a constant coefficient 3 order differential operator

$$A(\partial_x, \partial_t) : C_c^\infty(\mathbb{R}^{d+1}) \mapsto C_c^\infty(\mathbb{R}^{d+1} \times S_0^d)$$

with the following properties.

i) For any $\phi \in C_c^\infty(\mathbb{R}^{d+1})$ $A(\partial_x, \partial_t)\phi$ is a subsolution (solution of the linear system).

ii) With $\tau = \frac{|a+b|^2}{2}, \xi = (2s^2)^{-\frac{1}{3}}(a+b)$, any plane wave $\phi(\xi \cdot x - \tau.x)$ is solution of the equation:

$$A(\partial_x, \partial_t)(\phi(\xi \cdot x - \tau.x)) = \lambda\phi'''(\xi \cdot x - \tau.x)$$

Localisation of oscillating plane waves

Near a point $(x_0, t_0) \in \mathcal{O}'$ introduce the localized oscillating function:

$$\epsilon^3 \psi(x,t) \sin(\frac{\xi \cdot x - \tau . t}{\epsilon})$$

and write:

$$(\tilde{v}, \tilde{u}) = (\overline{v}, \overline{u}) + \epsilon^3 A(\partial_x, \partial_t)(\psi(x,t) \sin(\frac{\xi \cdot x - \tau . t}{\epsilon}))$$

$$= (\overline{v}, \overline{u}) + \lambda \psi(x,t) \cos(\frac{\xi \cdot x - \tau . t}{\epsilon}) + (O(\epsilon))\,.$$

Remarks

With similar proof that one can construct wild and admissible solutions of the isentropic Euler equations **in more than $1d$.**

$$\partial_\rho + \nabla \cdot (\rho u) = 0$$
$$\partial_t(\rho u) + \nabla(\rho u \otimes u + p(\rho)) = 0\,,$$
$$\int \rho(x,t)(\frac{|u|^2}{2} + S(\rho(x,t))dx = e(t)!!!$$

More Nash-Moser type

To prove the $C^{\frac{1}{10}}$ *regularity* Start from sub solutions

$$\frac{|v_k|^2}{2} < \overline{e}\,, \quad v_k \otimes v_k - \overline{u} < \frac{2}{d}\overline{e}I \text{ in } \quad \mathbb{R}^n \times (0, T)$$
$$\partial_t v_k + \nabla(v_k \otimes v_k) + \nabla p_k = -\nabla R_k$$
$$R_k = u_k - (v_k \otimes v_k - \frac{2}{d}\overline{e}I_k) \geq 0$$

$$(v_k, p_k, R_k) \to (v_k + w_k, p_{k+1}, R_{k+1})$$

To decrease:

$$\sup_t \int (e(x,t) - \frac{|v_k|^2}{2})dx > 0$$

and to decrease in H^{-1}

$$P_\sigma(R_k + (\partial_t w + \nabla(w \otimes w) + \nabla(v \otimes w + w \otimes v))$$

Eventually prove the $C^{\frac{1}{10}}$ convergence of $\sum w_k$ and the H^{-1} convergence of the R_k .

Ansatz

$$w = \sum_l P_\sigma(\phi_l(x,t)W_l(\lambda_k x, \lambda_k t))$$

The W are Beltrami flows:

$$\nabla \cdot W = 0 \quad \nabla(W \otimes W) = \nabla \frac{|W|^2}{2}$$

The ϕ_l and W_l are computed in term of of regularized (at the scale l k-data v_k and R_k) ..This is like Nash-Moser.

Open Problems

Below I quote a “pot pourri" of open problems that may have a reasonable solution but I indicate with starts how difficult I consider them.

****Prove the existence of “ wild solutions " that will not conserve energy but will be in C^α with $\alpha = \frac{1}{3} - \epsilon$. Since the most recent (to the best of my knowledge) improvement (due to Philip Isett and shown to me by Rafael Granero) is 171 pages long and produce solutions in $C^{\frac{1}{5}}$ further progress are not going to be easy.

****Show that the fact of being a weak limit of sequences of solutions of Navier-Stokes equation is a selection criteria among the (may be many) weak and admissible solutions of the Euler equations. This has already been done is some very specific examples. It is a reasonable conjecture that this would be generally true.

** Euler equation are well posed (locally in time in $3d$ and globally in $2d$ for data in $C^{1,\alpha}$ or in $2d$ with vorticity in L^∞ and not in C^α. Find more examples (the shear flow is one of them) for which the problem is either well or badly posed in the critical space C^1.

* In the paper “On admissibility criteria for weak solutions of the Euler equations" (*Arch. Ration. Mech. Anal.*, 195 (1) (2010), 225–260.) Camillo and Laszlo extended their construction of “wild solutions " to the isentropic compressible Euler in more than 1 dimension. They did not provided explicit examples of initial data for which there exist an infinite set of solution with non constant (or non decreasing energy). Would it be possible to produce explicit example based on the construction made by L. Székelyhidi. “Weak

solutions to the incompressible Euler equations with vortex sheet initial data", *C.R. Math. Acad. Sci. Paris*, 349:19-20 (2011), 1063–1066.

** The Proposition 2.5 in the paper of C. De Lellis and L. Székelyhidi, Jr., “The h-principle and the equations of fluid dynamics", *Bull. Amer. Math. Soc. (N.S.)*, 49 (2012), no. 3, 347–375 indicate that the construction of subsolution is in fact a local issue. Would it be possible to use this remark for initial data that are on locally like a flat vortex sheet (and anything else elsewhere) or (following work in preparation Bardos- Székelyhidi) locally like an arc of circle.. Then that would lead to quite general vortex sheet. With Delort and other improvements (cf. M. C. Lopes Filho, H. J. Nussenzveig Lopes, and Z. Xin. “Existence of vortex sheets with reflection symmetry in two space dimensions." *Arch. Ration. Mech. Anal.,* 158 (3) (2001) 235-257) weak admissible solution do exist for this type of this type of initial data and some of them may be limit of solutions of Navier-Stokes with the same initial data.

**** However continuing the previous issue except for very special (analytic, with compatibility conditions in the initial data) there will be in general no weak solution with vorticity supported by a vortex sheet. So the uniqueness of viscous limit seem really difficult to prove.

***** Even in the case of analytic initial data generating a solution in form of a vortex sheet (Kelvin -Helmholtz type solution) there is absolutely no proof of the fact the viscous limit of solutions of Navier-Stokes with such initial data will converge to this Kelvin-Helmholtz solution (even for a short time). There are related results of Marchioro and Pulvirenti or Lombardo, Caflisch-Sammartino, but none of them close enough to the above problem.

*********In the presence of Dirichlet boundary condition even with smooth initial data the situation is much worse for the zero viscosity limit. If the

Kato criteria is not satisfied weak limit may not be weak solution (in spite of the fact that the energy satisfies the admissibility condition). Then a non trivial reynolds stress tensor and non trivial energy dissipation may appear. There no explicit of formal construction of such behavior which seems conform to the intuition.

One cannot exclude scenario where the viscosity limit is a weak admissible solution but at variance with what is proven in the absence of boundary this does implies that this is a dissipative solution..Bad behavior may accumulate near the boundary...It is possible to construct wild solution of with this type of behavior but there no indication that they would be limit of solution of Navier-Stokes with Dirichle boundary condition.

References

This is also a “pot pourri" of references not organizes in alphabetic order but some of them contain a huge set of complementary references

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See also the notes of Laszlo Székelyhidi on his home page.

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