LIFTING OF THE AUTOMORPHISM GROUP OF POLYNOMIAL ALGEBRAS

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Abstract. Let K be an arbitrary field of zero characteristics. The main results of the paper are

Theorem 1.2. *Any* Ind-*scheme automorphism* φ *of* $\mathrm{NAut}(K[x_1,\dots,x_n])$ *for* $n \geq 3$ *is inner.*

Theorem 3.3. *Any* Ind-*scheme automorphism* φ *of* $\mathrm{NAut}(K\langle x_1,\dots,x_n\rangle)$ *for* $n \geq 3$ *is semi-inner.*

Theorem 1.1. *Let* K *be an infinite field of arbitrary characteristics. There exists no subgroup* H *of* $\mathrm{Aut}(K\langle x_1,\dots,x_n\rangle)$ *such that* $\mathrm{NAut}(K[x_1,\dots,x_n])$ *is isomorphic to* H *for* $n > 2$ *induced by the abelinization.*

We also establish Bialickii-Birula theorem for free algebras:

Theorem 3.1. *Any effective action of torus* $\mathbb{T}^n$ *on* $\mathbb{K}\langle x_1,\dots,x_n\rangle$ *is linearizable. That is, it is conjugated to a standard one for finfinite fields of arbitrary characteristics.*

Group NAut of nice automorphisms is a group of automorphisms approximated by tame ones. In the case of characteristic zero it coincides with group of all automorphisms (see definition 1.10).

1. Introduction and main results

In 2004, the famous Nagata conjecture over a field K of characteristic 0 was proved by Shestakov and Umirbaev [22, 23] and a stronger version of the conjecture was proved by Umirbaev and Yu [25]. That is, let K be a field of characteristic zero. Every wild $K[z]$-automorphism (wild $K[z]$-coordinate) of $K[z][x,y]$ is wild viewed as a K-automorphism (K-coordinate) of $K[x,y,z]$. In particular, the Nagata automorphism $(x - 2y(y^2+xz) - (y^2+xz)^2 z, y + (y^2+xz)z, z)$ (Nagata coordinates $x -$

1991 *Mathematics Subject Classification.* Primary 13S10, 16S10. Secondary 13F20, 14R10, 16W20, 16Z05.

Key words and phrases. Automorphisms, polynomial algebras, free associative algebras, lifting problem, tame and wild automorphisms, coordinates, Nagata conjecture, linearization.

The research of Jie-Tai Yu was partially supported by an RGC-GRF Grant.

$2y(y^2+xz)-(y^2+xz)^2z$ and $y+(y^2+xz)z)$ is (are) wild. In [25], a related question was raised:

The lifting problem. Whether or not a wild automorphism (wild coordinate) of the polynomial algebra $K[x,y,z]$ over a field K can be lifted to an automorphism (coordinate) of the free associative $K\langle x,y,z\rangle$?

In this paper we prove that the automorphism group of free associative algebra over an arbitrary field K cannot be isomorphic to any subgroup of automorphism group of polynomial algebra induced by the natural abelnization.

Theorem 1.1. *Let K be an arbitrary field, $G = \mathrm{NAut}(K[x_1,\ldots,x_n])$ and $n > 2$. Then G cannot be isomorphic to any subgroup H of $\mathrm{Aut}(K\langle x_1,\ldots,x_n\rangle)$ induced by the natural abelianization. The same is true for $\mathrm{NAut}(K[x_1,\ldots,x_n]$.*

This theorem is obtained as a consequence of a systematic study for the structure of automorphism group.

Theorem 1.2. *Any Ind-scheme automorphism φ of $\mathrm{NAut}(K[x_1,\ldots,x_n])$ for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.*

Theorem 1.3. *Any Ind-scheme automorphism φ of $\mathrm{NAut}(K\langle x_1,\ldots,x_n\rangle)$ for $n \geq 3$ is semi-inner (see definition 1.6).*

For the group of automorphisms of semigroup the similar results on set-theoretical level were obtained previously by A.Belov, R.Lipyanskii and I.Berzinsh [3, 2]. All these questions (including Aut(Aut) investigations) are closely related to Universal Algebraic Geometry and were proposed by B.Plotkin. Equivalence of two algebras have same generalized identities and isomorphism of first order means semmi-inner properties of automorphisms (see [3, 2] for details).

Regarding tame automorphism group, we have following results.

Theorem 1.4. *Any automorphism φ of $\mathrm{TAut}(K[x_1,\ldots,x_n])$ (in group theoretical sense) for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.*

Let $G_n \subset \mathrm{TAut}(K[x_1,\dots,x_n]), E_n \subset \mathrm{TAut}(K\langle x_1,\dots,x_n\rangle)$ be tame automorphism groups, preserving origin,

Theorem 1.5. *Any automorphism φ of G_n (in group theoretical sense) for $n \geq 3$ is inner, i.e. is a conjugation via some automorphism.*

Definition 1.6. *Antiautomorphism* Ψ of k-algebra B is automorphism B as a vector space such that $\Psi(ab) = \Psi(b)\Psi(a)$. Transposition of matrices is aniautomorphism. Antiautomorphism of free associative algebra A is *mirror* if it sends $x_i x_j$ to $x_j x_i$ for some fixed i and j. Automorphism of $\mathrm{Aut}(A)$ is *semi-inner* if it can be obtained as a composition of an inner automorphism and conjugation by a mirror antiautomorphism.

Theorem 1.7. *a) Any automorphism φ of $\mathrm{Aut}(K\langle x_1,\dots,x_n\rangle)$ (in group theoretical sense) for $n \geq 4$ is semiinner, i.e. is a conjugation via some automorphism and/or mirror anitautomorphism.*

b) The same is true for E_n, $n \geq 4$.

Theorem 1.8. *a) Let $\mathrm{Char}(k) \neq 2$. Then $\mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}(k\langle x,y,z\rangle)$ generated by conjugation on automorphism or mirror antiautomorphism.*

b) The same is true for $\mathrm{Aut}_{\mathrm{Ind}}(E_3)$.

By TAut we denote tame automorphism group, $\mathrm{Aut}_{\mathrm{Ind}}$ is group of Ind-scheme automorphisms (see section 1.1).

In spirit of that, we have following questions.

(1) Is it true that any automorphism φ of $\mathrm{Aut}(K\langle x_1,\dots,x_n\rangle)$ (in group theoretical sense) for $n = 3$ is semiinner, i.e. is a conjugation via some automorphism and/or mirror anitautomorphism.
(2) Is it true that $\mathrm{Aut}(K\langle x_1,\dots,x_n\rangle)$ is generated by linear automorphisms and automorphism $x_n \to x_n + x_1x_2, x_i \to x_i, i \neq n$? For $n \geq 5$ it seems to be more easy and answer suppose to be positive, for $n = 3$ answer may be negative, for $n = 4$ answer presumably positive.
(3) Is it true that $\mathrm{Aut}(K[x_1,\dots,x_n])$ is generated by linear automorphisms and automorphism $x_n \to x_n + x_1x_2, x_i \to x_i, i \neq n$? For $n \geq 4$ it seems to be more easy and answer suppose to be positive, for $n = 4$ answer presumably positive.

(4) Is any automorphism φ of $\mathrm{Aut}(K\langle x_1,\dots,x_n\rangle)$ (in group theoretical sense) for $n=3$ is semiinner?
(5) Is it true that conjugations in theorems 1.4 and 1.7 are done by tame automorphism? Suppose $\psi^{-1}\varphi\psi$ is tame automorphism for any tame φ. Does it follow that ψ is tame?
(6) Prove result of the theorem 1.8 for $\mathrm{Char}(k)=2$. Does it hold on the set theoretical level, i.e. $\mathrm{Aut}(\mathrm{TAut}(k\langle x,y,z\rangle))$ are generated by conjugation on automorphism or mirror antiautomorphism?

Similar questions can be posed for nice automorphisms.

1.1. Ind-**schemes and Approximation.**

Definition 1.9. An Ind-*variety* M is a direct limit of algebraic varieties $M=\varinjlim M_1\subseteq M_2\dots$ An Ind-*scheme* is an Ind-variety which is a group such that group inversion is a morphism $M_i\to M_{j(i)}$, and the group multiplication induces a morphism from $M_i\times M_j$ to $M_{k(i,j)}$. A map φ is a *morphism* of Ind-variety M to Ind-variety N, if $\varphi(M_i)\subseteq N_{j(i)}$ and restriction φ on M_i is morphism for all i. *Monomorphism, epimorphism and isomorphism* can be defined similarly in the natural way.

Example. Let M be the group of automorphisms of an affine space, and M_j be set of all automorphisms in M with degree $\leq j$.

There is interesting

Question. *Investigate* growth function *on* Ind-*varieties. For example, dimension of varieties of polynomial automorphisms of degree* $\leq n$.

For example, coincidence of growth functions for $\mathrm{Aut}(W_n)$ and $\mathrm{Sympl}(\mathbb{C}^{2n})$ imply Kontsevich-Belov conjecture [5]:

KB_n: Does $\mathrm{Aut}(W_n)\simeq \mathrm{Sympl}(\mathbb{C}^{2n})$.

Similar conjecture can be stated for endomorphisms

KB_n: Does $\mathrm{End}(W_n)\simeq \mathrm{Sympl}\,\mathrm{End}(\mathbb{C}^{2n})$.

If Jacobian conjecture JC_{2n} is true, then these two conjectures are equivalent. $W_n = \mathbb{C}[x_1, \ldots, x_n; \partial_1, \ldots, \partial_n]$ is *Weil algebra* of differential operators.

It is natural to approximate automorphisms by tame ones. There exists such approximation up to terms of any order not only in the situation of polynomial automorphisms, but also for automorphisms of Weil algebra, symplectomorphisms etc. However, naive approach fails.

It is known that $\mathrm{Aut}(W_1) \equiv \mathrm{Aut}_1(K[x,y])$ where Aut_1 means the jacobian is one. However, considerations from [20] shows that Lie algebra of the first group is derivations of W_1 and hence has no identities apart ones which have free Lie algebra, another consistant of vector fields with divergent to zero and has polynomial identity. They cannot be isomorphic [5, 4]. In other words, this group has two coordinate system non-smooth with respect to each other. The group $\mathrm{Aut}(W_n)$ can be embedded into $\mathrm{Sympl}(\mathbb{C}^{2n})$, for any n. But Lie algebra $\mathrm{Der}(W_n)$ has no polynomial identities apart ones which have free Lie algebra, another consistent of vector fields preserving symplectic form and has polynomial identity. (In the paper [20] functionals on $\mathfrak{m}/\mathfrak{m}^2$ where considered in order to define Lie algebra structure. It is not quite clear, why these spaces have non-zero limit.)

In his remarkable paper, Yu. Bodnarchuck [11] established Theorem 1.2 by using the Shafarevich results for tame automorphism group and for case when automorphism of Ind-scheme is regular in following sense: sent polynomials on coordinate functions (coordinate – coefficient before corresponding monomial) to polynomial coordinate functions. In this case tame approximation works (as well as for the symplectic case as well). For this case his method is similar to ours, but we display it for reader convenience and also to treat free associative case. But in general case – for regular functions, if the approximation via Shafarevich approach is correct, then the Kontsevich-Belov conjecture (for isomorphism between $\mathrm{Aut}(W_n)$ and $\mathrm{Sympl}(K^n)$) would follow easily.

We have to mention also the very recent paper of H. Kraft and I. Stampfli [16]. They show that every automorphism of the group $\mathcal{G}_n := \mathrm{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of complex affine n-space

$\mathbb{A}^n = \mathbb{C}^n$ is inner up to field automorphisms when restricted to the subgroup $T\mathcal{G}_n$ of tame automorphisms. They play on conjugation with translation. This generalizes a result of J.Deserti [12] who proved this for dimension two where all automorphisms are tame: $T\mathcal{G}_2 = \mathcal{G}_2$. Our method is slightly different. We calculate automorphism of tame automorphism group preserving origin. In this case we can not play on translations. We also establish these results for free associative case. We always treat dimension more than two.

We do not assume regularity in the sense of [11] but only assume that restriction on any subvariety is a morphism. Note that morphisms of Ind-schemes $\mathrm{Aut}(W_n) \to \mathrm{Sympl}(\mathbb{C}^{2n})$ has this property, but not regular in the sense of Bodnarchuk [11].

In order to make approximation work, we use the idea of singularity which allows us to prove the augmentation group structure preserving, so approximation works in the case (not in all situations, in a much more complicated way).

Consider the isomorphism $\mathrm{Aut}(W_1) \cong \mathrm{Aut}_1(K[x,y])$. It has some strange property. Let us add a small parameter t. Then an element arbitrary close to zero with respect to t^k does not go to zero arbitrarily, so it is impossible to make tame limit! There is a sequence of convergent product of elementary automorphisms, which is not convergent under this isomorphism. Exactly same situation happens for W_n. These effects cause problems in quantum field theory.

Note that the Aut(Aut) issue comes from the proof of independent choice of an infinite large prime in order to construct an homomorphism from $\mathrm{Aut}(W_n) \to \mathrm{Sympl}(\mathbb{C}^n)$. (One way using one prime, another way using another then compare). We want to propose the following

Conjecture. *All automorphisms of* $\mathrm{Sympl}(\mathbb{C}^n)$ *as* Ind-*scheme are inner.*

The same conjecture can be proposed for $\mathrm{Aut}(W_n)$.

It is result of D.Anick that any automorphism of $K[x_1, \ldots, x_n]$ if $\mathrm{Char}(K) = 0$ can be approximated by tame ones respect to augmentation subgroups H_n.

Definition 1.10. Endomorphism $\varphi \in \mathrm{End}(K[x_1, \dots, x_n])$ is *good* if for any m there exist $\psi_m \in \mathrm{End}(K[x_1, \dots, x_n])$ and $\phi_m \in \mathrm{Aut}(K[x_1, \dots, x_n])$ such that

- $\varphi = \psi_m \phi_m$
- $\psi(x_i) = x_i + P_i$; $P_i \in \mathrm{Id}(x_1, \dots, x_m)^n$.

Automorphism $\varphi \in \mathrm{Aut}(K[x_1, \dots, x_n])$ is *nice* if for any m there exist $\psi_m \in \mathrm{Aut}(K[x_1, \dots, x_n])$ and $\phi_m \in \mathrm{TAut}(K[x_1, \dots, x_n])$ such that

- $\varphi = \psi_m \phi_m$
- $\psi(x_i) = x_i + P_i$; $P_i \in \mathrm{Id}(x_1, \dots, x_m)^n$, i.e. $\Psi \in H_m$.

D.Anick shown that if $\mathrm{Char}(K) = 0$ any automorphism is nice. However, this is unclear in positive characteristic.

Question. *Is any automorphism over arbitrary field nice?.*

One can formulate a question generalizing Jacobian conjecture for any characteristics.

Jacobian conjecture for arbitrary fields. *Is any good endomorphism over arbitrary field an automorphism?.*

Each good automorphism has Jacobian 1, and all such automorphisms are good (even nice) when $\mathrm{Char}(K) = 0$. Naive formulation is not good because of example of mapping $x \to x - x^p$ in characteristic p.

Similar notions can be formulate for free associative algebra case. Hence we have similar questions.

Question. *Is any automorphism of free associative algebra over arbitrary field nice?.*

Question. *Is any good endomorphism of free associative algebra over arbitrary field an automorphism?.*

2. The automorphism of automorphisms group

2.1. **The automorphisms of tame automorphisms group of $K[x_1, \dots, x_n]$.** We need the following theorem of Byalickii-Birula [10, 9]:

Theorem 2.1 (Byalickii-Birula)**.** *Any effective action of torus $\mathbb{T}^n$ on $\mathbb{C}^n$ is linearizable. That is, it is conjugated to a standard one.*

Remark. An effective action of $\mathbb{T}^{n-1}$ on $\mathbb{C}^n$ is linearizable [10, 9]. There is a conjecture whether an action of $\mathbb{T}^{n-2}$ on $\mathbb{C}^n$ is linearizable, established for $n = 3$. For codimensions more than 2, counterexamples were constructed [1].

Consider the standard action of torus T^n on C^n: $x_i \to \lambda_i x_i$, let H be the image of T^n under φ. Then by Theorem 2.1, H is conjugated to the standard torus via some automorphism ψ. Composing φ with conjugation with respect to ψ, we come to the case when φ is the identity on the maximal torus. Then we have the following

Corollary 2.2. *Without loss of generality it is enough to prove Theorem 1.2 for the case when $\varphi|_{\mathbb{T}} = \mathrm{Id}$.*

Now we are in the situation when φ preserves all linear mappings $x_i \to \lambda_i x_i$. We have to prove that it is identity.

Proposition 2.3 (E.Rips, private communication)**.** *Let $n > 2$ and let φ preserves the standard torus action for a free associative algebra or a polynomial algebra. Then φ preserves all elementary transformations.*

Corollary 2.4. *Let φ satisfies the conditions of the proposition 2.3. Then φ preserves all tame automorphisms.*

Proof of Proposition 2.3. We need several lemmas. First of all, we need to see compositions of given automorphisms with action of maximal torus.

Lemma 2.5. *Consider the diagonal $\mathbb{T}^1$ automorphisms: $\alpha : x_i \to \alpha_i x_i$, $\beta : x_i \to \beta_i x_i$. Let $\psi : x_i \to \sum_{i,J} a_{iJ} x^J, i = 1, \ldots, n$; $J = (j_1, \ldots, j_n)$ – multyindex, $x^J = x^{j_1} \cdots x^{j_n}$. Then*

$$\alpha \circ \psi \circ \beta : x_i \to \sum_{i,J} \alpha_i a_{iJ} x^J \beta^J,$$

In particular,

$$\alpha \circ \psi \circ \alpha^{-1} : x_i \to \sum_{i,J} \alpha_i a_{iJ} x^J \alpha^{-J}.$$

Applying the lemma 2.5 and comparing coefficients we get the following

Lemma 2.6. *Consider the diagonal $\mathbb{T}^1$ action: $x_i \to \lambda x_i$. Then the set of automorphisms commuting with this action is exactly linear automorphisms.*

Similarly we obtain the lemmas 2.7, 2.9, 2.10:

Lemma 2.7. *a) Consider the following $\mathbb{T}^2$ action: $x_1 \to \lambda\delta x_1$, $x_2 \to \lambda x_2$, $x_3 \to \delta x_3, x_i \to x_i, i > 3$. Then the set S of automorphisms commuting with this action generated with following automorphisms $x_1 \to x_1 + \beta \cdot x_2 x_3, x_i \to \varepsilon_i x_i, i > 1$, $(\alpha, \beta, \varepsilon \in K)$.*

b) Consider the following $\mathbb{T}^2$ action: $x_1 \to \lambda^I x_1$, $x_j \to \lambda_j x_j, j > 1$. Then the set S of automorphisms commuting with this action generated with following automorphisms $x_1 \to x_1 + \beta \cdot \prod_{j=2}^n x_j^{i_j}$, $(\lambda = (\lambda_2, \ldots, \lambda_n), \beta, \lambda_j \in K)$.

Remark. The similar statement for free associative case is true, but one has to consider the set $\hat{S}$ of automorphisms $x_1 \to x_1 + H, x_i \to \varepsilon_i x_i, i > 1$, ($\varepsilon \in K$, polynomial $H \in k < x_2, \ldots, x_n >$ has multydegree J).

Corollary 2.8. *Let $\varphi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{Aut}(K(x_1, \ldots, x_n)))$ stabilizing all elements from $\mathbb{T}$. Then $\varphi(S) = S$.*

Lemma 2.9. *Consider the following $\mathbb{T}^1$ action: $x_1 \to \lambda^2 x_1$, $x_2 \to \lambda x_2$, $x_i \to x_i, i \neq 1, 2$. Then the set S of automorphisms commuting with this action generated with following automorphisms $x_1 \to x_1 + \beta \cdot x_2^2, x_i \to \lambda_i x_i, i > 2$, $(\beta, \lambda_i \in K)$.*

Lemma 2.10. *Consider the set S defined in the previous lemma. Then $[S, S] = \{uvu^{-1}v^{-1}\}$ consists of the following automorphisms $x_1 \to x_1 + \beta \cdot x_2 x_3, x_2 \to x_2, x_3 \to x_3$, $(\beta \in K)$.*

Lemma 2.11. *Let $n \geq 3$. Consider the following set of automorphisms $\psi_i : x_i \to x_i + \beta_i x_{i+1} x_{i+2}, \beta_i \neq 0, x_k = x_k, k \neq i$ for $i = 1, \ldots, n-1$. (Numeration is cyclic, so for example $x_{n+1} = x_1$). Let $\beta_i \neq 0$ for all i. Then all of ψ_i simultaneously conjugated by torus action to $\psi_i' : x_i \to x_i + x_{i+1} x_{i+2}, x_k = x_k, k \neq i$ for $i = 1, \ldots, n$ in a unique way.*

Proof. Let $\alpha : x_i \to \alpha_i x_i$, then by the lemma 2.5 we obtain

$$\alpha \circ \psi_i \circ \alpha^{-1} : x_i \to x_i + \beta_i x_{i+1} x_{i+2} \alpha_{i+1}^{-1} \alpha_{i+2}^{-1} \alpha_i$$

and

$$\alpha \circ \psi_i \circ \alpha^{-1} : x_k \to x_k$$

for $k \neq i$.

Comparing coefficients before quadratic terms, we see that it is sufficient to solve the system:

$$\beta_i \alpha_{i+1}^{-1} \alpha_{i+2}^{-1} \alpha_i = 1, i = 1, \ldots, n-1.$$

because $\beta_i \neq 0$ for all i, this system has unique solution.

Remark. In free associative case, instead of $\beta x_2 x_3$ one has to consider $\beta x_2 x_3 + \gamma x_3 x_2$.

Lemma 2.12 (Rips)**.** *Linear transformations and ψ_i' generate all tame automorphism group.*

Proposition 2.3 if $\mathrm{Char}(K) \neq 2$ and $|K| = \infty$ follows from Lemmas 2.6, 2.7, 2.9, 2.10, 2.11, 2.12. Note that we have proved analogue of the theorem 1.2 for tame automorphisms.

Proof of the lemma 2.12. We need several statements.

Lemma 2.13. *Linear transformations and $\psi : x \to x, y \to y, z \to z + xy$ generate all the mappings of the following form $\psi_n^b(x,y,z) : x \to x, y \to y, z \to z + bx^n, b \in K$.*

Proof of the lemma 2.13. We proceed by induction. Suppose we have automorphism $\phi_{n-1}^b(x,y,z) : x \to x, y \to y, z \to z + bx^{n-1}$. Conjugating by linear transformation we obtain the automorphism $\phi_{n-1}^b(x,z,y) : x \to x, y \to y + bx^{n-1}, z \to z$. Composing this with ψ from the right we get the automorphism $\varphi(x,y,z) : x \to x, y \to y + bx^{n-1}, z \to z + yx + x^n$. Note that $\phi_{n-1}(x,y,z)^{-1} \circ \varphi(x,y,z) : x \to x, y \to y, z \to z + xy + bx^n$. Now we see that $\psi^{-1}\phi_{n-1}(x,y,z)^{-1} \circ \varphi(x,y,z) = \phi_n^b$ and lemma is proven.

Corollary 2.14. *Let $\mathrm{Char}(K) = 0$ or coprime with n. Let $|K| = \infty$. Then G contains all the transformations $z \to z + bx^k y^l, y \to y, x \to x$ such that $k + l = n$.*

Proof. For any invertible linear transformation $\varphi : x \to a_{11}x + a_{12}y; y \to a_{21}x + a_{22}y, z \to z; a_{ij} \in K$ we have $\varphi^{-1}\phi^b n\varphi : x \to x, y \to y, z \to z + b(a_{11}x + a_{12}y)^n$. Note that sums of such expressions contains all the terms of the form bx^ky^l. Corollary is proven.

Lemma 2.15. *If* $\mathrm{Char}(K) \neq 2$ *then linear transformations and* $\psi : x \to x, y \to y, z \to z + xy$ *generate all the mappings of the following form* $\alpha_n^b(x, y, z) : x \to x, y \to y, z \to z + byx^n, b \in K$.

Proof of the lemma 2.15. Note that $\alpha = \beta \circ \phi_n^b(x, z, y) : x \to x + by^n, y \to y + x + by^n, z \to z$ where $\beta : x \to x, y \to x + y, z \to z$. Then $\gamma : \alpha^{-1}\psi\alpha : x \to x, y \to y, z \to z + xy + 2bxy^n + by^{2n}$. Composing with ψ^{-1} and $\psi_{2n}{-2b}$ we get needed $\alpha_n^{2b}(x, y, z) : x \to x, y \to y, z \to z + 2byx^n, b \in K$.

Corollary 2.16. *Let* $\mathrm{Char}(K) = 0$ *or coprime with* n*. Let* $|K| = \infty$*. Then* G *contains all the transformations* $z \to z + bx^ky^l, y \to y, x \to x$ *such that* $k + l = n + 1$.

The **proof** is similar to the proof of the corollary 2.14. Note that either n or $n + 1$ is coprime with p so we have

Lemma 2.17. *If* $\mathrm{Char}(K) \neq 2$ *then linear transformations and* $\psi : x \to x, y \to y, z \to z + xy$ *generate all the mappings of the following form* $\alpha_P : x \to x, y \to y, z \to z + P(x, y), P(x, y) \in K[x, y]$.

We have proven 2.12 for three variable case. In order to treat the general case we need one more statement.

Lemma 2.18. *Let* $M(\vec{x}) = a\prod x_i^{k_i}$*,* $a \in K, |K| = \infty$*,* $\mathrm{Char}(K) = 0$ *or* $\mathrm{Char}(K)$ *is coprime with some of* k_i*. Consider linear transformations* $f : x_i \to y_i \sum a_{ij}x_j, \det(a_{ij}) \neq 0$*, and monomials* $M_f = M(\vec{y})$*. Then sums of* M_f *for different* f *contains all polynomials respect to* x_i*.*

The lemma 2.12 follows from the lemma 2.18 similarly as in the proofs of the corollaries 2.14, 2.16.

Proof of the proposition 2.3 for general case.

Let $M = a\prod_{i=1}^{n-1} x_i^{k_i}$ be monomial, $a \in K$. For polynomial $P(x, y) \in K[x, y]$ we define elementary automorphism $\psi_P : x_i \to x_i, i = 1, \ldots, n-$

$1, x_n \to x_n + P(x_1, \dots, x_{n-1})$. $P = \sum M_i$ and ψ_P uniquely decomposes as a product of ψ_{M_i}. Let $\Psi \in \mathrm{Aut}(\mathrm{Aut}(K[x,y,z]))$ stabilizing linear mappings and ϕ. Then according to the corollary 2.8 $\Psi(\psi_P) = \prod \Psi(\psi_{M_i})$. If $M = ax^n$ then due to lemma 2.13

$$\Psi(\psi_M) = \psi_M.$$

We have to prove the same for other type of monomials.

Let $M = a \prod_{i=1}^{n-1} x_i^{k_i}$. Consider automorphism $\alpha : x_i \to x_i + x_1, i = 2, \dots, n-1; x_1 \to x_1, x_n \to x_n$. Then

$$\alpha^{-1}\psi_M\alpha = \psi_{x_1^{k_1}\prod_{i=2}^{n-1}(x_i+x_1)^{k_i}} = \psi_Q\psi_{ax_1^{\sum k_i}}$$

Polynomial $Q = x_1^{k_1}\prod_{i=2}^{n-1}(x_i + x_1)^{k_i} - ax_1^{\sum k_i}) = \sum N_i$, N_i are monomials, no one of them proportional to power of x_1.

Due to the corollary 2.8 $\Psi(\psi_M) = \psi_{bM}$ for some $b \in K$. We have just to prove that $b = 1$. Suppose the contrary, $b \neq 1$. Then

$$\Psi(\alpha^{-1}\psi_M\alpha) = \prod \Psi(\psi_{N_i}) \circ \Psi(\psi_{ax_1^{\sum k_i}}) = \prod \psi_{b_iN_i} \circ \psi_{ax_1^{\sum k_i}}$$

for some $b_i \in K$. From the other hand

$$\Psi(\alpha^{-1}\psi_M\alpha) = \alpha^{-1}\Psi(\psi_M)\alpha = \alpha^{-1}\psi_{bM}\alpha = \prod \Psi(\psi_{bN_i}) \circ \Psi(\psi_{bax_1^{\sum k_i}})$$

It reminds to note that $b \neq 1$ hence $bax_1^{\sum k_i} \neq ax_1^{\sum k_i}$ and we got contradiction. Proposition 2.3 is proven.

2.2. **The automorphisms of tame automorphisms group of $K\langle x_1, \dots, x_n\rangle$, $n > 3$.** Now consider the case of free associative algebra.

For free associative algebra, we note that any automorphism preserving torus action preserves also symmetric $x_1 \to x_1 + \beta(x_2x_3 + x_3x_2), x_i \to x_i, i > 1$ and skew symmetric $x_1 \to x_1 + \beta(x_2x_3 - x_3x_2), x_i \to x_i, i > 1$ elementary automorphisms. First property follows from Lemma 2.9. Second follows from the fact that skew symmetric automorphisms commute with automorphisms of following type $x_2 \to x_2 + x_3^2, x_i \to x_i, i \neq 2$ and this property define them among elementary automorphisms of the type $x_1 \to x_1 + \beta x_2x_3 + \gamma x_3x_2, x_i \to x_i, i > 1$.

Theorem 1.3 follows from the fact that only forms $\beta x_2x_3 + \gamma x_3x_2$ corresponding to multiplication preserving the associative law when either $\beta = 0$ or $\gamma = 0$ and the approximation issue (see section 3).

Proposition 2.19. *Group G, containing all linear transformations and mappings $x \to x, y \to y, z \to z + xy, t \to t$ contains also all the transformations of form $x \to x, y \to y, z \to z + P(x, y), t \to t$.*

Proof. It is enough to prove that G contains all transformations of the following form $x \to x, y \to y, z \to z + aM, t \to t; a \in K$, M is monomial.

Step 1. Let $M = a\prod_{i=1}^{n} x^{k_i}y^{l_i}$ or $M = a\prod_{i=1}^{n} y^{l_0}x^{k_i}y^{l_i}$ or $M = a\prod_{i=1}^{n} x^{k_i}y^{l_i}$ or $M = a\prod_{i=1}^{n} x^{k_i}y^{l_i}x^{k_{n+1}}$. $H(M)$ – *Height of* M is number of different powers of variables needed to compose M. (For example, $H(a\prod_{i=1}^{n} x^{k_i}y^{l_i}x^{k_{n+1}}) = 2n + 1$. Using induction on H one can reduce situation to the case when $M = x^k y$. Let $M = M'x^k$ such that $H(M') < H(M)$. (Case when $M = M'y^l$ is similar.) Let

$$\phi : x \to x, y \to y, z \to z + M', t \to t.$$

$$\alpha : x \to x, y \to y, z \to z, t \to t + zx^k.$$

Then

$$\phi^{-1} \circ \alpha \circ \phi : x \to x, y \to y, z \to z, t \to t + M + zx^k.$$

It is composition of automorphisms $\beta : x \to x, y \to y, z \to z, t \to t + M$ and $\gamma : x \to x, y \to y, z \to z, t \to t + zx^k$. β is conjugated to automorphism $\beta' : x \to x, y \to y, z \to z + M, t \to t$ by linear automorphism $x \to x, y \to y, z \to t, t \to z$, similarly γ is conjugated to automorphism $\gamma' : x \to x, y \to y, z \to z + yx^k, t \to t$. We reduced to the case when $M = x^k$ or $M = yx^k$.

Step 2. Consider automorphisms $\alpha : x \to x, y \to y + x^k, z \to z, t \to t$ and $\beta : x \to x, y \to y, z \to z, t \to t + azy$. Then $\alpha^{-1} \circ \beta \circ \alpha : x \to x, y \to y, z \to z, t \to t + azx^k + azy$. It is composition of automorphism $\gamma : x \to x, y \to y, z \to z, t \to t + azx^k$ which is conjugate to needed automorphism $\gamma' : x \to x, y \to y, z \to z + yx^k, t \to t$, and automorphism $\delta : x \to x, y \to y, z \to z, t \to t + azy$ which is conjugate to the automorphism $\delta' : x \to x, y \to y, z \to z + axy, t \to t$ and then to the automorphism $\delta'' : x \to x, y \to y, z \to z + xy, t \to t$ (using similarities).

Step 3. Obtaining automorphism $x \to x, y \to y + x^n, z \to z, t \to t$. Similar to the commutative case of $k[x_1, \dots, x_n]$.

The proposition 2.19 is proven.

Let us formulate remark after the lemma 2.7 as a statement:

Lemma 2.20. *Consider the following $\mathbb{T}^2$ action: $x_1 \to \lambda^I x_1$, $x_j \to \lambda_j x_j, j > 1$. Then the set S of automorphisms commuting with this action generated with following automorphisms $x_1 \to x_1 + H, x_i \to x_i; i > 1$, H is homogenous polynomial of the same degree as $\prod_{j=2}^n x_j^{i_j}$ ($\lambda = (\lambda_2, \dots, \lambda_n), \beta, \lambda_j \in K$).*

Proposition 2.19 and lemma 2.20 imply

Corollary 2.21. *Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}(K\langle x_1, \dots, x_n\rangle))$ stabilizing all elements of torus and linear automorphisms, $\phi_P : x_n \to x_n + P(x_1, \dots, x_{n-1}), x_i \to x_i, i = 1, \dots, n-1$. Let $P = \sum_I P_I$, P_I – homogenous component of P of multydegree I. Then*

a) $\Psi(\phi_P) : x_n \to x_n + P^\Psi(x_1, \dots, x_{n-1}), x_i \to x_i, i = 1, \dots, n-1$.

b) $P^\Psi = \sum_I P_I^\Psi$; P_I^Ψ – homogenous of multydegree I.

c) If I has positive degree respect to one or two variables, then $P_I^\Psi = P_I$.

Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}(K\langle x_1, \dots, x_n\rangle))$ stabilizing all elements of torus and linear automorphisms, $\phi : x_n \to x_n + P(x_1, \dots, x_{n-1}), x_i \to x_i, i = 1, \dots, n-1$.

Let $\varphi_Q : x_1 \to x_1, x_2 \to x_2, x_i \to x_i + Q_i(x_1, x_2), i = 3, \dots, n-1, x_n \to x_n$; $Q = (Q_3, \dots, Q_{n-1})$. Then $\Psi(\varphi_Q) = \varphi_Q$ due to proposition 2.19.

Lemma 2.22. *a) $\varphi_Q^{-1} \circ \phi_P \circ \varphi_Q = \phi_{P_Q}$, where $P_Q(x_1, \dots, x_{n-1}) = P(x_1, x_2, x_3 + Q_3(x_1, x_2), \dots, x_{n-1} + Q_{n-1}(x_1, x_2))$.*

b) Let $P_Q = P_Q^{(1)} + P_Q^{(2)}$, $P_Q^{(1)}$ consists of all terms containing one of the variables $x_3, \dots, x_{n-1}$, $P_Q^{(1)}$ consists of all terms containing just variables x_1, x_2. Then $P_Q^\Psi = P_Q^\Psi = P_Q^{(1)\Psi} + P_Q^{(2)\Psi} = P_Q^{(1)\Psi} + P_Q^{(2)}$.

Lemma 2.23. *If $P_Q^{(2)} = R_Q^{(2)}$ for all Q then $P = R$.*

Proof. It is enough to prove that if $P \neq 0$ then $P_Q^{(2)} \neq 0$ for appropriate $Q = (Q_3, \dots, Q_{n-1})$. Let $m = \deg(P), Q_i = x_1^{2^{i+1}m} x_2^{2^{i+1}m}$. Let

$\hat{P}$ highest component of P, then $\hat{P}(x_1, x_2, Q_3, \dots, Q_{n-1})$ is the highest component of $P_Q^{(2)}$. It is enough to prove that $\hat{P}(x_1, x_2, Q_3, \dots, Q_{n-1}) \neq 0$. Let $x_1 \prec x_2 \prec x_2 \prec \cdots \prec x_{n-1}$. Consider lexicographically minimal term M of $\hat{P}$. It is easy to see that the term $M|_{Q_i \to x_i}, i = 3, n-1$ can not be cancel with any other term $N|_{Q_i \to x_i}, i = 3, n-1$ of $\hat{P}(x_1, x_2, Q_3, \dots, Q_{n-1})$. Hence $\hat{P}(x_1, x_2, Q_3, \dots, Q_{n-1}) \neq 0$.

Lemmas 2.22, 2.23 imply

Corollary 2.24. *Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}(K\langle x_1, \dots, x_n\rangle))$ stabilizing all elements of torus and linear automorphisms. Then $P^\Psi = P$ and Ψ stabilizes all elementary automorphisms, hence $\mathrm{TAut}(K\langle x_1, \dots, x_n\rangle)$.*

We get following

Proposition 2.25. *Let $n \geq 4$. Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}(K\langle x_1, \dots, x_n\rangle))$ stabilizing all elements of torus and linear automorphisms and automorphism $EL : x_i \to x_i; i = 1, \dots, x_{n-1}, x_n \to x_n + x_1 x_2$. Then $\Psi = \mathrm{Id}$.*

Let $n \geq 4$. Let $\Psi \in \mathrm{Aut}(\mathrm{TAut}(K\langle x_1, \dots, x_n\rangle))$ stabilizing all elements of torus and linear automorphisms. We have to prove that $\Psi(EL) = EL$ or $\Psi(EL) : x_i \to x_i; i = 1, \dots, x_{n-1}, x_n \to x_n + x_2 x_1$. In the last case Ψ is conjugation with mirror antiautomorphism of $K\langle x_1, \dots, x_n\rangle$. In any case $\Psi(EL) : x_i \to x_i; i = 1, \dots, x_{n-1}, x_n \to x_n + x_1 * x_2$, where $x * y = axy + byx; a, b \in K$.

Next lemma can be obtained by direct computation:

Lemma 2.26. *$*$ is associative iff $a = 0$ or $b = 0$. Moreover, $x^2 * y = (x * (x * y))$ iff $ab = 0$.*

It mean that $*$ is either associative or not alternative operation.

Consider automorphisms $\alpha : x \to x, y \to y, z \to z + xy, t \to t$, $\alpha : x \to x, y \to y, z \to z + xy, t \to t + xz$, $h : x \to x, y \to y, z \to z, t \to t - xz$. Then $\gamma = h\alpha^{-1}\beta\alpha : (x \to x, y \to y, z \to z, t \to t + xxy)$ and $\Psi(\gamma) : x \to x, y \to y, z \to z, t \to t + x * (x * y)$.

Let $\delta : x \to x, y \to y, z \to z + x^2, t \to t$, $\epsilon : x \to x, y \to y, z \to z, t \to t + zy$. Then $\epsilon^{-1}\delta^{-1}\epsilon\delta : x \to x, y \to y, z \to z, t \to t + x^2 y$. From the other hand $\varepsilon = \Psi(\epsilon^{-1}\delta^{-1}\epsilon\delta) : x \to x, y \to y, z \to z, t \to t + (x^2) * y$. We also have $\varepsilon = \gamma$. Equality $\Psi(\varepsilon) = \Psi(\gamma)$ is equivalent to the equality $x * (x * y) = x^2 y$. Theorem is proven.

2.3. **Automorphisms of** $\mathrm{TAut}(k\langle x, y, z\rangle)$**.** In this section we shall determine $\mathrm{TAut}(k\langle x, y, z\rangle)$ on Ind-scheme level and prove theorem 1.8. We use approximation results of the section 3. We suppose here that $\mathrm{Char}(k) \neq 2$. $\{a, b, c\}_*$ denotes associator of a, b, c respect to operation $*$, i.e. $\{a, b, c\}_* = (a * b) * c - a * (b * c)$. Let $\Psi \in \mathrm{TAut}(k\langle x, y, z\rangle)$ Ind-scheme automorphism, stabilizing linear automorphisms.

Lemma 2.27. *Let* $\Psi \in \mathrm{Aut}_{\mathrm{Ind}}(\mathrm{TAut}(k\langle x, y, z\rangle))$ *stabilizing linear automorphisms. Let* $\phi : x \to x, y \to y, z \to z + xy$*. Then* $\Psi(\phi) : x \to x, y \to y, z \to z + axy$ *or* $\Psi(\phi) : x \to x, y \to y, z \to z + ayx$*.*

Proof. Consider automorphism $t : x \to x, y \to y, z \to z + xy$. Then $\Psi(t) : x \to x, y \to y, z \to z + x * y$, $x * y = axy + byx$. Due to conjugation on the mirror antiautomorphism and coordinate exchange one can suppose that $x * y = xy + \lambda yx$. We have to prove that $\lambda = 0$. In that case $\Psi = \mathrm{Id}$.

Lemma 2.28. *Let* $A = k\langle x, y, z\rangle$*. Let* $a*b = ab+\lambda ba$*. Then* $\{a, b, c\}_* = \lambda[b, [a, c]]$*.*

In particular $\{a, b, a\}_* = 0$, $a * (a * b) - (a * a) * b = -\{a, a, b\}_* = \lambda[a, [a, b]]$, $(b * a) * a - b * (a * a) = \{b, a, a\}_* = \lambda[a, [a, b]]$.

Proof. $\{a, b, c\}_* = (a*b)*c-a*(b*c) = (ab+\lambda ba)c+\lambda c(ab+\lambda ba) = \lambda(bac + cab - acb - bca) = \lambda([b, ac] + [ca, b]) = \lambda[b, [ac]]$.

Lemma 2.29. *Let* $\varphi_1 : x \to x + yz, y \to y, z \to z$, $\varphi_2 : x \to x, y \to y, z \to z + yx$, $\varphi = \varphi_2^{-1}\varphi_1^{-1}\varphi_2\varphi_1$*. Then modulo terms of order* ≥ 4 *we have:*

$\varphi : x \to x-y^2x, y \to y, z \to z+y^2z$ *and* $\Psi(\varphi) : x \to x-y*(y*x), y \to y, z \to z + y * (y * z)$*.*

Proof. Direct computation.

Lemma 2.30. *a) Let* $\phi_l : x \to x, y \to y, z \to z + y^2x$*. Then* $\Psi(\phi) : x \to x, y \to y, z \to z + y * (y * x)$*.*

b) Let $\phi_r : x \to x, y \to y, z \to z + xy^2$*. Then* $\Psi(\phi) : x \to x, y \to y, z \to z + (x * y) * y$*.*

Proof. According results of the previous section we have $\Psi(\phi_l)$: $x \to x, y \to y, z \to z+P(y,x)$ where $P(y,x)$ is polynomial homogenous of degree 2 respect to y and degree 1 respect to x. We have to prove that $H(y,x) = P(y,x) - y*(y*x) = 0$.

Let $\tau : x \to z, y \to y, z \to x; \tau = \tau^{-1}, \phi' = \tau\phi_l\tau^{-1} : x \to x+y^2z, y \to y, z \to z$. Then $\Psi(\phi_l') : x \to x + P(y,z), y \to y, z \to z$.

Let $\phi_l" = \phi_l\phi_l' : x \to x + P(y,z), y \to y, z \to z + P(y,x)$ modulo terms of degree ≥ 4.

Let $t : x \to x - z, y \to y, z \to z$, φ_2, φ – automorphisms described in the lemma 2.29.

Then $T = t^{-1}\phi_l^{-1}t\phi_l'' : x \to x, y \to y, z \to z$ modulo terms of order ≥ 4.

From the other hand $\Psi(T) : x \to x + H(y,z) - H(y,x), y \to y, z \to z + P$ modulo terms of order ≥ 4. Because $\deg_y(H(y,x) = 2, \deg_x(H(y,x)) = 1$ we get $H = 0$.

the proof of part b) is pretty similar.

Lemma 2.31. *a) Let $\psi_1 : x \to x + y^2, y \to y, z \to z$, $\psi_2 : x \to x, y \to y, z \to z + x^2$. Then $[\psi_1, \psi_2] = \psi_2^{-1}\psi_1^{-1}\psi_2\psi_1 : x \to x, y \to y, z \to z+y^2x+xy^2$, $\Psi([\psi_1,\psi_2]) : x \to x, y \to y, z \to z+(y*y)*x+x*(y*y)$.*

b) $\phi_l^{-1}\phi_r^{-1}[\psi_1,\psi_2] : x \to x, y \to y, z \to z$ modulo terms of order ≥ 4 but $\Psi(\phi_l^{-1}\phi_r^{-1}[\psi_1,\psi_2]) : x \to x, y \to y, z \to z+(y*y)*x+x*(y*y) - (x*y)*y - y*(y*x) = z + 4\lambda[x[x,y]]$ modulo terms of order ≥ 4

Proof. a) can be obtained by direct computation. b) follows from a) and the lemma 2.28.

Lemma 2.27 follows from the lemma 2.31.

Consider an automorphism

$\phi : x \to x, y \to y, z \to z + P(x,y)$. let $\Psi \in \mathrm{TAut}(k\langle x,y,z\rangle)$. Then $\Psi(\phi) : x \to x, y \to y, z \to z + Q(x,y)$. We denote $\Psi(P) = Q$. We have to prove that $\Psi(P) = P$ for all P if Ψ stabilizes linear automorphisms and $\Psi(xy) = xy$.

Lemma 2.32.

$$\Psi(x^ky^l) = x^ky^l$$

Proof. Let $\phi : x \to x, y \to y, z \to z + x^k y^l$, $\varphi_1 : x \to x + y^l, y \to y, z \to z$, $\varphi_2 : x \to x, y \to y + x^k, z \to z$, $\varphi_3 : x \to x, y \to y, z \to z + xy$, $h : x \to x, y \to y, z \to z - x^{k+1}$. Then

$$h\varphi_3^{-1}\varphi_1^{-1}\varphi_2^{-1}\varphi_3\varphi_1\varphi_2 : x \to x, y \to y, z \to z + x^k + y^l + N;$$

N is sum of terms of degree strictly more then $k + l$.

Applying Ψ we get the result because $\Psi(\varphi_i) = \varphi_i, i = 1, 2, 3$ and $\varphi(H_n) \subseteq H_n$. The lemma is proven.

Let

$$M_{k_1,\dots,k_s} = x^{k_1} y^{k_2} \cdots y^{k_s}$$

for even s and

$$M_{k_1,\dots,k_s} = x^{k_1} y^{k_2} \cdots x^{k_s}$$

for odd s, $k = \sum_{i=1}^n k_i$. Then

$$M_{k_1,\dots,k_s} = M_{k_1,\dots,k_{s-1}} y^{k_s}$$

for even s and

$$M_{k_1,\dots,k_s} = M_{k_1,\dots,k_{s-1}} x^{k_s}$$

for odd s.

We have to prove that $\Psi(M_{k_1,\dots,k_s}) = M_{k_1,\dots,k_s}$. We can suppose by induction that $\Psi(M_{k_1,\dots,k_{s-1}}) = M_{k_1,\dots,k_{s-1}}$.

For any monomial $M = M(x, y)$ we shall define an automorphism $\varphi_M : x \to x, y \to y, z \to z + M$.

We also define automorphisms $\phi_k^e : x \to x, y \to y + zx^k, z \to z$ and $\phi_k^o : x \to x + zy^k, y \to y, z \to z$. We shall treat case of even s. Odd case is similar.

Let $D^e_{zx^k}$ be derivation of $k\langle x, y, z\rangle$ such that $D^e_{zx^k}(x) = 0$, $D^e_{zx^k}(y) = zx^k$, $D^e_{zx^k}(z) = 0$. Similarly $D^o_{zy^k}$ be derivation of $k\langle x, y, z\rangle$ such that $D^o_{zy^k}(y) = 0$, $D^o_{zx^k}(x) = zy^k$, $D_{zy^k}(z)^o = 0$.

Next lemma can be obtained via direct computation:

Lemma 2.33. *Let*

$$u = \phi_{k_s}^{e\ -1}\varphi(M_{k_1,\dots,k_{s-1}})^{-1}\phi_{k_s}^e\varphi(M_{k_1,\dots,k_{s-1}})$$

for even s and

$$u = {\phi^o_{k_s}}^{-1}\varphi(M_{k_1,\dots,k_{s-1}})^{-1}\phi^o_{k_s}\varphi(M_{k_1,\dots,k_{s-1}})$$

for add s. Then

$$u : x \to x, y \to y + M_{k_1,\dots,k_s}, z \to z + D^e_{zx^k}(M_{k_1,\dots,k_{s-1}}) + N$$

for even s and

$$u : x \to x + M_{k_1,\dots,k_s}, y \to y, z \to z + D^o_{zx^k}(M_{k_1,\dots,k_{s-1}}) + N$$

for odd s.

where N is sum of terms of degree strictly more then $k = \sum_{i=1}^{s} k_i$.

Let $\psi(M_{k_1,\dots,k_s}) : x \to x, y \to y, z \to z + M_{k_1,\dots,k_s}$, $\alpha_e : x \to x, y \to y - z, z \to z$, $\alpha_o : x \to x - z, y \to y, z \to z$, $P_M = \Psi(M) - M$.

Next two lemmas also can be obtained via direct computation:

Let

$$v = \alpha_e^{-1}\psi(M_{k_1,\dots,k_s})^{-1}\alpha_e\psi(M_{k_1,\dots,k_s})u$$

for even s and

$$v = \alpha_o^{-1}\psi(M_{k_1,\dots,k_s})^{-1}\alpha_o\psi(M_{k_1,\dots,k_s})u$$

for odd s.

Lemma 2.34. *a)*

$$v : x \to x, y \to y + H, z \to z + H_1 + H_2$$

for even s and

$$v : x \to x + H, y \to y, z \to z + H_1 + H_2$$

for odd s

b)

$$\Psi(v) : x \to x, y \to y + P_{M_{k_1,\dots,k_s}} + H, z \to z + H_1 + H_2$$

for even s and

$$\Psi(v) : x \to x + P_{M_{k_1,\dots,k_s}} + H, y \to y, z \to z + H_1 + H_2$$

for odd s

where H_2 is sum of terms of degree strictly more then $k = \sum_{i=1}^{s} k_i$, H_1 is sum of terms of degree k and degree 1 respect to z.

Let $\beta : x \to x, y \to y, z \to 2z$, $w = v^{-2}\beta^{-1}v\beta$.

Lemma 2.35. *a)*

$$w : x \to x, y \to y - R, z \to z + H_1 + T$$

for even s and

$$w : x \to x + R, y \to y, z \to z + H_1 + T$$

for odd s.

b)

$$\Psi(w) : x \to x, y \to y - P_{M_{k_1,\dots,k_s}} + R, z \to z + H_1 + T$$

for even s and

$$\Psi(w) : x \to x - P_{M_{k_1,\dots,k_s}} + R, y \to y, z \to z + H_1 + T$$

for odd s.

where R and T is sums of terms of degree strictly more then k.

Main result of this section follows from the lemma 2.35 and the following lemma:

Lemma 2.36. *Let P is sum of terms of degree $> d$, Q is sum of terms of degree $\geq d$, $\gamma \in \mathrm{TAut}(k\langle x, y, z\rangle)$ such that*

$$\gamma : x \to x, y \to y + P, z \to z + Q$$

.

Then $\Psi(\gamma) : x \to x, y \to y + P_1, z \to z + Q_1$ where P_1 is sum of terms of degree $> d$, Q_1 is sum of terms of degree $\geq d$.

Proof. Let $\delta : x \to x, y \to y, z \to z + y$. Then δ commutes with γ modulo H_d and δ does not commutes with γ modulo H_d if P_1 contains term of degree $\leq d$ (because results of the section 3).

3. The approximation issue

Now we have to use approximations. This is most important tool of the paper. In order to do it we have to prove that φ preserves structure of augmentation subgroups. We treat here affine case. For symplectomorphisms situation is more complicate.

Theorem 3.1. *$\varphi(H_n) \subseteq H_n$ where H_n is subgroup of elements identity modulo ideal $(x_1, \ldots, x_k)^n$*

Theorem 3.2. *$\varphi(H_n) \subseteq H_n$ where H_n is subgroup of elements identity modulo ideal $(x_1, \ldots, x_k)^n$ also for free associative case.*

Corollary 3.3. $\varphi = \mathrm{Id}$.

Proof. Every automorphism can be approximated via tame ones. i.e. for any ψ and any n there exists a tame automorphism ψ'_n such that $\psi{\psi'_n}^{-1} \in H_n$.

In fact this theorem implies group none lifting, because elementary actions determine a coordinates and we have an approximations.

So the main point is **Why** $\varphi(H_n) \subseteq H_n$**.**

Proof of the Theorem 3.1. Consider matrix $A(t)$ dependent on parameter t such that eigenvalues are t^{n_i} and $n_i k \leq n_j$. $\varphi(A(t)) = A(t)$, because φ preserves linear transformations.

Definition 3.4. The ideal I generated by variables x_i is the *augmentation ideal.* The *augmentation subgroup* H_n is group of all automorphisms φ such that $\varphi(x_i) \equiv x_i \mod I^n$. The set $G_n \supset H_n$ is a group of automorphisms whose linear part is scalar, and $\varphi(x_i) \equiv \lambda x_i \mod I^n$ (λ does not dependant on i).

This follows from the next two lemmas.

Lemma 3.5. *Let M be an automorphism of free associative-commutative algebra. Then $A(t)MA(t)^{-1}$ has no singularities i.e. is affine curve for $t = 0$ for any $A(t)$ with property*

$A(t)$ dependent on parameter t such that eigenvalues are t^{n_i} and $n_i k \leq n_j$ $\qquad (*)$

iff $M \in \hat{H}_n$ where $\hat{H}_n$ is homothety modulo the augmentation ideal.

Proof. The 'If' part is obvious, because the sum $\sum_{j=1}^{k} n_{i_j}$ is greater then n_m and homothety commutes with linear map hence conjugation of the homothety via linear map is itself.

We have to prove that if linear part of φ does not satisfy condition $(*)$ then $A(t)MA(t)^{-1}$ has a singularity in $t = 0$.

Case 1. The linear part $\bar{M}$ of M is not a scalar matrix. Then after basis change it is not a diagonal matrix and has a non zero coefficient in i, j position E_{ij}. Consider diagonal matrix $A(t) = D(t)$ such that on all position on main diagonal except j-th it has t^{n_i} and on j-th position t^{n_j}. Then $D(t)\bar{M}D^{-1}(t)$ has (i, j) entry with coefficient $\lambda t^{n_i - n_j}$ and if $n_j > n_i$ it has singularity at $t = 0$.

Let also $n_i < 2n_j$. Then non-linear part of M does not produce singularity and can not compensate with linear part singularity so we are done in the case 1.

Case 2. The linear part $\bar{M}$ of M is a scalar matrix. Then conjugation of linear part can not produce singularities and we are interested just in smallest non linear term. Let $\varphi \in H_k \backslash H_{k+1}$. Due to linear base exchange we can assume that $\varphi(x_1) = \lambda \cdot x_1 + \delta x_2^k + S$, where S is sum of monomials of degree $\geq k$ different from x_2^k with coefficients in K.

Let $A(t) = D(t)$ be a diagonal matrix of the form $(t^{n_1}, t^{n_2}, t^{n_1}, \ldots, t^{n_1})$. Let $(k+1) \cdot n_2 > n_1 > k \cdot n_2$. Then in $A^{-1}MA$ term δx_2^k will be transformed in $\delta x_2^k t^{kn_2 - n_1}$ all other terms produce power $t^{ln_2 + sn_1 - n_1}$ such that $(l, s) \neq (1, 0), l, s > 0$. In this case $ln_2 + sn_1 - n_1 > 0$ and we are done with the proof of Lemma 3.5.

The next lemma can be proved by concrete calculations:

Lemma 3.6.

a) $[G_n, G_n] \subset H_n, n > 2$. *There exist elements* $\varphi \in H_n \backslash H_{n+k-1}, \psi_1 \in G_k, \psi_2 \in G_n$, *such that* $\varphi = [\psi_1, \psi_2]$.

b) $[H_n, H_k] \subset H_{n+k-1}$.

c) Let $\varphi \in G_n \backslash H_n$, $\psi \in H_k \backslash H_{k+1}$, $k > n$. *Then* $[\varphi, \psi] \in H_k \backslash H_{k+1}$.

Proof. a) Let $\psi_1 : x \to x + y^k, y \to y$, $\psi_2 : x \to x, y \to y + x^n$, $\psi_i, i = 1, 2$ stable on other variables. Then $\varphi = [\psi_1, \psi_2] = \psi_1^{-1}\psi_2^{-1}\psi_1\psi_2 : x \to x + (y + x^n)^k - (y + x^n - (x + (y + x^n)^k)^n)^k, y \to y + x^n - (x + (y + x^n)^k)^n$.

It is easy to see that if either k or n coprime with $\mathrm{Char}(K)$, then all the terms of degree $k+n-1$ does not cancel and $\varphi \in H_{n+k-1} \backslash H_{n+k}$.

Now suppose that n is not coprime with $\mathrm{Char}(K)$, then $n-1$ is coprime with $\mathrm{Char}(K)$. Consider mappings $\psi_1 : x \to x + y^k, y \to y$, $\psi_2 : x \to x, y \to y + x^{n-1}z$, $\psi_i, i = 1, 2$ stable on other variables. Then $\varphi = [\psi_1, \psi_2] = \psi_1^{-1}\psi_2^{-1}\psi_1\psi_2 : x \to x + (y + zx^{n-1})^k - (y + zx^{n-1} - (x + z(y + x_n)^k)^{n-1})^k, y \to y + zx^{n-1} - z(x + (y + zx^{n-1})^k)^{n-1} = y + (n-1)y^k x^{n-1} + o$. o means sum of terms of degree $\geq n+k$. We see that $\varphi \in H_{n+k-1} \backslash H_{n+k}$.

b) Let $\psi_1 : x_i \to x_i + f_i; \psi_2 : x_i \to x_i + g_i; i = 1, \dots, n; \deg(f_i) \geq n, \deg(g_i) \geq k$. Then modulo terms of degree $\geq n+k$ we have $\psi_1\psi_2 : x_i \to x_i + f_i + g_i + \partial f_i/\partial x_j g_j$, and modulo terms of degree $\geq n+k-1$ we have $\psi_1\psi_2 : x_i \to x_i + f_i + g_i$ and $\psi_2\psi_1 : x_i \to x_i + f_i + g_i$, i.e. ψ_1, ψ_2 commute.

Corollary 3.7. *Let $\Psi \in \mathrm{Aut}(\mathrm{Aut}(K[x_1, \dots, x_n]))$. Then $\Psi(G_n) = G_n$, $\Psi(H_n) = H_n$.*

Corollary 3.7 and Proposition 2.3 imply Theorem 3.1 because every automorphism can be approximated via tame ones.

4. Lifting of the automorphism group

Theorem 4.1. *Any effective action of torus $\mathbb{T}^n$ on $\mathbb{K}\langle x_1, \dots, x_n\rangle$ is linearizable. That is, it is conjugated to a standard one.*

Proof. Similar to the proof of Theorem 2.1.

As a consequence of the above theorem, we get

Proposition 4.2. *Let T^n be standard torus action. Let $\widehat{T}^n$ its lifting to automorphism group of the free algebra. Then $\widehat{\mathbb{T}}^n$ is also standard torus action.*

Proof. Consider the roots $\widehat{x_i}$ of this action. They are liftings of the coordinates x_i. We have to prove that they generate the whole associative algebra.

Due to the reducibility of this action, all elements are product of eigenvalues of this action. Hence it is enough to prove that eigenvalues of this action can be presented as linear combination of this action. This can be done like Byalitsky Birula paper [10]. Note that all propositions of previous section holds for free associative algebra. Proof of the theorem 3.2 is pretty similar. Hence we have the following

Theorem 4.3. *Any* Ind-*scheme automorphism* φ *of* $\mathrm{Aut}(K\langle x_1, \dots, x_n\rangle)$ *for* $n \geq 3$ *is inner, i.e. is a conjugation via some automorphism.*

Hence the group lifting (under the sense of isomorphism induced by the natural abelianization) implies the analogue of Theorem 3.1.

This also implies that the group lifting satisfies the approximation properties.

Proposition 4.4. *Let* $H = \mathrm{Aut}(K[x_1, \dots, x_n])$, $G = \mathrm{Aut}(K\langle z_1, \dots, z_n\rangle)$. *Suppose* $\Psi : H \to G$ *be a group homomorphism such that its composition with natural projection is the identity map. Then*

(1) *After some coordinate change* ψ *provide correspondence between standard torus actions* $x_i \to \lambda_i x_i$ *and* $z_i \to \lambda_i z_i$.
(2) *Images of elementary automorphisms*

$$x_j \to x_j, j \neq i,\ x_i \to x_i + f(x_1, \dots, \widehat{x_i}, \dots, x_n)$$

are elementary automorphisms of the form

$$z_j \to z_j, j \neq i,\ z_i \to z_i + f(z_1, \dots, \widehat{z_i}, \dots, z_n).$$

(Hence image of tame automorphism is tame automorphism).
(3) $\psi(H_n) = G_n$. *Hence* ψ *induces map between completion of the groups of* H *and* G *respect to augmentation subgroup structure.*

Proof of Theorem 1.1

Any automorphism, including the Nagata automorphism can be approximated via product of elementary automorphisms with respect to augmentation topology. In the case of the Nagata automorphism corresponding to

$$\mathrm{Aut}(K\langle x_1, \dots, x_n\rangle),$$

all such elementary automorphisms fix all coordinates except x_1, x_2, Due to (2) and (3) of Proposition 4.4, the lifted automorphism would be an automorphism induced by automorphism of $K\langle x_1, x_2, x_3\rangle$ fixing z_3. However, it is impossible to lift the Nagata automorphism to such an automorphism due to the main result of [6]. Therefore, Theorem 1.1 is proved.

5. Acknowledgement

The authors would like to thank to J.P. Furter, T. Kambayashi, R.Lipyanski, Boris Plotkin, Eugeny Plotkin, Andrey Regeta, Eliahu Rips and M. Zaidenberg for stimulating discussion.

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