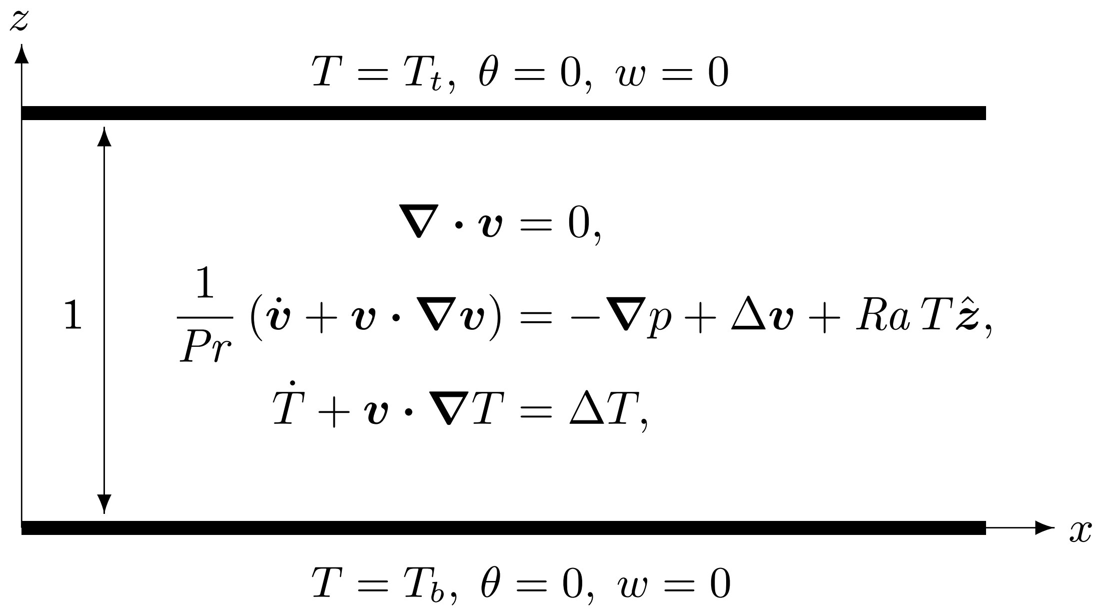
Rayleigh-Bénard convection with free-slip boundaries:



$$T = T_t, \ \theta = 0, \ w = 0$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0,$$

$$\frac{1}{Pr}\left(\dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \Delta \boldsymbol{v} + Ra\, T \hat{\boldsymbol{z}},$$

$$\dot{T} + \boldsymbol{v} \cdot \boldsymbol{\nabla} T = \Delta T,$$

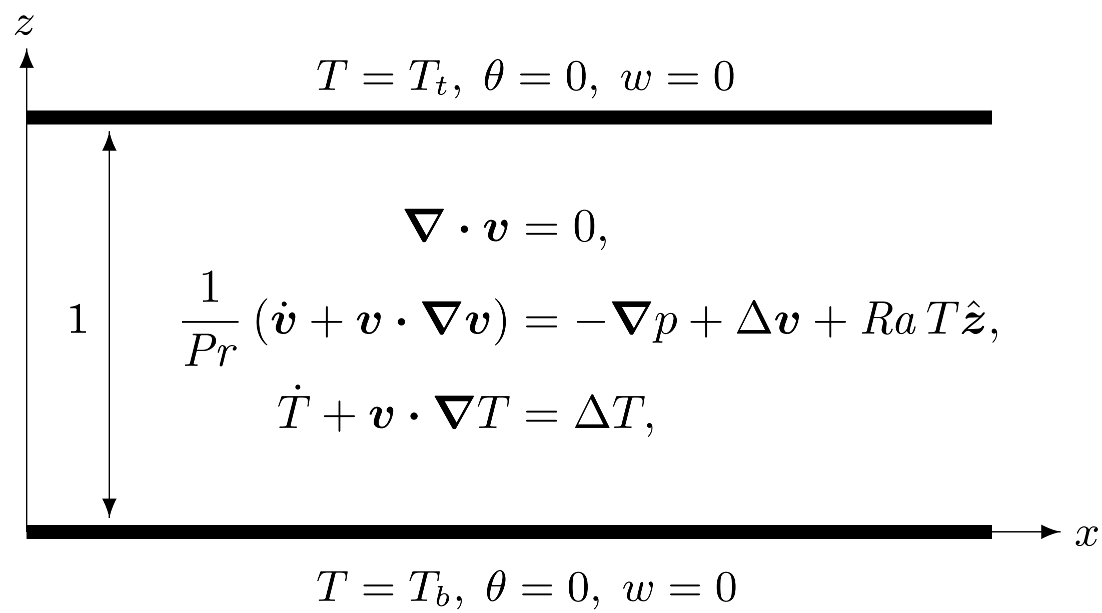
$$T = T_b, \ \theta = 0, \ w = 0$$

$$\left\langle |\boldsymbol{\nabla} \boldsymbol{v}|^2 \right\rangle = Ra \ \langle wT \rangle$$

$$Pe^2 = Ra\,(Nu - 1)$$

Optimal: $Nu_{\max}(Ra, \mathit{\Gamma}) = 1 + (K(\mathit{\Gamma}))^{4/3}\, Ra^{1/3}$

Rayleigh-Bénard convection with free-slip boundaries:



$$Nu_{\mathrm{MAX}}(Pe) = 1 + 0.2175\, Pe^{0.58}$$

$$0.5882\cdots = 10/17 \ \mathbf{?}$$

If so, then: $Nu_{\mathrm{MAX}}(Ra) = 1 + 0.1152\, Ra^{5/12}$

$$\Gamma_{\mathrm{opt}}(Ra) \sim Ra^{-0.25}$$

PHYSICS OF FLUIDS **21**, 083603 (2009)

Large Rayleigh number thermal convection: Heat flux predictions and strongly nonlinear solutions

Gregory P. Chini[1,a)] and Stephen M. Cox[2]
[1]*Department of Mechanical Engineering, University of New Hampshire, Durham, New Hampshire 03824, USA*
[2]*School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom*

Nu ~ $\mathrm{Ra}^{1/3}$ for ***steady*** stress-free RB convection

PHYSICS OF FLUIDS **27**, 051702 (2015)

Heat transport by coherent Rayleigh-Bénard convection

Fabian Waleffe,[1,2,a)] Anakewit Boonkasame,[1,b)] and Leslie M. Smith[1,2,c)]
[1]*Department of Mathematics, UW-Madison, Madison, Wisconsin 53706, USA*
[2]*Department of Engineering Physics, UW-Madison, Madison, Wisconsin 53706, USA*

Nu ~ Ra$^{.31}$ for steady ***no-slip*** RB convection

Heat transport by coherent Rayleigh-Bénard convection

Fabian Waleffe,[1,2,a)] Anakewit Boonkasame,[1,b)] and Leslie M. Smith[1,2,c)]
[1]*Department of Mathematics, UW-Madison, Madison, Wisconsin 53706, USA*
[2]*Department of Engineering Physics, UW-Madison, Madison, Wisconsin 53706, USA*

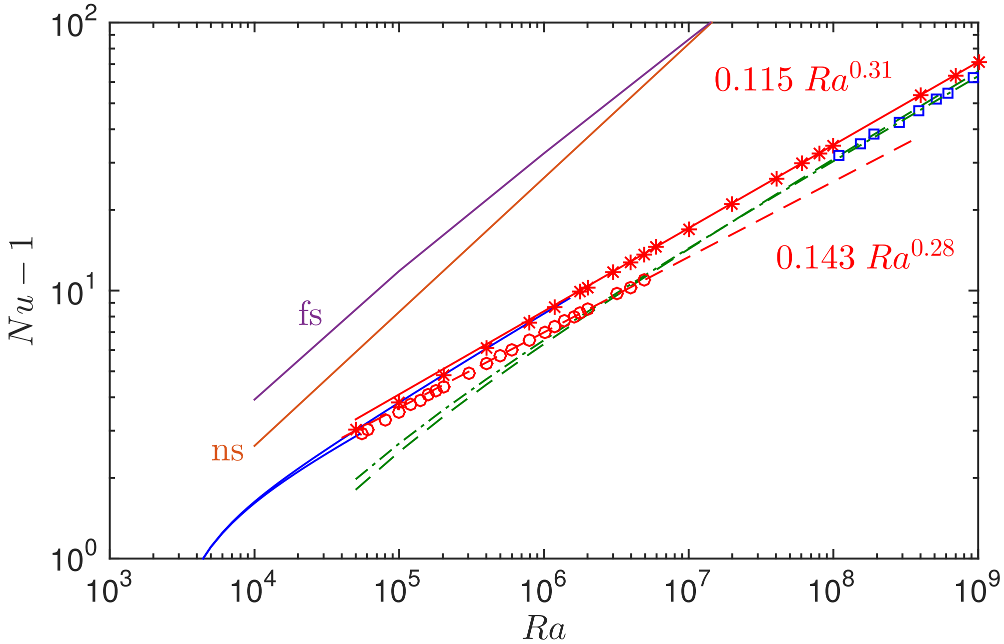


FIG. 2. $Nu - 1$ vs. Ra for *primary branch* ($\alpha = \pi/2$, red $\circ$'s) with $Nu - 1 \sim 0.143\,Ra^{0.28}$ (red dashed line) and *optimum branch* (red $*$'s) with $Nu - 1 \sim 0.115\,Ra^{0.31}$ (least square fit in $10^7 < Ra \leq 10^9$, red solid). Lower (green) dashed line is the 3D turbulent data fit[18] $Nu \sim 0.088\,Ra^{0.32}$ and (green) dashed-dotted line is the 3D turbulent data fit[16] $Nu \sim 0.105\,Ra^{0.312}$, both for domain aspect ratio 1/2. The blue $\square$'s for $Ra > 10^8$ is the aspect ratio 4 data[18] (Table 1, Nu_{corr}). Line "fs" is the best free-slip upper bound[29] $Nu - 1 \lesssim 0.106\,Ra^{5/12}$. Line "ns" is the best no-slip upper bound[23] $Nu - 1 \lesssim 0.026\,34\,Ra^{1/2}$.

Wall to Wall Optimal Transport

- What are the next steps?

- **Time dependent flows ...**

Physics Letters A 379 (2015) 518–523

Contents lists available at ScienceDirect

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Maximal transport in the Lorenz equations

Andre N. Souza [a], Charles R. Doering [a,b,c,*]

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ABSTRACT

We derive rigorous upper bounds on the transport $\langle XY \rangle$ where $\langle \cdot \rangle$ indicates time average, for solutions of the Lorenz equations without assuming statistical stationarity. The bounds are saturated by nontrivial steady (albeit often unstable) states, and hence they are sharp. Moreover, using an optimal control formulation we prove that no other flow protocol of the same strength, i.e., no other function of time $X(t)$ driving the $Y(t)$ and $Z(t)$ variables while satisfying the basic balance $\langle X^2 \rangle = \langle XY \rangle$, produces higher transport.

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$$\dot{X} = \sigma Y - \sigma X$$

$$\dot{Y} = rX - Y - XZ$$

$$\dot{Z} = XY - bZ \qquad Nu \sim \langle XY \rangle$$

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$$X(t) = \text{control w}/Pe^2 = \left\langle X^2 \right\rangle$$

$$\dot{Y} = rX - Y - XZ$$

$$\dot{Z} = XY - bZ \qquad Nu \sim \left\langle XY \right\rangle$$

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Maximal transport in the Lorenz equations

Theorem: *Steady flows produce absolute optimal transport … for Lorenz and control!*

Andre N. Souza [a], Charles R. Doering [a,b,c,*]

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Higher order reduced models

Phys. Fluids **8** (7), July 1996

Energy-conserving truncations for convection with shear flow

Jean-Luc Thiffeault[a)] and Wendell Horton[b)]
Institute for Fusion Studies and University of Texas, Austin, Texas 78712-1060

(Received 23 October 1995; accepted 27 March 1996)

7-mode model ...

JOURNAL OF THE ATMOSPHERIC SCIENCES VOLUME 59

Selection of Modes in Convective Low-Order Models

ALEXANDER GLUHOVSKY, CHRISTOPHER TONG, AND ERNEST AGEE
Department of Earth and Atmospheric Sciences, Purdue University, West Lafayette, Indiana

(Manuscript received 1 November 2000, in final form 2 October 2001)

8-mode model ...

Physica D 308 (2015) 26–33

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Transport bounds for a truncated model of Rayleigh–Bénard convection

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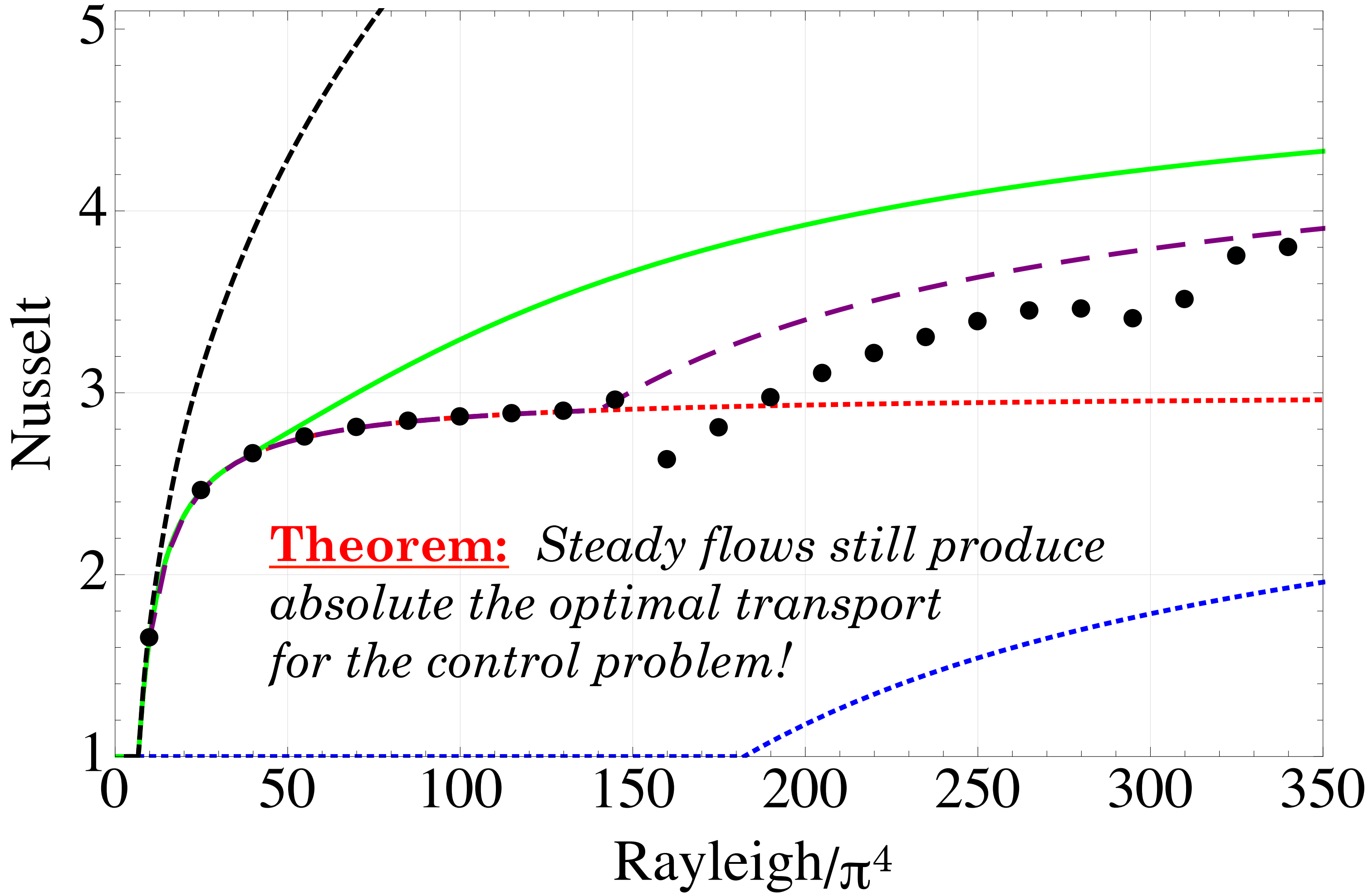
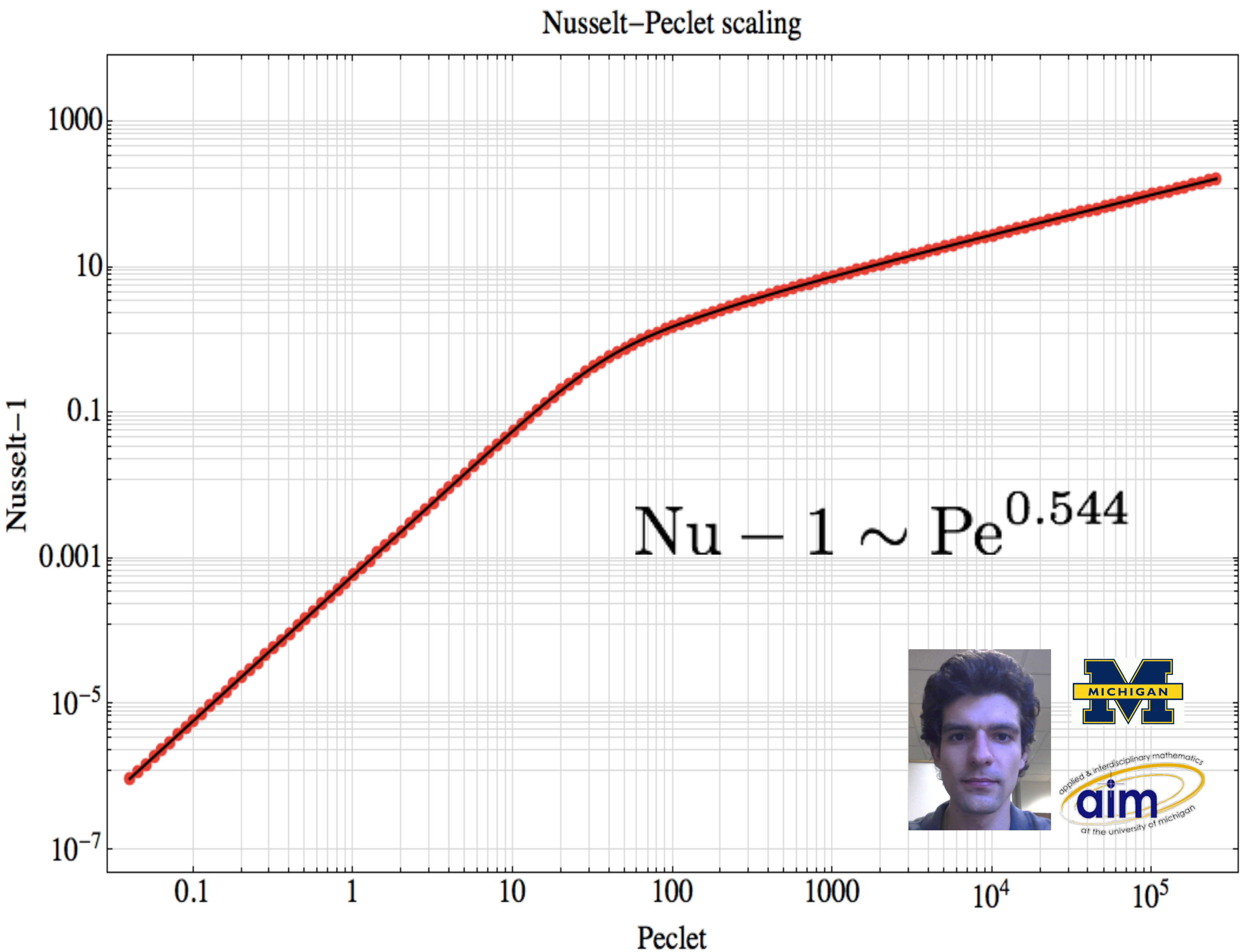


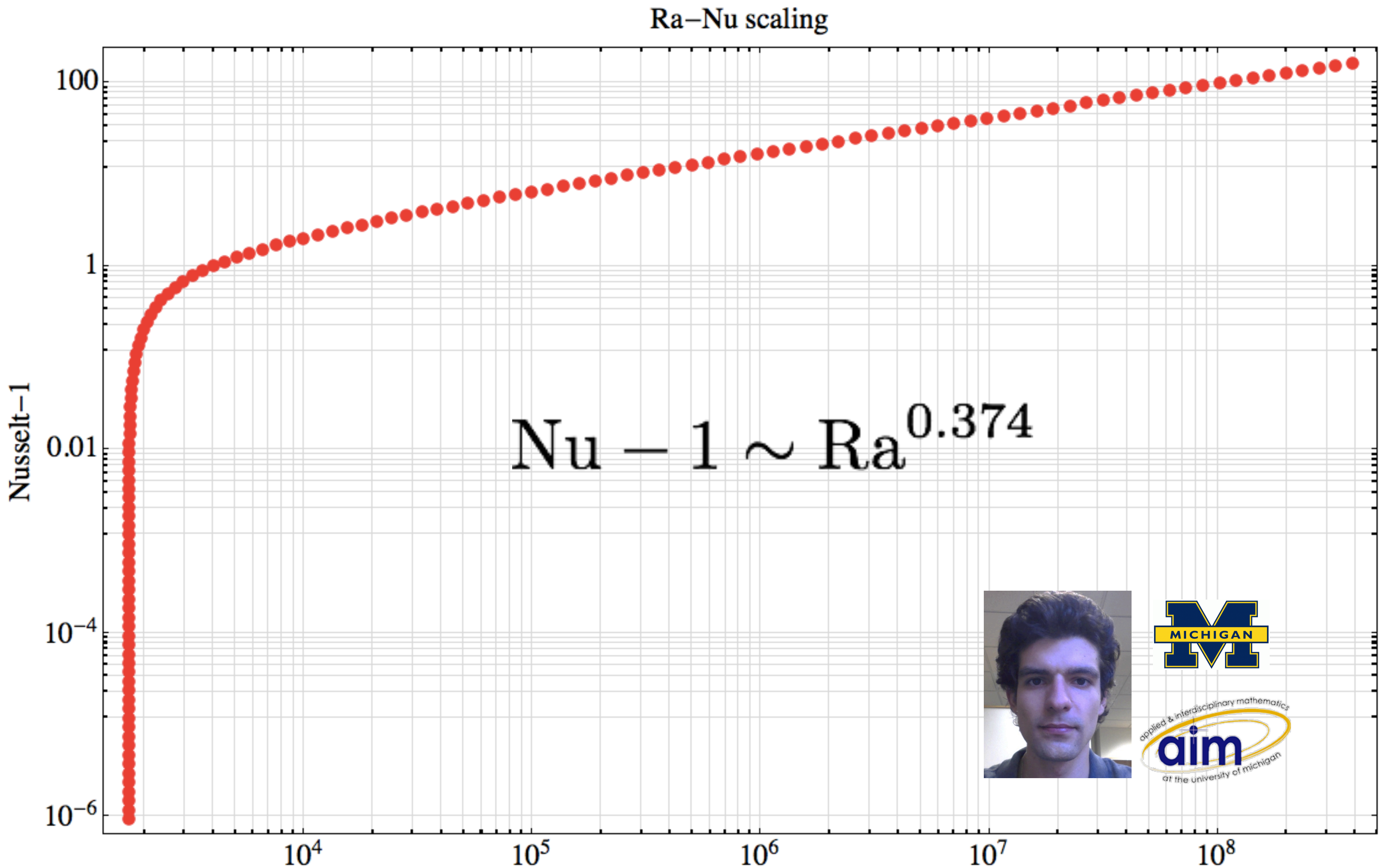
Wall to Wall Optimal Transport

- What are the next steps?

- Time dependent flows ...

- **No-slip boundary conditions ...**


Nusselt–Peclet scaling
Nusselt−1
1000
10
0.1
0.001
10^{-5}
10^{-7}
0.1
1
10
100
1000
10^4
10^5
Peclet
$\mathrm{Nu} - 1 \sim \mathrm{Pe}^{0.544}$
MICHIGAN
applied & interdisciplinary mathematics
aim
at the university of michigan



$$\mathrm{Nu} - 1 \sim \mathrm{Ra}^{0.374}$$

Wall to Wall Optimal Transport

- What are the next steps?

- Time dependent flows …

- No-slip boundary conditions …

Wall to Wall Optimal Transport

- What are the next steps?
- Time dependent flows ...
- No-slip boundary conditions ...
- **Consider 3-dimensional flow fields** ...

Wall to Wall Optimal Transport

- What are the next steps?

- Time dependent flows …

- No-slip boundary conditions …

- Consider 3-dimensional flow fields …

- ***Analysis*: do true optimal flows satisfy Euler-Lagrange equations?**

Wall to Wall Optimal Transport

- What are the next steps?

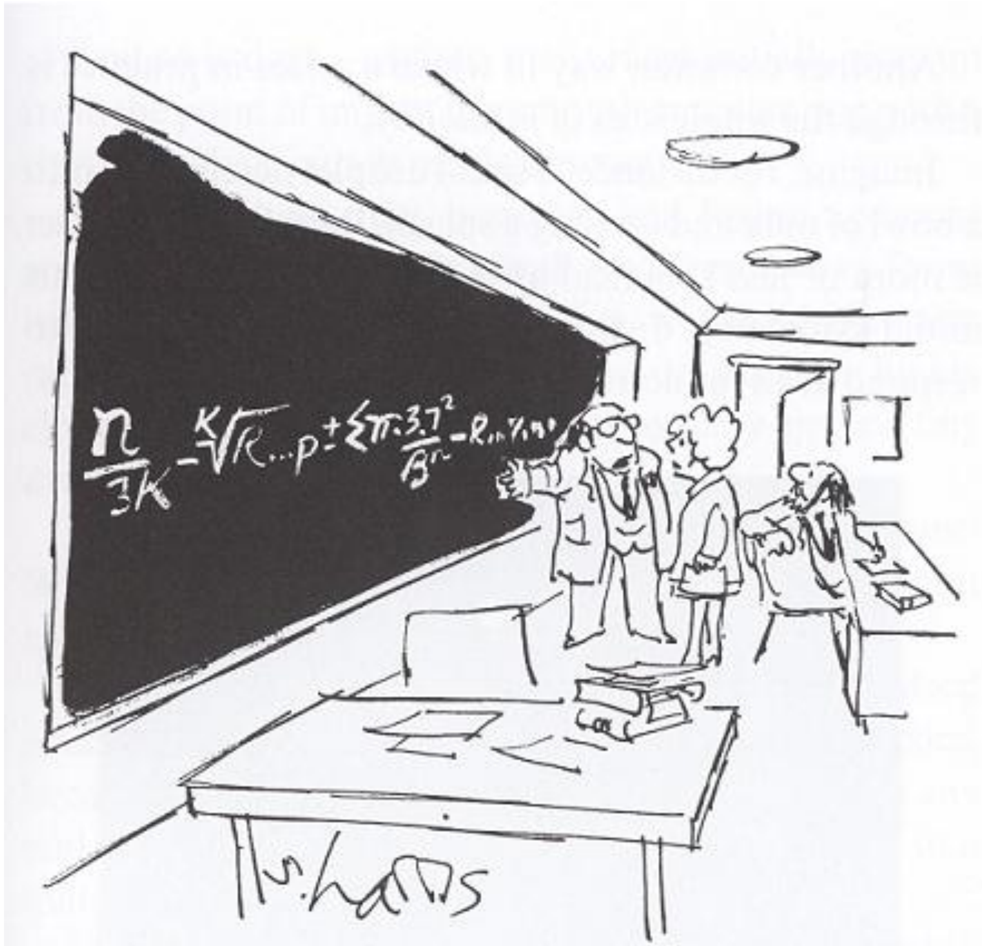
- Time dependent flows …

- No-slip boundary conditions …

- Consider 3-dimensional flow fields …

- *Analysis*: do true optimal flows satisfy Euler-Lagrange equations? **If so, estimates?**

THANKS FOR YOUR ATTENTION



"BUT THIS IS THE SIMPLIFIED VERSION FOR THE GENERAL PUBLIC."