THE SEPARATING VARIETY FOR THE BASIC REPRESENTATIONS OF THE ADDITIVE GROUP

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ABSTRACT. For a group G acting on an affine variety X, the separating variety is the closed subvariety of $X \times X$ encoding which points of X are separated by invariants. We concentrate on the indecomposable rational linear representations V_n of dimension $n+1$ of the additive group of a field of characteristic zero, and decompose the separating variety into the union of irreducible components. We show that if n is odd, divisible by four, or equal to two, the closure of the graph of the action, which has dimension $n+2$, is the only component of the separating variety. In the remaining cases, there is a second irreducible component of dimension $n+1$.

1. INTRODUCTION

Let $\Bbbk$ be an algebraically closed field, and let G be an algebraic group acting rationally on an irreducible affine variety X. This action induces an action on $\Bbbk[X]$, the ring of regular functions on X, via $(\sigma * f)(u) = f(\sigma^{-1} * u)$. The *ring of invariants* is the subalgebra $\Bbbk[X]^G \subseteq \Bbbk[X]$ formed by the elements fixed by G, or equivalently, the subalgebra formed by the elements which are constant on the orbits. Thus, for $x, y \in X$ and $f \in \Bbbk[X]^G$, having $f(x) \neq f(y)$ implies that x and y belong to distinct orbits. In this situation, we say that the invariant f *separates* x and y. A *separating set* is a set of invariants which separate any two points which are separated by some invariant (see [1, Definition 2.3.8]).

The *separating variety*

$$\mathcal{S}_G := \{(x, y) \in X \times X \mid f(x) = f(y) \text{ for all } f \in \Bbbk[X]^G\}$$

provides an alternative characterization of separating sets. Namely, if $\delta : \Bbbk[X] \to \Bbbk[X] \times \Bbbk[X]$ is the map defined by $\delta(f) := f \otimes 1 - 1 \otimes f$, then $E \subseteq \Bbbk[X]^G$ is a separating set if and only if $\mathcal{V}_{X\times X}(\delta(E)) = \mathcal{S}_G = \mathcal{V}_{X\times X}(\delta(\Bbbk[X]^G))$, where $\mathcal{V}$ denotes the common zero set of a set of polynomials. The separating variety encodes which points can be separated using invariants. In the case of finite groups, the invariants

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Date: October 28, 2012.

2010 *Mathematics Subject Classification.* 13A50,14L30.

Key words and phrases. Invariant Theory, separating invariants, locally nilpotent derivations, basic actions, Weitzenböck derivations.

separate the orbits, and so the separating variety is in fact equal to the *graph* of the G-action:

$$\Gamma_G := \{(x, \sigma \cdot x)) \in X \times X \mid x \in X,\ \sigma \in G\}.$$

This fact played a central role in the proof that, when X is a representation of a finite group G, if there exists a polynomial separating algebra, then the action of G on X is generated by reflections (see [2, Theorem 1.1]).

In general, we have $\Gamma_G \subseteq \mathcal{S}_G$. Moreover, as $\mathcal{S}_G$ is Zariski-closed, we also have $\overline{\Gamma_G} \subseteq \mathcal{S}_G$. Even for reductive groups, this inclusion can be strict (see [5, Example 2.1]). The invariants may not always separate orbits (as for the natural action of the multiplicative group on a vector space), but in the case of reductive groups, they do separate disjoint orbit closures (see [6, Corollary 3.5.2]). Exploiting this, Kemper gives an algorithm to compute the separating variety and then a separating set (see [5, Algorithm 2.9]), which is the first step in his algorithm to compute the invariants of reductive groups in arbitrary characteristic (see [5, Algorithm 1.9]).

Our motivation is to better understand the separating variety in the case of non-reductive groups. We concentrate on what is perhaps the simplest situation: algebraic actions of the additive group $\mathbb{G}_a = (\Bbbk, +)$ on an irreducible affine variety X, where $\Bbbk$ is a field of characteristic zero.

Actions of the additive group on X are in one to one correspondence with locally nilpotent derivations (abbreviated LND) on $\Bbbk[X]$. Recall that a *locally nilpotent derivation* D is a $\Bbbk$-linear map $\Bbbk[X] \to \Bbbk[X]$ such that $D(ab) = aD(b) + bD(a)$ for all $a, b \in \Bbbk[X]$ and, for all $a \in \Bbbk[X]$, there exists an $m \geq 1$ such that $D^m(a) = 0$. A locally nilpotent derivation D on $\Bbbk[X]$ induces an action $* : \mathbb{G}_a \times \Bbbk[X] \to \Bbbk[X]$ via

$$(-t) * f := \exp(tD)f = \sum_{k=0}^{\infty} \frac{t^k}{k!} D^k(f) \text{ for } t \in \mathbb{G}_a,\ f \in \Bbbk[X].$$

The invariant ring $\Bbbk[X]^{\mathbb{G}_a}$ coincides with the kernel of D and is denoted by $\Bbbk[X]^D$. We write $\mathcal{S}_D = \mathcal{S}_{\mathbb{G}_a}$ to denote the separating variety corresponding to the action induced by the locally nilpotent derivation D, and Γ_D to denote the graph of the corresponding $\mathbb{G}_a$-action.

An important contribution of the LND approach is van den Essen's algorithm to compute the kernel of a LND, and thus the invariants of a $\mathbb{G}_a$-action (see [7]). An element $s \in \Bbbk[X]$ such that $Ds \neq 0$ and $D^2 s = 0$ is a *local slice*. By the Slice Theorem (which is in fact the first step of the algorithm, see [7, Section 3]), for a local slice s and any $f \in \Bbbk[X]$, the element $\pi(f) := \exp(tD)f|_{t:=-s/Ds}$ is in $\Bbbk[X]^D_{Ds}$, and the algebra homomorphism π maps $\Bbbk[X]$ onto $\Bbbk[X]^D_{Ds}$. We are particularly interested in the *plinth ideal* $\mathrm{pl}(D)$, that is, the ideal of $\Bbbk[X]^D$ formed by the images Ds of all local slices s together with zero.

In Section 2, we give a rather rough description of the separating variety for additive group actions.

In Section 3, we focus further on the basic actions of the additive group, that is, the finite dimensional indecomposable rational linear representations of $\mathbb{G}_a$. We use the separating set constructed in [4] to compute the separating variety and write it as the union of irreducible components (Theorem 3.1).

Remark 1.1. This document in an abridged version of the paper [3] by the same authors.

2. Separation properties of invariants

We give two general results on separating properties of invariants of additive group actions.

Proposition 2.1. *If* $S \subseteq \sqrt{\mathrm{pl}(D)\Bbbk[X]}$, *then the invariants separate orbits outside* $\mathcal{V}_X(S)$, *that is,*

$$\mathcal{S}_D \setminus (\mathcal{V}_X(S) \times \mathcal{V}_X(S)) \subseteq \Gamma_D.$$

Proposition 2.2. *Let* $I \subseteq \sqrt{\mathrm{pl}(D)\Bbbk[X]}$ *be an ideal of* $\Bbbk[X]$, *and consider the canonical projection* $\tau : \Bbbk[X] \to \Bbbk[X]/I$, *given by* $f \mapsto f + I$. *Let* $A \subseteq \Bbbk[X]^D$ *be a separating algebra. If* $h_1, \ldots, h_r$ *are elements of* $\Bbbk[X]$ *such that* $\Bbbk[\tau(h_1), \ldots, \tau(h_r)] = \tau(A)$, *then the separating variety decomposes as*

$$\mathcal{S}_D = \Big(\mathcal{V}_{X\times X}(\delta(h_1), \ldots, \delta(h_r)) \cap (\mathcal{V}_X(I) \times \mathcal{V}_X(I))\Big) \cup \Gamma_D.$$

3. The basic actions

Basic actions are induced by the Weitzenböck derivations $D_n = x_0\frac{\partial}{\partial x_1} + \ldots + x_{n-1}\frac{\partial}{\partial x_n}$ on the polynomial rings $\Bbbk[x_0, \ldots, x_n] = \Bbbk[V_n]$. We recall some results and notation from [4], where separating sets for the basic actions were first constructed. Define the invariants

$$f_m := \sum_{k=0}^{m-1}(-1)^k x_k x_{2m-k} + \frac{1}{2}(-1)^m x_m^2 \in \ker D_n \quad \text{for } m = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

and $f_0 := x_0$. For $m = 0, \ldots, \lfloor \frac{n-1}{2} \rfloor$, [4, Equation (3)] also gives polynomials s_m such that $D_n s_m = f_m$. It follows that

$$I_n := (x_0, \ldots, x_{\lfloor \frac{n-1}{2} \rfloor}) = \sqrt{(f_0, \ldots, f_{\lfloor \frac{n-1}{2} \rfloor})} \subseteq \sqrt{\mathrm{pl}(D_n)\Bbbk[V_n]}.$$

Consider the projection $\tau : \Bbbk[V_n] \to \Bbbk[V_n]/I_n$. We can reformulate [4, Proposition 3.1] as follows:

$$\tau(\Bbbk[V_n]^{D_n}) = \begin{cases} \Bbbk & \text{for } 2 \nmid n, \\ \Bbbk[\tau(x_m^2)] & \text{for } n = 2m,\ 2 \nmid m, \\ \Bbbk[\tau(x_m^2), \tau(x_m^3)] & \text{for } n = 2m,\ 2 \mid m. \end{cases}$$

Proposition 2.2 then implies that the separating variety $\mathcal{S}_{D_n}$ is

(1)
$$\begin{cases} \big(\mathcal{V}_{V_n}(I_n) \times \mathcal{V}_{V_n}(I_n)\big) \cup \overline{\Gamma_{D_n}}, & \text{if } 2 \nmid n, \\ \big(\mathcal{V}_{V_n\times V_n}(\delta(x_m^2)) \cap (\mathcal{V}_{V_n}(I_n) \times \mathcal{V}_{V_n}(I_n))\big) \cup \overline{\Gamma_{D_n}}, & \text{if } n = 2m,\ 2 \nmid m, \\ \big(\mathcal{V}_{V_n\times V_n}(\delta(x_m)) \cap (\mathcal{V}_{V_n}(I_n) \times \mathcal{V}_{V_n}(I_n))\big) \cup \overline{\Gamma_{D_n}}, & \text{if } n = 2m,\ 2 \mid m. \end{cases}$$

Theorem 3.1.

(a) *If n is odd, divisible by four, or equal to 2, then the separating variety is equal to the Zariski closure of the graph of the $\mathbb{G}_a$-action, that is, $\mathcal{S}_{D_n} = \overline{\Gamma_{D_n}}$.*

(b) *If $n = 2m$ and $m \geq 3$ is odd, then the separating variety has two irreducible components:*
   - *$\overline{\Gamma_{D_n}}$, which has dimension $n+2$,*
   - *and a second of dimension $n+1$:*

$$\mathcal{V}_{V_n\times V_n}\big(x_m \otimes 1 - 1 \otimes x_m\big) \cap \big(\mathcal{V}_{V_n}(I_n) \times \mathcal{V}_{V_n}(I_n)\big).$$

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