Tame and co-tame automorphisms of $\mathbb{C}^3$.

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Let K be a field of characteristic 0.

We denote by $G = \mathrm{GA}_3(K)$ the group of all automorphisms of the K-algebra $K[x, y, z]$, $A = \mathrm{Aff}_3(K)$ the affine automorphisms sub-group, $B = \mathrm{BA}_3(K)$ the triangular automorphisms sub-group and $T =< A, B >_G= \mathrm{TA}_3(K)$ the tame automorphisms sub-group.

We denote by $s = (y, x, z)$ (resp. $t = (x, z, y)$) the automorphism which exchanges x and y (y and z).

Definition 1 Let $\sigma \in G$ we say that σ is *co-tame* if $T \subset< A, \sigma >_G$. In other words, σ is co-tame, if every tame automorphism is in the sub-group generated by σ and all affine automorphisms.

Remarks Let $\sigma \in G$ be an automorphism.
1) If σ is tame, we have $T \supset< A, \sigma >_G$ (this is the origin of our terminology).
2) If σ is both tame and co-tame, we have $T =< A, \sigma >_G$.
3) If σ is affine then σ is tame but is not co-tame.
4) If σ is co-tame then all automorphisms in $A\sigma A$ are also co-tame. "To be tame" and "to be co-tame" are properties of the orbits $A\sigma A$ ($\sigma \in G$).
5) If there exists an affine automorphism $\alpha \in A$ such that $\sigma\alpha\sigma^{-1}$ is co-tame then σ is co-tame.

Derksen first proved the existence of a co-tame automorphism. In fact, the simplest non-affine automorphism is co-tame!

Theorem 1 (Derksen, 1997) *The automorphism* $(x + y^2, y, z)$ *is co-tame.*

Bodnarchuk improved this result to a large class of tame automorphisms.

Theorem 2 (Bodnarchuk, 2004) *Let* $P = \{\sigma \in G\,;\, \sigma(y), \sigma(z) \in K[y, z]\}$ *be the sub-group of parabolic automorphisms. If* $\sigma \in (P \cup PAP) \smallsetminus A$ *then* σ *is co-tame.*

Remark We have: $P = <t, B>$.

Both Derksen and Bodnarchuk work in dimension $n \geq 3$, but here we focus in dimension 3.

Bodnarchuk asked the following important question:

Question 1. Is A is a maximal sub-group of T?

With our terminology this question can be formulate as:

Question 1'. Are all non-affine tame automorphisms co-tame?

This question is still open.

On the other hand, the theory developed by Shestakov and Umirbaev implies that there exist non tame automorphisms.

Theorem 3 (Shestakov, Umirbaev, 2004)

$$\{\sigma \in G \,;\, \sigma(z) = z\} \cap T = <s, B> .$$

Remark. The sub-group $< s, B >$ is the image in G of $\mathrm{TA}_2(K[z])$ by the map $(f, g) \mapsto (f, g, z)$.

Using the Jung-van der Kulk theorem, it's very easy to check if a given $\sigma \in G$ such that $\sigma(z) = z$ is in $<s, B>$ or not. In this way, the Shestakov-Umirbaev theorem can be use to prove that some automorphisms are non-tame.

It turns out that some non-tame automorphisms are co-tame.

Theorem 4 (Edo) *Let $\sigma \in G \setminus A$ be a non-affine automorphism. We assume that $\sigma(z) = z$ and $\sigma(y) \in \mathcal{R}_1 \cup \mathcal{R}_2$ then σ is co-tame, where*

$\mathcal{R}_1 = \{\, p_1(z)x + Q_1(y, z) \,;\, p_1 \in \mathbb{C}(z), Q_1 \in \mathbb{C}(z)[y]\}$,

$\mathcal{R}_2 = \{\, p_2(z)y + Q_2(p_1(z)x + Q_1(y, z), z) \,;\, p_1, p_2 \in \mathbb{C}(z), Q_1, Q_2 \in \mathbb{C}(z)[y]\}$.

The following question is still open.

Question 2. Are all non-tame automorphisms co-tame?