FOUNDATIONS OF INVARIANT THEORY FOR THE DOWN OPERATOR

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Abstract. This paper lays out the basic theory of the down operator D of the infinite polynomial ring $R = \mathbf{k}[x_0, x_1, x_2, ...]$, defined by $Dx_i = x_{i-1}$ ($i \geq 1$) and $Dx_0 = 0$. Here, $\mathbf{k}$ is any field of characteristic zero. The only linear invariant is x_0, and the quadratic invariants are well known and easily described. One of the paper's main results, *Thm. 6.2*, gives a complete description of the cubic invariants, ordered according to bi-degree and the number of variables involved. The distinction between core and compound invariants is introduced, and quartic and quintic invariants are studied relative to this property. As an application of the theory, *Thm. 8.2* gives a new family of counterexamples to Hilbert's Fourteenth Problem; the proof of non-finite generation is much simpler than for previously known examples.

1. Introduction

One goal of classical invariant theory was to understand the invariants of the natural action of the group $SL_2(\mathbb{C})$ on the vector space of binary forms of degree n, together with its semi-invariants, which are the invariants of the subgroup $\mathbb{G}_a$. Writing in 1906, Elliott [15] referred to "the old severe question" of finding minimal generating sets of these invariant and semi-invariant rings. In the intervening century, our knowledge of these generating sets has improved but little over what was known at the time. Indeed, the SL_2-invariants are currently known only for $n \leq 10$. The cases $n \leq 6$ were completed by Gordan in 1868, and the case $n = 8$ by Shioda in 1967; the case $n = 7$ was settled in 1986 by Dixmier and Lazard; and the cases $n = 9, 10$ were completed in 2010 by Brouwer and Popoviciu.

Our main interest is in the $\mathbb{G}_a$-action, where the situation is even more opaque: These invariants are known only for $n \leq 8$. Gordan gave generators for $n \leq 6$; the case $n = 8$ was done by Shioda; and the case $n = 7$ was completed by Cröni in 2002. Unlike the SL_2-invariants, the $\mathbb{G}_a$-invariants satisfy $A_n \subset A_{n+1}$ for each n. It is important to understand these rings for reasons that go beyond invariant theory.

One difficulty of the subject is that many generators for A_n, typically found as the result of lengthy calculations, become superfluous in higher dimensions. Thus, existing algorithms for calculating these invariants are not progressive, that is, knowing generators for A_{n-1} may be of little use in finding generators of A_n. From another perspective, this is not surprising: The partial derivative $\partial/\partial x_n$ restricts to A_n and its kernel is A_{n-1}. In general, we do not expect the generators of the kernel of a locally nilpotent derivation of a ring to form a subset of generators for the ambient ring.

Given $n \geq 2$, let $\mu(n)$ denote the minimal number of homogeneous generators of A_n as a $\mathbb{C}$-algebra, and let $\delta(n)$ be the highest degree occurring within a minimal generating set. As seen in *Table 1*, these two functions exhibit seemingly erratic behavior, at least based on the few values we know.

Motivated by these considerations, this paper investigates invariants of the locally nilpotent derivation induced by the *down operator* D of the infinite polynomial ring $R = \mathbf{k}[x_0, x_1, x_2, ...]$,

Table 1. Known values of $\mu(n)$ and $\delta(n)$

n	2	3	4	5	6	7	8
$\mu(n)$	2	4	5	23	26	147	69
$\delta(n)$	2	4	3	18	15	30	12

defined by:

$$Dx_i = x_{i-1} \quad (i \geq 1) \quad \text{and} \quad Dx_0 = 0$$

Here, $\mathbf{k}$ is any field of characteristic zero. If A denotes the kernel of D, then $A_n \subset A$ for each $n \geq 0$. The overarching goal of this approach is to describe a homogeneous generating set of A which is minimal in some appropriately defined sense.

Using the infinite polynomial ring R enables us to introduce a single natural mapping which unifies the whole theory. In Section 3, we define the operator $\theta : R \to A$, which is the main tool used in constructing invariants. *Theorem 3.1* asserts that the sequence of A-modules

$$R \xrightarrow{\theta} R_+ \xrightarrow{D} R_+ \to 0$$

is exact, where R_+ denotes the ideal of polynomials which vanish at 0. Equivalently, every homogeneous polynomial of positive degree lies in the image of D, and every homogeneous invariant of positive degree lies in the image of θ.

The theory is applied in Sections 5 and 6 to give a complete description of the cubic invariants of D. One of the main results of this paper is *Thm. 6.2*, which gives a basis for a space of irreducible cubic invariants complementary to the space of reducible cubics. This basis is ordered in such a way that cubics in A_n precede those in $A_{n+1} - A_n$. With this description, one can immediately identify all cubic generators in A_n for any given value of n. No algorithm is required.

Section 7 considers compound and core generators in higher degrees. Section 8 uses properties of the down operator to construct a new family of counterexamples to Hilbert's Fourteenth Problem; the theory provides a way to give a much simpler and shorter proof than proofs for previous counterexamples.

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1.1. **Background.** Interest in the invariants and semi-invariants of SL_2 dates back to at least the work of Boole, Cayley, Eisenstein, and Hesse. Cayley came to believe that the ring A_7 was not finitely generated. Subsequently, Gordan showed that both the invariant and semi-variant rings must, in fact, be finitely generated, and calculated generators for these rings up to $n = 6$ [20]. Gordan's work inspired numerous attempts in the following decades to establish generating sets for these rings beyond $n = 6$, but most of these attempts resulted in proposed generating sets which were either incomplete or overdetermined, due to the size and complexity of the polynomials involved. For the case $n = 8$, Sylvester and Franklin (1879) and von Gall (1880) made important contributions, but the first to determine and prove the minimal number of generators for the invariants and semi-invariants was Shioda (1967) [17, 38, 34]. The reader is referred to [10, 26, 29, 31] for accounts of these developments from the Nineteenth Century.

The first accurate calculation of a minimal generating set for A_7 is due to Cröni in 2002 [11]. In 2009, Bedratyuk, apparently unaware of Cröni's results, produced an equivalent generating set for A_7 [2]. In addition, Cerezo, Cröni and Bedratyuk each confirmed the results of Shioda for A_8 [8, 1, 11]. For $n = 9, 10, 12$, certain lower bounds are known. Cröni showed that $\mu(9) \geq 474$ and $\delta(9) \geq 20$. These bounds were improved by Brouwer and Popoviciu, who also gave bounds for $n = 10$ and $n = 12$ [4, 5, 6]. Their results are summarized in *Table 2*.

Table 2. Brouwer-Popoviciu Lower Bounds

n	9	10	11	12
$\mu(n)$	≥ 476	≥ 510	open	≥ 989
$\delta(n)$	≥ 22	≥ 21	open	≥ 17

In 1879, Jordan showed that $\delta(n) \leq 2n^6$. This is still the best available upper bound for degrees, but is too large to be of practical use in calculating generators for A_n. Kraft and Weyman give a modern proof for Jordan's bound in [27].

Many of the results in *Table 1* and *Table 2* were originally found using the symbolic method, which Weyl called "the great war-horse of Nineteenth Century invariant theory" (see [26]). The reader is referred to [14, 26, 32] for details about the symbolic method and classical techniques for constructing invariants.

1.2. **Cubic Invariants.** In Lecture XIX of Hilbert's 1897 course in invariant theory at Göttingen, Hilbert set out to explicitly identify all quadratic and cubic covariants of the SL_2-actions (equivalently, all quadratic and cubic invariants of the down operator). A basis for the space of quadratic invariants is given by the images $\theta(x_n)$ for even $n \geq 0$, and Hilbert lists these. Turning his attention to cubics, Hilbert states:

> Regarding the *covariants of degree three*, they all have odd weight $p = 2\pi + 1$ and are those which occur in the following expression. ([23], pp 62-63)

He then displays the cubic polynomial $\theta(x_1 x_{p-1})$ as the leading coefficient of the corresponding covariant. This is clearly a mistake – for example, the generating set for A_4 calculated by Cayley includes a cubic of weight 6. Lecture XIX concludes:

> If we now add covariants $f \cdot f_p$, where f_p runs through the covariants of degree two for even p, then we have the complete in- and covariant system of degree three. (p 64)

Corollary 3.2(b) below shows that there are, in fact, many other cubic invariants of the down operator not accounted for in Hilbert's description.

Hilbert's stated goal in considering the quadratic and cubic invariants is the following.

> ...we want to show that every in- and covariant of a form can be expressed as a polynomial function of the in- and covariants of degrees two and three – aside from the base form itself. (p 61)

In Lecture XX, Hilbert succeeds in showing that A_n is rationally generated over $\mathbb{C}(x_0)$ by the quadratic and cubic invariants which he defined in Lecture XIX, namely,

$$\theta(x_2), \theta(x_1 x_2), \theta(x_4), \theta(x_1 x_4), \cdots, \theta(x_1 x_{n-1}) \text{ or } \theta(x_n)$$

the latter depending on whether n is odd or even, respectively. This fact was first shown by Stroh [35].

In general, work on cubic $\mathbb{G}_a$-invariants is sparsely represented in the literature. A terse symbolic description of these was given by Grace and Young in 1903 [21] (§260). In §6 of their paper, *op. cit.*, Kraft and Weyman offer a more detailed description of cubic invariants in terms of their symbolic representations, giving spanning sets for cubic invariants of a given weight for a binary form of a specified degree. An analysis of cubics of the type carried out by Kraft and Weyman is given by Hagedorn and Wilson in [22]. In it, the authors determine an explicit basis for a space of irreducible cubics complementary to the subspace of reducible cubics in symbolic notation. Their paper also recognizes the error in the statement about cubics appearing in Hilbert's lecture notes.

2. Preliminaries

We assume throughout that $\mathbf{k}$ is a field of characteristic zero. Given an integer $m \geq 0$, $\mathbf{k}^{[m]}$ denotes the polynomial ring in m variables over $\mathbf{k}$.

2.1. **Vector Algebras.** Let V be a vector space over $\mathbf{k}$. Then $\dim V$ indicates the dimension of V as a vector space over $\mathbf{k}$. The operator $\Delta \in \mathrm{End}(V)$ is *locally nilpotent* if, to each $v \in V$, there exists a positive integer n with $\Delta^n(v) = 0$. The set of locally nilpotent operators on V is denoted $\mathrm{LN}(V)$. Note that, when $\dim V$ is finite, locally nilpotent operators are nilpotent.

Definition 2.1. By a *vector algebra* we mean a $\mathbf{k}$-vector space V equipped with a bilinear *product map* $\pi : V \times V \to V$.

The vector algebra consisting of vector space V and product π is denoted (V, π). If $W \subset V$ is a vector subspace and π restricts to $W \times W$, then (W, π) is a *vector subalgebra* of (V, π).

Definition 2.2. The vector algebra (V, π) is:

1. *trivial* if $\pi(u, v) = 0$ for all $u, v \in V$
2. *commutative* if $\pi(u, v) = \pi(v, u)$ for all $u, v \in V$
3. *associative* if $\pi(\pi(u, v), w) = \pi(u, \pi(v, w))$ for all $u, v, w \in V$

Definition 2.3. A *derivation* of the vector algebra (V, π) is a $\mathbf{k}$-linear map $\delta : V \to V$ such that, for all $u, v \in V$:

$$\delta\pi(u, v) = \pi(\delta u, v) + \pi(u, \delta v)$$

The set of derivations of (V, π) is denoted $\mathrm{Der}(V, \pi)$.

Definition 2.4. Let (V, π) be a vector algebra. Elements of the set

$$\mathrm{LND}(V, \pi) := \mathrm{Der}(V, \pi) \cap \mathrm{LN}(V)$$

are *locally nilpotent derivations* of (V, π).

In the present work, the vector algebras used are those induced by locally nilpotent derivations. Their products are commutative or anti-commutative, but not associative. For details regarding the theory of locally nilpotent derivations on commutative $\mathbf{k}$-domains, the reader is referred to [18].

2.2. **Degree Closed Subalgebras.** Let B be a commutative $\mathbf{k}$-algebra with degree function

$$\deg : B \to \mathbb{N} \cup \{-\infty\}$$

and induced filtration:

$$B = \cup_{d \geq 0} B_d \quad \text{where} \quad B_d = \{f \in B \mid \deg f \leq d\}$$

If $A \subset B$ is a subalgebra and $d \geq 0$, set $A_d = A \cap B_d$. We make the following definitions.

- $A \subset B$ is *degree closed* in B if and only if, for every $d \geq 0$:

$$A \cap \mathbf{k}[B_d] = \mathbf{k}[A_d]$$

- Given $A \subset B$, the *degree closure* of A in B is the intersection of all degree closed subalgebras of B containing A, denoted $\overline{\deg}(A)$.
- $f \in B_d - B_{d-1}$ is *compound* if and only if $f \in \mathbf{k}[B_{d-1}]$. Otherwise, f is a *core* element of B.

Example 2.1. If $B = \mathbf{k}[x_1, ..., x_n]$ is a polynomial ring with standard degree function, then every variable x_i is a core element of B, and every coordinate subring $\mathbf{k}[x_1, ..., x_i]$ is degree closed in B $(1 \leq i \leq n)$.

It is easy to check the following properties.

1. $A = \mathbf{k}[A_d]$ for some $d \geq 0 \quad \Rightarrow \quad A \cap \mathbf{k}[B_e] = \mathbf{k}[A_e]$ for every $e \geq d$
2. A and A' are degree closed in $B \quad \Rightarrow \quad A \cap A'$ is degree closed in B
3. $\overline{\deg}(A)$ is degree closed in B

Suppose C is a commutative $\mathbf{k}$-algebra with a degree function, and A, B are subalgebras with $A \subset B \subset C$.

4. A is degree closed in $C \quad \Rightarrow \quad A$ is degree closed in B
5. A is degree closed in B and B is degree closed in $C \quad \Rightarrow \quad A$ is degree closed in C

2.3. **Products Induced by Derivations.** Let R be a commutative $\mathbf{k}$-algebra. The set of $\mathbf{k}$-derivations of R is denoted $\mathrm{Der}_{\mathbf{k}}(R)$, and $\mathrm{LND}(R)$ is the set of locally nilpotent derivations. We show how any $D \in \mathrm{Der}_{\mathbf{k}}(R)$ induces a product on R which generalizes the classical transvectant. According to Olver and Sanders:

> The transvectants are the most important computational tool in the classical invariant theory of binary forms....In the symbolic calculus of classical invariant theory, the transvectants are based on a fundamental differential operator, known as Cayley's omega process. ([30], p 252)

As in the classical era, the generalization presented here is the main tool for constructing invariants (i.e., kernel elements) of D when D is locally nilpotent. The crux of the matter is found in *Prop. 2.2(d)*.

R is a $\mathbf{k}$-vector space equipped with a product π, and as such it is a vector algebra (R,π). Suppose that $D \in \mathrm{Der}_{\mathbf{k}}(R)$ is non-zero. Then D is a linear operator on the vector space R. For each $n \geq 0$, define the binary operation $\phi_n^D : R \times R \to R$ by:

$$\phi_n^D(f,g) = (f, Df, ..., D^n f) \cdot ((-1)^n D^n g, ..., -Dg, g) = \sum_{i=0}^{n} (-1)^i D^i f D^{n-i} g$$

It is easy to see that ϕ_n^D is bilinear over $\mathbf{k}$, meaning that (R, ϕ_n^D) is a vector algebra. Observe that $\phi_0^D = \pi$.

We will also use the notation $\phi_n^D(f,g) = [f,g]_n^D$, or more simply $\phi_n(f,g) = [f,g]_n$ when the underlying derivation is understood. Note that $[f,1]_n = D^n f$.

Proposition 2.1. *The following properties hold for ϕ_n.*

(a) *ϕ_n is bilinear over* $\ker D$

(b) *$[g,f]_n = (-1)^n [f,g]_n$ for all $f,g \in R$ and $n \geq 0$*

(c) *Given $f \in R$ and $n \geq 1$, $[f,f]_n = 0$ if n is odd; if $n \geq 2$ is even, then*:

$$[f,f]_n = 2D^n f \cdot f - [Df, Df]_{n-2} = 2\left(\sum_{i=0}^{\frac{n}{2}-1} (-1)^i D^i f D^{n-i} f\right) + (-1)^{\frac{n}{2}} (D^{\frac{n}{2}} f)^2$$

(d) *$D \in \mathrm{Der}(R, \phi_n)$ for each $n \geq 0$*

Proof. Parts (a)-(c) follow easily from the definition of ϕ_n. For part (d): From the product rule for inner products (see p. 79 of [18]), we have:

$$\begin{aligned} D([f,g]_n) &= D((f, Df, ..., D^n f) \cdot ((-1)^n D^n g, ..., -Dg, g)) \\ &= D(f, Df, ..., D^n f) \cdot ((-1)^n D^n g, ..., -Dg, g) \\ &\quad + (f, Df, ..., D^n f) \cdot D((-1)^n D^n g, ..., -Dg, g) \\ &= (Df, D^2 f, ..., D^{n+1} f) \cdot ((-1)^n D^n g, ..., -Dg, g) \\ &\quad + (f, Df, ..., D^n f) \cdot ((-1)^n D^{n+1} g, ..., -D^2 g, Dg) \\ &= [Df, g]_n + [f, Dg]_n \end{aligned}$$

Therefore $D \in \mathrm{Der}(R, \phi_n)$. ∎

Next, assume that $D \in \mathrm{LND}(R)$. Then $D \in \mathrm{LN}(R)$, and *Prop. 2.1(d)* implies $D \in \mathrm{LND}(R, \phi_n)$ for each $n \geq 0$. The degree function $\deg_D$ on R is defined by:

$$\deg_D(f) = \min\{n \geq 0 \mid D^{n+1} f = 0\} \quad (f \neq 0)$$

A *local slice* of D is any $t \in R$ with $\deg_D t = 1$. This degree function induces the filtration:

$$R = \cup_{n \in \mathbb{Z}} R^{(n)}, \quad R^{(n)} = \{r \in R \mid \deg_D r \leq n\}$$

Note that $R^{(0)} = \ker D$ and $R^{(n)} = \{0\}$ for $n < 0$.

Proposition 2.2. *Let $D \in \mathrm{LND}(R)$.*

- **(a)** *For all $r, s \in \mathbb{Z}$:*
$$\phi_n : R^{(r)} \times R^{(s)} \to R^{(r+s-n)}$$
- **(b)** *If $g \in R^{(m)}$ and $m < n$, then for all $f \in R$:*
$$[f, g]_n = [D^{n-m} f, g]_m$$
- **(c)** *If $f, g \in R^{(m)}$ and $m < n \leq 2m$, then:*
$$[f, g]_n = (-1)^{3m-n}[D^{n-m} f, D^{n-m} g]_{2m-n}$$
- **(d)** *$\phi_n : R^{(n)} \times R^{(n)} \to R^{(0)}$ for each $n \geq 0$*
- **(e)** *If $m \geq 1$ is odd, then:*
$$\phi_m : R^{(n)} \times R^{(n)} \to R^{(2n-m-1)}$$
- **(f)** *If $n \geq 2$ is even, then:*
$$\phi_{n-1} : R^{(n)} \times R^{(n)} \to R^{(n)}$$
- **(g)** (Wronskian) *Given $n \geq 1$, if $f_0, f_1, ..., f_n \in R^{(n)}$, then:*
$$W_D(f_0, f_1, ..., f_n) = [\cdots [\,[f_0, f_1]_1, f_2]_2, \cdots, f_n]_n$$

Proof. **(a)** This follows by definition of ϕ_n.

- **(b)** If $g \in R^{(m)}$, then:
$$\begin{aligned} [f, g]_n &= (f, Df, ..., D^n f) \cdot (0, ..., 0, (-1)^m D^m g, ..., -Dg, g) \\ &= (D^{n-m} f, ..., D^n f) \cdot ((-1)^m D^m g, ..., -Dg, g) \\ &= [D^{n-m} f, g]_m \end{aligned}$$
- **(c)** This follows by two applications of part (b).
- **(d)** Let $f, g \in R^{(n)}$. Then part (b) implies:
$$\begin{aligned} [Df, g]_n &= (-1)^n [g, Df]_n \\ &= (-1)^n [Dg, Df]_{n-1} \\ &= (-1)^n (-1)^{n-1} [Df, Dg]_{n-1} \\ &= -[Df, Dg]_{n-1} \end{aligned}$$
In the same way, since $f \in R^{(n)}$, we obtain $[f, Dg]_n = [Df, Dg]_{n-1}$. It follows from the product rule that:
$$D([f, g]_n) = [Df, g]_n + [f, Dg]_n = 0$$
- **(e)** Let $t \in R$ be a local slice of D, and let $K = \mathrm{frac}(\ker D)$. Given $F \in R$, write $F = \sum_{i \geq 0} a_i t^i$ for $a_i \in K$. Given $i \geq 0$, define $\epsilon_i(F) = a_i t^i$. Let $r, s, k \geq 0$ be given, with $F \in R^{(r)}$, and $G \in R^{(s)}$. Then generally we have:
$$\epsilon_{r+s-k}([F, G]_k) = [\epsilon_r(F), \epsilon_s(G)]_k$$
Suppose that $f = \sum_{0 \leq i \leq n} u_i t^i$ and $g = \sum_{0 \leq j \leq n} v_j t^j$. Since m is odd, we have:
$$\epsilon_{2n-m}([f, g]_m) = [\epsilon_n(f), \epsilon_n(g)]_m = [u_n t^n, v_n t^n]_m = u_n v_n [t^n, t^n]_m = 0$$
Therefore, $\deg_D [f, g]_m < 2n - m$.
- **(f)** This is a special case of part (e).
- **(g)** This follows by induction on n using properties of the Wronskian found in [18], Cor. 2.20.

□

Recall that any local slice t of D induces an algebra map π_t from R to the localization of $\ker D$ at Dt, called the *Dixmier map* induced by t; see [18].

Proposition 2.3. *Let $t \in R$ be a local slice of D.*

(a) *For all $f \in R$ and $n \geq 0$:*

$$[f, t^n]_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} D^i f (Dt)^{n-i} t^i$$

(b) (Dixmier map) *Given $n \geq 0$, if $f \in R^{(n)}$, then $[f, t^n]_n = n!(Dt)^n \pi_t(f)$*

(c) *Given $m, n \geq 0$, $f \in R^{(n)}$, and $g \in R^{(m)}$:*

$$(n+m)![f, t^n]_n [g, t^m]_m = n!m![fg, t^{n+m}]_{n+m}$$

Proof. **(a)**

$$\begin{aligned} [f, t^n]_n &= \sum_{i=0}^{n} (-1)^i D^i f D^{n-i}(t^n) \\ &= \sum_{i=0}^{n} (-1)^i D^i f n(n-1)\cdots(n-(n-i)+1) t^{n-(n-i)} (Dt)^{n-i} \\ &= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} D^i f (Dt)^{n-i} t^i \end{aligned}$$

(b) This follows from part (a) and the definition of π_t .

(c) This follows from part (b) and the fact that π_t is an algebra homomorphism.

□

3. The Down Operator on the Infinite Polynomial Ring

3.1. **Basic Definitions.** Let $\mathcal{V}$ be a vector space with a countably infinite basis $\{x_0, x_1, x_2, ...\}$, and define the *down operator* $D \in \mathrm{End}(\mathcal{V})$ by:

$$Dx_m = x_{m-1} \text{ for } m \geq 1 \text{ , and } Dx_0 = 0$$

Then $D \in \mathrm{LN}(\mathcal{V})$.

The symmetric algebra $R = S(\mathcal{V}) = \mathbf{k}[x_0, x_1, x_2, ...]$ is the polynomial ring in a countably infinite set of variables. The down operator extends to a derivation $D \in \mathrm{Der}_{\mathbf{k}}(R)$. Note that $D \in \mathrm{LND}(R)$. In addition, for all $n \geq 0$, we have:

$$[\partial/\partial x_n, D] = \frac{\partial}{\partial x_{n+1}} \tag{1}$$

Let $A = \ker D$, the kernel of D as a derivation. Define ideals $R_+ \subset R$ and $A_+ \subset A$ by:

$$R_+ = (x_0, x_1, x_2, ...) \quad \text{and} \quad A_+ = A \cap R_+$$

The *standard* $\mathbb{Z}$-grading $\mathfrak{r}$ of R is that for which x_n is homogeneous and $\deg_{\mathfrak{r}} x_n = 1$ for each $n \geq 0$. Relative to this grading, D is homogeneous and $\deg_{\mathfrak{r}} D = 0$. Given $r \geq 0$, let $V_r \subset R$ denote the vector space of r-forms, and set $W_r = A \cap V_r$.

3.2. **The Function θ.** Define the map of A-modules $\theta : R \to A$ as follows: Given $f \in R$:

$$\theta(f) = \sum_{i \geq 0} (-1)^i D^i(f) x_i$$

If $d = \deg_D(f)$, then $\theta(f) = [f, x_d]_d$. By *Prop. 2.2(d)*, it follows that $\theta(R) \subset A$, as asserted.

Lemma 3.1. *If $r \geq 1$ and $f \in W_r$, then :*

$$\theta\left(\frac{\partial f}{\partial x_0}\right) = rf$$

Consequently, $\theta(R) = A_+$.

Proof. Equation (1) implies that, for all $i \geq 0$:

$$D^i\left(\frac{\partial f}{\partial x_0}\right) = (-1)^i \frac{\partial f}{\partial x_i} \tag{2}$$

Thus, by Euler's Lemma, it follows that:

$$rf = \sum_{i \geq 0} x_i \frac{\partial f}{\partial x_i} = \sum_{i \geq 0} x_i (-1)^i D^i\left(\frac{\partial f}{\partial x_0}\right) = \theta\left(\frac{\partial f}{\partial x_0}\right)$$

□

Theorem 3.1. *The sequence of A-modules*

$$R \xrightarrow{\theta} R_+ \xrightarrow{D} R_+ \to 0$$

is exact.

In order to prove this, several preliminaries are required.

3.3. **Compatible $\mathbb{Z}$-Gradings.** Let $\mathfrak{g}$ denote a $\mathbb{Z}$-grading on R, and let $\deg_{\mathfrak{g}}$ denote the corresponding degree function. Then $\mathfrak{g}$ is said to be *compatible* if it satisfies the following two conditions:

1. x_n is homogeneous for each $n \geq 0$
2. $\deg_{\mathfrak{g}} x_n$ is a linear function of n

Note that condition 2 is equivalent to either of the following conditions.

2.′ The difference $\deg_{\mathfrak{g}} x_{n+1} - \deg_{\mathfrak{g}} x_n$ does not depend on n
2.″ D is homogeneous relative to $\mathfrak{g}$

When these conditions are satisfied, the fact that $D^n x_n = x_0$ gives the linear relation:

$$n \deg_{\mathfrak{g}} D + \deg_{\mathfrak{g}} x_n = \deg_{\mathfrak{g}} x_0$$

Given a compatible $\mathbb{Z}$-grading $\mathfrak{g}$, define $E, U \in \mathrm{End}(\mathcal{V})$ as follows. For each $n \geq 0$:

Euler operator

$$E x_n = (\deg_{\mathfrak{g}} x_n)\, x_n$$

Up operator

$$U x_n = \omega_n\, x_{n+1} \quad \text{where} \quad \omega_n = \sum_{i=0}^{n} \deg_{\mathfrak{g}} x_i \tag{3}$$

Extend E and U to derivations $E, U \in \mathrm{Der}_{\mathbf{k}}(R)$. Then E and U are homogeneous, where $\deg_{\mathfrak{g}} E = 0$ and $\deg_{\mathfrak{g}} U = -\deg_{\mathfrak{g}} D$. Note that, for each $\mathfrak{g}$-homogeneous $f \in R$, we have the Euler identity:

$$Ef = (\deg_{\mathfrak{g}} f) f$$

The following relations are easily verified:

$$[D, U] = E\ , \quad [D, E] = -(\deg_{\mathfrak{g}} D) D\ , \quad [U, E] = (\deg_{\mathfrak{g}} D) U \tag{4}$$

In addition, for each $n \geq 0$:

$$[\partial/\partial x_n, U] = \begin{cases} 0 & n = 0 \\ \omega_{n-1}(\partial/\partial x_{n-1}) & n \geq 1 \end{cases} \tag{5}$$

Another key fact is the following integration property.

Lemma 3.2. *If $f \in A$ is $\mathfrak{g}$-homogeneous and $n \geq 1$, then*

$$D^n U^n(f) = c_1 \cdots c_n f$$

where the sequence $c_i \in \mathbb{Z}$ $(1 \leq i \leq n)$ is defined by:

$$c_i = i \deg_{\mathfrak{g}} f - \tfrac{i(i-1)}{2} \deg_{\mathfrak{g}} D$$

Proof. We first show that, for $n \geq 1$:

$$DU^n f = c_n U^{n-1} f \tag{6}$$

We proceed by induction on n.

By Euler's lemma, we have $Ef = (\deg_{\mathfrak{g}} f)f$. It follows that:

$$DU(f) = [D, U](f) = Ef = (\deg_{\mathfrak{g}} f)f = c_1 f$$

Therefore, equation (6) is valid when $n = 1$.

Assume (6) holds for $n \geq 1$. Then:

$$[D, U](U^n f) = DU(U^n f) - UD(U^n f) = DU^{n+1}(f) - U(c_n U^{n-1} f) = DU^{n+1} f - c_n U^n f$$

In addition:

$$[D, U](U^n f) = E(U^n f) = (\deg_{\mathfrak{g}} U^n f)U^n f = (\deg_{\mathfrak{g}} f - n \deg_{\mathfrak{g}} D)U^n f$$

Combining these two equalities yields:

$$DU^{n+1} f = (\deg_{\mathfrak{g}} f - n \deg_{\mathfrak{g}} D)U^n f + c_n U^n f = (\deg_{\mathfrak{g}} f - n \deg_{\mathfrak{g}} D + c_n)U^n f = c_{n+1} U^n f$$

Therefore, equation (6) holds for all $n \geq 1$.

It follows that, for $n \geq 1$:

$$D^n U^n f = D^{n-1}(DU^n f) = D^{n-1}(c_n U^{n-1} f) = c_n D^{n-1} U^{n-1} f$$

By applying this equality iteratively, the equality asserted in the lemma is proved. $\square$

Example 3.1. The standard $\mathbb{Z}$-grading $\mathfrak{r}$ of R is compatible. If U is the up derivation induced by $\mathfrak{r}$, then $Ux_n = (n+1)x_{n+1}$ for each $n \geq 0$. By *Lemma 3.2*, we have

$$D^n U^n f = n!(\deg_{\mathfrak{r}} f)^n f$$

for each homogeneous $f \in A$ and $n \geq 0$.

3.4. **Proof of Thm. 3.1.** We need to show:

$$\operatorname{im} D = R_+ \quad \text{and} \quad \operatorname{im} \theta = A_+$$

The second of these equalities was already established in *Lemma 3.1.* For the first equality, it will suffice to show that, for each $r \geq 1$, the map $D : V_r \to V_r$ is surjective. Given $r \geq 1$, we show by induction on $m \geq 1$ that every element of $\ker D^m \cap V_r$ lies in the image of D.

Let U and E denote the up and Euler derivations, respectively, induced by the standard $\mathbb{Z}$-grading $\mathfrak{r}$ or R. Given non-zero $g \in \ker D \cap V_r$, we have:

$$D(Ug) = [D, U](g) = Eg = rg$$

So g is in the image of D in this case. Therefore, D surjects onto $\ker D \cap V_r$.

Given $m \geq 1$, assume that D surjects onto $\ker D^m \cap V_r$. Let $g \in \ker D^m \cap V_r$ be given. By *Lemma 3.2*, we have

$$D^{m+1} U^{m+1}(D^m g) = (m+1)! r^{m+1} D^m g$$

meaning that:

$$h := DU^{m+1}D^m g - (m+1)!r^{m+1}g \in \ker D^m \cap V_r$$

By the inductive hypothesis, there exists $p \in V_r$ such that $Dp = h$. Therefore:

$$(m+1)!r^{m+1}g = DU^{m+1}D^m g - Dp = D(U^{m+1}D^m g - p)$$

It follows by induction that D surjects onto $\ker D^{m+1} \cap V_r$. Therefore, $D : V_r \to V_r$ is surjective for each $r \geq 1$. □

3.5. **$\mathbb{Z}^2$-Grading.** Define the $\mathbb{Z}$-grading $\mathfrak{s}$ of R by setting

$$\deg_{\mathfrak{s}} x_n = n \quad (n \geq 0)$$

where each x_n is homogeneous. Then $\mathfrak{s}$ is a compatible $\mathbb{Z}$-grading. If $(\mathfrak{r}, \mathfrak{s})$ denotes the $\mathbb{Z}^2$-grading of R defined by $\mathfrak{r}$ and $\mathfrak{s}$, then D is bi-homogeneous and $\mathrm{bideg}\, D = (0, -1)$.

Given $r, s \geq 0$, let $V_{(r,s)}$ denote the vector space of bi-homogeneous elements of R of degree (r, s), and let $W_{(r,s)} = A \cap V_{(r,s)}$. Accordingly, we have:

$$V_r = \oplus_{s \geq 0} V_{(r,s)}$$

For notational convenience, let $V_{(r,s)} = \{0\}$ if $r < 0$ or $s < 0$, and $V_r = \{0\}$ if $r < 0$.

Given $k \geq 0$, let ϕ_k denote the product map on R induced by D. Since D is bi-homogeneous, ϕ_k is bi-homogeneous for each $k \geq 0$:

$$\phi_k : V_{(r,s)} \times V_{(u,v)} \to V_{(r+u,s+v-k)}$$

Recall from the preceding section that R is also filtered by $\deg_D$. Given $r, s \geq 0$, we have:

$$\deg_D x_s = s \quad \text{and} \quad V_{(r,s)} \subset \mathbf{k}[x_0, ..., x_s] \cap R^{(s)}$$

From *Prop. 2.2(d)*, it follows that:

$$\phi_s : V_{(r,s)} \times V_{(u,s)} \to W_{(r+u,s)} \tag{7}$$

Note that, for $s \geq 2$, x_s is *not* homogeneous relative to the $\mathbb{Z}$-grading of R induced by D.

Let $f \in V_{(r,s)}$ be given, and set $d = \deg_D f$. Then $d \leq s$ and:

$$\theta(f) = [f, x_d]_d = [f, x_s]_s \in A \cap V_{(r+1,s)} = W_{(r+1,s)}$$

Therefore, θ is bi-homogeneous, with $\mathrm{bideg}\, \theta = (1, 0)$ and:

$$\theta : V_{(r,s)} \to W_{(r+1,s)} \tag{8}$$

Theorem 3.1 implies the following.

Corollary 3.1. **(a)** *For each $r \geq 0$, the sequence of vector spaces*

$$V_r \xrightarrow{\theta} V_{r+1} \xrightarrow{D} V_{r+1} \to 0$$

is exact.

(b) *For each $r, s \geq 0$, the sequence of finite-dimensional vector spaces*

$$V_{(r,s)} \xrightarrow{\theta} V_{(r+1,s)} \xrightarrow{D} V_{(r+1,s-1)} \to 0$$

is exact.

Proof. This result follows from *Thm. 3.1*, using the fact that θ is bi-homogeneous of degree $(1, 0)$, and D is bi-homogeneous of degree $(0, -1)$. □

3.6. **Kernel Decomposition.** We next give a structure theorem for the vector spaces $W_{(r,s)}$.

Define the *shift map* to be the **k**-algebra endomorphism $\sigma : R \to R$ defined by $\sigma(x_i) = x_{i+1}$. Note that σ is an isomorphism of R with $\sigma(R) = \bar{R}$, where:

$$\bar{R} = R/x_0R = \mathbf{k}[x_1, x_2, ...] \subset R$$

Define the map of **k**-algebras $\epsilon : R \to \bar{R}$ by $\epsilon(x_0) = 0$, that is:

$$\epsilon(f(x_0, x_1, ..., x_n)) = f(0, x_1, ..., x_n)$$

Then ϵ is called the *evaluation map.* If $\bar{D} = \epsilon D$, then:

$$\bar{D} = \sigma D \sigma^{-1} \quad \text{and} \quad \sigma(A) = \ker(\bar{D})$$

Note that $\epsilon(A) \subset \sigma(A)$, but $x_1 \in \sigma(A) - \epsilon(A)$.

Lemma 3.3. *$A \cap \bar{R} = \mathbf{k}$, and consequently $A \cap \sigma(A) = \mathbf{k}$.*

Proof. Suppose $f \in W_r$ for $r \geq 1$. Then:

$$\frac{\partial f}{\partial x_0} = 0 \quad \Rightarrow \quad rf = \theta\left(\frac{\partial f}{\partial x_0}\right) = 0 \quad \Rightarrow \quad f = 0$$

□

Theorem 3.2. **(a)** *The sequence of A-modules*

$$0 \to x_0 A \hookrightarrow A_+ \xrightarrow{\sigma^{-1}\epsilon} A_+ \to 0$$

is exact.

(b) *For each $r, s \geq 0$, the sequence of finite-dimensional vector spaces*

$$0 \to x_0 W_{(r-1,s+r)} \hookrightarrow W_{(r,s+r)} \xrightarrow{\sigma^{-1}\epsilon} W_{(r,s)} \to 0$$

is split exact.

(c)

$$\dim W_{(r,s)} = \dim W_{(r-1,s)} + \dim W_{(r,s-r)}$$

Proof. Parts (a) and (c) are implied by part (b). In order to prove part (b), it will suffice to construct a section for $\sigma^{-1}\epsilon$.

If $f \in A_+$ is non-zero, then $\bar{D}\sigma(f) = 0$, but by *Lemma 3.3*, $D\sigma(f) \neq 0$. Therefore, $D\sigma$ maps A_+ injectively into $x_0 R_+$.

Assume that $\{f_1, ..., f_k\}$ is a basis for $W_{(r,s)}$, where $k = \dim W_{(r,s)}$. Since D maps $V_{(r-1,s+r)}$ onto $V_{(r-1,s+r-1)}$ by *Cor. 3.1*, we may choose, for each i, a preimage $g_i \in V_{(r-1,s+r)}$ such that:

$$Dg_i = \tfrac{1}{x_0} D\sigma(f_i) \quad (1 \leq i \leq k)$$

Define the map $\tau : W_{(r,s)} \to W_{(r,s+r)}$ by:

$$\tau(f_i) = x_0 g_i - \sigma(f_i)$$

Then τ is a section for $\sigma^{-1}\epsilon$. □

Corollary 3.2. *Let $s \geq 0$ be given, and let $t \geq 0$ be such that $0 \leq s - 6t \leq 5$.*

(a)

$$\dim W_{(2,s)} = \begin{cases} 1 & s \text{ even} \\ 0 & s \text{ odd} \end{cases}$$

(b)

$$\dim W_{(3,s)} = \begin{cases} t & s \equiv 1 \pmod 6 \\ t+1 & \text{otherwise} \end{cases}$$

Proof. Let $k \geq 0$ be such that $0 \leq s - 2k \leq 1$. *Thm. 3.2(c)* implies:

$$\dim W_{(2,s)} = \sum_{i=0}^{k} \dim W_{(1,s-2i)}$$

Since $W_1 = W_{(1,0)} = \mathbf{k} \cdot x_0$, part (a) is clear.

For part (b), let $m \geq 0$ be such that $0 \leq s - 3m \leq 2$. Then *Thm. 3.2(c)* implies:

$$\dim W_{(3,s)} = \sum_{i=0}^{m} \dim W_{(2,s-3i)} \tag{9}$$

By part (a), the sum in (9) is a sum of $m+1$ terms in alternating values 0 and 1. There are four cases to consider.

(i) If m is even and s is even, the sum is:

$$(1+0) + \cdots + (1+0) + 1 = \frac{m}{2} + 1 = \frac{m+2}{2}$$

(ii) If m is even and s odd, the sum is:

$$(0+1) + \cdots + (0+1) + 0 = \frac{m}{2}$$

(iii) If m is odd and s even, the sum is:

$$(1+0) + \cdots + (1+0) = \frac{m+1}{2}$$

(iv) If m odd and s is odd, the sum is:

$$(0+1) + \cdots + (0+1) = \frac{m+1}{2}$$

We have thus shown the following:

$$\dim W_{(3,s)} = \begin{cases} \frac{m+2}{2} & (s, m \text{ even}) \\ \frac{m+1}{2} & (m \text{ odd}) \\ \frac{m}{2} & (s \text{ odd}, m \text{ even}) \end{cases}$$

This is equivalent to the equality asserted in part (b). □

3.7. **Quadratic Invariants.** Decompose $V_1 = V_1^+ \oplus V_1^-$, where:

$$V_1^+ = \oplus_{i \geq 0} \mathbf{k} \cdot x_{2i} \quad \text{and} \quad V_1^- = \oplus_{i \geq 0} \mathbf{k} \cdot x_{2i+1}$$

The surjective map $\theta : V_1 \to W_2$ has kernel V_1^-, meaning that $\theta : V_1^+ \to W_2$ is an isomorphism. By *Cor. 3.2(a)*, we have:

Corollary 3.3. $W_2 = \oplus_{n \geq 0} \mathbf{k} \cdot \theta(x_{2n})$

3.8. **An Irreducibility Criterion.** Recall that A is factorially closed in B. Therefore, given $f \in A$, if f is irreducible in A, then f is also irreducible in B. This property allows us to formulate the following simple criterion for irreducibility of elements of A.

Lemma 3.4. *Given $f \in A_n$ for $n \geq 2$, write*

$$f = \sum_{i=0}^{m} \alpha_i x_n^i$$

where $m \geq 0$ and $\alpha_i \in R_{n-1}$ for each i.

(a) $\alpha_m \in A_{n-1}$

(b) *If α_m is irreducible, then f is irreducible.*

Proof. For part (a), since $0 = Df = D\alpha_m x_n^m +$ (lower-degree x_n-terms), it follows that $D\alpha_m = 0$.

For part (b), it will suffice to show that A has no element of the form:

$$g = x_k^t + \sum_{i=0}^{t-1} \beta_i x_k^i \ , \ (k \geq 1 \, , \, t \geq 1 \, , \, \beta_i \in R_{k-1}) \tag{10}$$

Assume to the contrary that $g \in A$ has the form specified in equation (10). Then $k \geq 2$, since $A_1 = \mathbf{k}[x_0]$.

Define the ideal $I \subset R_{k-1}$ by $I = x_0 R_{k-1} + \cdots + x_{k-2} R_{k-1}$. Then $D(R_{k-1}) \subset I$. Since $Dg = 0$, it follows that $-tx_{k-1} = D\beta_{t-1} \in I$, a contradiction. Therefore, A contains no such element g. $\square$

Remark 3.1. In the vocabulary of Nineteenth Century invariant theory, the *degree* of a homogeneous invariant $f \in W_{(r,s)}$ is its $\mathfrak{r}$-degree, the *weight* is its $\mathfrak{s}$-degree, and the *extent* is the smallest integer n such that $f \in \mathbf{k}[x_0, ..., x_n]$. The *order* is a degree function on $\mathbf{k}[x_0, ..., x_n]$ in which elements of $W_{(r,s)}$ have order $nr - 2s$. Thus, in the current context, the order of $f \in W_{(r,s)}$ is not well-defined, since $f \in \mathbf{k}[x_0, ..., x_n]$ for all sufficiently large n.

Remark 3.2. If $\mathfrak{g}$ is such that $\deg_{\mathfrak{g}} D \neq 0$, the relations in (4) show that D, U and E form the Lie algebra $\mathfrak{sl}_2$ over $\mathbf{k}$. The corresponding Lie group SL_2 is reductive, represented by 2×2 matrices with unit determinant. We may thus view R as an SL_2-module, where the $\mathbb{G}_a$-action on R defined by D is a restriction of the SL_2-action.

If $\mathfrak{g}$ is such that $\deg_{\mathfrak{g}} D = 0$, then $\mathfrak{g} = k\mathfrak{r}$ for some $k \in \mathbb{Z}$. If $k \neq 0$, then D, U and E form the Lie algebra $\mathfrak{h}_3$ represented by 3×3 upper-triangular matrices with zero diagonal. The corresponding Lie group $\mathcal{H}_3$ is the Heisenberg group, which is unipotent, represented by 3×3 upper-triangular matrices with unit diagonal. In this case, we may view R as an $\mathcal{H}_3$-module, where the $\mathbb{G}_a$-action on R defined by D is a restriction of the $\mathcal{H}_3$-action.

4. The Standard n-compatible $\mathbb{Z}$-Grading

Given $n \geq 0$, let $\mathcal{V}_n \subset \mathcal{V}$ denote the vector subspace spanned by $x_0, ..., x_n$, noting that D restricts to each subspace $\mathcal{V}_n$. Define subrings $R_n \subset R$ by

$$R_n = S(\mathcal{V}_n) = \mathbf{k}[x_0, ..., x_n] = \mathbf{k}^{[n+1]}$$

as well as subrings $A_n := A \cap R_n$. Let $\mathfrak{g}$ be a compatible $\mathbb{Z}$-grading of R. Then each subring R_n is a $\mathfrak{g}$-homogeneous subring.

The first property to observe in this regard is that the partial derivative $\partial/\partial x_n$ commutes with the restriction of D to R_n; see equation (1) above. The following lemma is an easy consequence of this property.

Lemma 4.1. *Fix $n \geq 0$.*

- **(a)** *$\partial/\partial x_n$ restricts to A_n*
- **(b)** *$[\partial/\partial x_n, \theta](f) = (-1)^n D^n(f)$ for all $f \in R_n$*
- **(c)** *$\dfrac{\partial}{\partial x_n} \theta(f) = (-1)^n D^n(f)$ for all $f \in R_{n-1}$*
- **(d)** *Given $k \geq 0$, let ϕ_k be the product on R_n determined by the locally nilpotent operator $D|_{R_n}$. Then for every $k \geq 0$:*

$$\frac{\partial}{\partial x_n} \in \mathrm{LND}(R_n, \phi_k)$$

Definition 4.1. For each $n \geq 0$, let $\partial_n \in \mathrm{LND}(A_n)$ denote the restriction of $\partial/\partial x_n$ to A_n.

Suppose that $\mathfrak{g}$ is a compatible $\mathbb{Z}$-grading of R, with induced Euler operator E and up operator U. Then E restricts to $\mathcal{V}_n$ for each $n \geq 0$. On the other hand, given $n \geq 0$, U restricts to $\mathcal{V}_n$ if and only if $Ux_n = 0$. In this case, $U \in \mathrm{LN}(\mathcal{V}_n)$, and the induced SL_2-action on R restricts to R_n.

Definition 4.2. Given an integer $n \geq 0$, a compatible $\mathbb{Z}$-grading $\mathfrak{g}$ of R is *n-compatible* if and only if:

$$\sum_{i=0}^{n} \deg_{\mathfrak{g}} x_i = 0$$

Definition 4.3. Given an integer $n \geq 0$, the *standard n-compatible $\mathbb{Z}$-grading* of R is $\mathfrak{p}_n$, defined by:

$$\deg_{\mathfrak{p}_n} x_i = n - 2i \quad (i \geq 0)$$

The *n-th standard up operator* on R is the induced up operator for $\mathfrak{p}_n$, denoted U_n.

Lemma 4.2. *Given $n \geq 0$, $\mathfrak{p}_n$ is n-compatible, and every n-compatible $\mathbb{Z}$-grading of R is proportional to $\mathfrak{p}_n$.*

Proof. Let $\mathfrak{g}$ be an n-compatible $\mathbb{Z}$-grading of R. Since $\mathfrak{g}$ is compatible,

$$\deg_{\mathfrak{g}} x_i = (-\deg_{\mathfrak{g}} D)i + \deg_{\mathfrak{g}} x_0$$

for each $i \geq 0$. Summing each side over all $i = 0, ..., n$ yields:

$$0 = (-\deg_{\mathfrak{g}} D)\frac{n(n+1)}{2} + (n+1)\deg_{\mathfrak{g}} x_0 \quad \Rightarrow \quad n\deg_{\mathfrak{g}} D = 2\deg_{\mathfrak{g}} x_0$$

Therefore:

$$n\deg_{\mathfrak{g}} x_i = (-n\deg_{\mathfrak{g}} D)i + n\deg_{\mathfrak{g}} x_0 = (n-2i)\deg_{\mathfrak{g}} x_0 = (\deg_{\mathfrak{g}} x_0)(\deg_{\mathfrak{p}_n} x_i)$$

$\square$

The following properties for $\mathfrak{p}_n$ and U_n are easily checked.

Lemma 4.3. **(a)** $\deg_{\mathfrak{p}_n} = n\deg_{\mathfrak{r}} - 2\deg_{\mathfrak{s}}$
In particular, if $f \in V_{(r,s)}$, then f is $\mathfrak{p}_n$-homogeneous, and $\deg_{\mathfrak{p}_n} f = nr - 2s$.

(b) $U_n x_i = (i+1)(n-i)x_{i+1} \quad (i \geq 0)$

(c) $\deg_{\mathfrak{p}_n} D = -\deg_{\mathfrak{p}_n} U_n = 2$

(d) $\deg_{\mathfrak{r}} U_n = 0$ *and* $\deg_{\mathfrak{s}} U_n = 1$

Restricting U_n to R_n, we also have:

Lemma 4.4. $\deg_{\mathfrak{p}_n} f = \deg_{U_n} f$ *for every $\mathfrak{p}_n$-homogeneous $f \in A_n$. Consequently:*

(a) $\deg_{U_n} f = nr - 2s \geq 0$ *for every non-zero $f \in A_n \cap W_{(r,s)}$*

(b) $A_n \cap W_{(r,s)} = \{0\}$ *if $nr - 2s < 0$*

(c) $A_n \cap \ker U_n = \{0\} \cup \{f \in A_n \mid \deg_{\mathfrak{p}_n} f = 0\}$

Proof. If $N = \deg_{U_n} f$, then $U_n^{N+1} f = 0$ and $U_n^N f \neq 0$. From equation (6), we have

$$0 = DU_n^{N+1} f = c_{N+1} U_n^N f$$

where:

$$c_{N+1} = (N+1)\deg_{\mathfrak{p}_n} f - \tfrac{N(N+1)}{2}\deg_{\mathfrak{p}_n} D$$

Therefore:

$$0 = c_{N+1} \quad \Rightarrow \quad 2\deg_{\mathfrak{p}_n} f = N\deg_{\mathfrak{p}_n} D = 2N$$

$\square$

Lemma 4.5. *Let n, k be integers with $1 \leq k < n$. Assume $f \in A_k$ is $\mathfrak{p}_k$-homogeneous, and set $d = \deg_{\mathfrak{p}_k} f$. Then $\theta U_k^n f \in A_n$ and:*

$$\partial_n \theta U_k^n f = \begin{cases} 0 & 0 \leq d \leq n-1 \\ (-1)^n \frac{n!d!}{(d-n)!} f & d \geq n \end{cases}$$

Proof. Note first that $U_k^n f \in R_k \cap R^{(n)}$, which implies $\theta U_k^n f \in A_n$. From *Lemma 4.1* it follows that:

$$\partial_n \theta U_k^n f = (-1)^n D^n U_k^n f$$

From this, *Lemma 3.2* implies

$$\partial_n \theta U_k^n f = (-1)^n c_1 \cdots c_n f$$

where the sequence $c_i \in \mathbb{Z}$ is defined by:

$$c_i = id - \tfrac{i(i-1)}{2} \deg_{\mathfrak{p}_k} D$$

Since $\deg_{\mathfrak{p}_k} D = 2$ by *Lemma 4.3*, it follows that:

$$c_i = i(d-i+1)\ ,\ 1 \leq i \leq n$$

Therefore, the product $c_1 \cdots c_n$ equals 0 if $d < n$, and equals $n!d!/(d-n)!$ if $d \geq n$. □

Proposition 4.1. ([7], Cor. 2.3) *Let $r, s \geq 0$ be given. Given $n \geq 1$, the mapping*

$$D : R_n \cap V_{(r,s+1)} \to R_n \cap V_{(r,s)}$$

is surjective if $2s < rn$, and injective if $2s \geq rn$.

Proof. Consider first the case that $2s < rn$. Given k with $0 \leq k \leq s$, set:

$$(R_n \cap V_{(r,s)})^{(k)} = \ker D^{k+1} \cap (R_n \cap V_{(r,s)})$$

This gives a nested sequence of subspaces of $R_n \cap V_{(r,s)}$, with:

$$(R_n \cap V_{(r,s)})^{(0)} = R_n \cap W_{(r,s)} \quad \text{and} \quad (R_n \cap V_{(r,s)})^{(s)} = R_n \cap V_{(r,s)}$$

We show by induction on k that D surjects onto $(R_n \cap V_{(r,s)})^{(k)}$ for each $k = 0, ..., s$.

Let non-zero $f \in R_n \cap W_{(r,s)}$ be given. Then $U_n f \in R_n \cap V_{(r,s+1)}$, since $\deg_{\mathfrak{s}} U_n = 1$. By *Lemma 3.2*, we have:

$$DU_n f = (\deg_{\mathfrak{p}_n} f) f = (nr - 2s) f \neq 0$$

This establishes the basis for induction.

Given k with $1 \leq k \leq s$, assume that D surjects onto $(R_n \cap V_{(r,s)})^{(k-1)}$. Let $g \in (R_n \cap V_{(r,s)})^{(k)}$ be given, and assume that $D^k g \neq 0$. By *Lemma 3.2*, we have

$$D^{k+1} U_n^{k+1} (D^k g) = c_1 \cdots c_k D^k g\ ,$$

where the constants c_i $(1 \leq i \leq k)$ are given by:

$$c_i = i \deg_{\mathfrak{p}_n}(D^k g) - \tfrac{i(i-1)}{2} \deg_{\mathfrak{p}_n} D = i(nr - 2(s-k) - i + 1) = i(nr - 2s + 2k - i + 1) > 0$$

Define:

$$h = DU_n^{k+1} D^k g - c_1 \cdots c_k g \in (R_n \cap V_{(r,s)})^{(k-1)}$$

By the inductive hypothesis, there exists $\eta \in R_n \cap V_{(r,s+1)}$ such that $D\eta = h$. It follows that:

$$c_1 \cdots c_k g = DU_n^{k+1} D^k g - D\eta = D(U_n^{k+1} D^k g - \eta)$$

By induction, we conclude that D surjects onto $(R_n \cap V_{(r,s)})^{(k)}$. Therefore, $D : R_n \cap V_{(r,s+1)} \to R_n \cap V_{(r,s)}$ is surjective if $2s < nr$.

Consider next the case $2s \geq rn$. By *Lemma 4.4(b)*:

$$nr - 2(s+1) < 0 \quad \Rightarrow \quad A_n \cap W_{(r,s+1)} = \{0\}$$

Therefore, the restriction of D to $R_n \cap V_{(r,s+1)}$ is injective in this case. □

Remark 4.1. Homogeneous elements $f \in A_n \cap \ker U_n$ have $\deg_{\mathfrak{p}_n} f = 0$, and these are precisely the homogeneous SL_2-invariants for R_n. For example, when n is even, these include the quadratic form $\theta(x_n)$, which is composed of monomials $x_i x_{n-i}$, $0 \leq i \leq n$.

5. Cubic Invariants

In this section, we determine a basis for $W_{(3,s)}$ for each $s \geq 0$, as described in *Thm. 5.1*. Given $l, n \geq 0$ with $n \geq 2l$, note that:

$$\deg_{U_n} \theta(x_{2l}) = 2(n-2l)$$

We therefore want to consider the integrals $U_n^k \theta(x_{2l})$ with $l \geq 0$ and $1 \leq k \leq 2(n-2l)$.

Recall from *Lemma 3.2* that

$$D^k U_n^k \theta(x_{2l}) = c_1 \cdots c_k \, \theta(x_{2l})$$

where:

$$c_i = i \deg_{\mathfrak{p}_n} \theta(x_{2l}) - \tfrac{i(i-1)}{2} \deg_{\mathfrak{p}_n} D = i(2n - 4l - i + 1) \quad (1 \leq i \leq k)$$

Suppose that $c_{i_0} \leq 0$ for some $i_0 \leq k$. Then

$$2n - 4l - i_0 + 1 \leq 0 \quad \Rightarrow \quad 2n - 4l \leq i_0 - 1 \leq k - 1 \leq 2n - 4l - 1$$

which is a contradiction. We have thus established the following fact.

Lemma 5.1. *Let $n, l, k \in \mathbb{Z}$ satisfy $l \geq 0$ and $1 \leq k \leq 2(n-2l)$.*

(a) $D^k U_n^k \theta(x_{2l}) = c_1 \cdots c_k \, \theta(x_{2l})$*, where* $c_i = i(2n - 4l - i + 1) > 0$ *for* $1 \leq i \leq k$

(b) $\deg_D U_n^k \theta(x_{2l}) = k$

Proposition 5.1. *Let $n, l, k \in \mathbb{Z}$ satisfy $l \geq 0$, $n \geq 2l$, and $0 \leq k \leq n$.*

(a) *If $k < n - 2l$ or $k > 2(n-2l)$, then:*

$$\frac{\partial}{\partial x_n} \theta U_n^k \theta(x_{2l}) = 0$$

(b) *If $0 \leq n - 2l \leq k \leq 2(n-2l)$, then*

$$\frac{\partial}{\partial x_n} \theta U_n^k \theta(x_{2l}) = \begin{cases} a_k \theta(x_{2l+k-n}) & k < n \\ (a_n + (-1)^n b)\theta(x_{2l}) & k = n \end{cases}$$

where:

$$a_k = 2\frac{n!k!(n-2l)!}{(2l)!(2n-4l-k)!} \quad \text{and} \quad b = \frac{n!(2n-4l)!}{(n-4l)!}$$

(c) *If $n - k$ is odd, or if $n = k = 4l + 1$, then:*

$$\frac{\partial}{\partial x_n} \theta U_n^k \theta(x_{2l}) = 0$$

Proof. Note first that $k \leq n$ implies $\theta U_n^k \theta(x_{2l}) \in R_n$. In addition, *Lemma 4.1(b)* implies:

$$\frac{\partial}{\partial x_n} \theta U_n^k \theta(x_{2l}) = \theta \left(\frac{\partial}{\partial x_n} U_n^k \theta(x_{2l}) \right) + (-1)^n D^n U_n^k \theta(x_{2l}) \tag{11}$$

Since

$$\deg_{U_n} \theta(x_{2l}) = 2(n-2l)$$

we see that:

$$U_n^k \theta(x_{2l}) = 0 \quad \text{if} \quad k > 2(n-2l)$$

Assume that $k < n - 2l$. In this case:

$$U_n^k \theta(x_{2l}) \in V_{(2,2l+k)} \subset R_{2l+k} \quad \text{and} \quad 2l + k < n \quad \Rightarrow \quad \frac{\partial}{\partial x_n} U_n^k \theta(x_{2l}) = 0$$

In addition, $k < n - 2l \leq n$ means that $D^n U_n^k \theta(x_{2l}) = 0$. From equation (11), we conclude that

$$\frac{\partial}{\partial x_n}\, \theta U_n^k \theta(x_{2l}) = 0$$

when $k < n - 2l$ or $k > 2(n - 2l)$. This proves part (a).

For part (b), assume that $0 \leq n - 2l \leq k \leq 2(n - 2l)$. If $n = 2l$, then $k = 0$, and it is easy to check that the stated equalities hold in this case. So assume that $n > 2l$.

Since $U_n^k \theta(x_{2l}) \in V_{(2,2l+k)}$, there exists $a_k \in \mathbb{Z}$ such that:

$$\frac{\partial}{\partial x_n}\, U_n^k \theta(x_{2l}) = a_k x_{2l+k-n}$$

If $k < n$, then $D^n U_n^k \theta(x_{2l}) = 0$. If $k = n$, then *Lemma 5.1* implies:

$$D^n U_n^n\, \theta(x_{2l}) = c_1 \cdots c_n \theta(x_{2l})$$

where $c_i = i(2n - 4l - i + 1) > 0$. Equation (11) thus becomes:

$$\frac{\partial}{\partial x_n}\, \theta U_n^k \theta(x_{2l}) = \begin{cases} a_k \theta(x_{2l+k-n}) & k < n \\ \left(a_n + (-1)^n (c_1 \cdots c_n)\right) \theta(x_{2l}) & k = n \end{cases}$$

It remains to determine the constants a_k.

Recall that

$$[\partial/\partial x_n, D] = [\partial/\partial x_0, U_n] = 0$$

when these derivations are restricted to R_n. Consider first the case $n - 2l = k$:

$$\frac{\partial}{\partial x_n}\, U_n^{n-2l} \theta(x_{2l}) = a_{n-2l} x_0 \quad \Rightarrow \quad U_n^k \theta(x_{2l}) = a_{n-2l} x_0 x_n + f$$

for some $f \in \mathbf{k}[x_1, ..., x_{n-1}]$. Therefore,

$$a_{n-2l} x_n = \frac{\partial}{\partial x_0}\, U_n^{n-2l} \theta(x_{2l}) = U_n^{n-2l} \frac{\partial}{\partial x_0} \theta(x_{2l}) = U_n^{n-2l}(2x_{2l}) = 2\omega_{2l} \cdots \omega_{n-1} x_n$$

where $\omega_i = (i + 1)(n - i)$. It follows that:

$$a_{n-2l} = 2\omega_{2l} \cdots \omega_{n-1} = \frac{2n!(n - 2l)!}{(2l)!}$$

Next, assume $k > n - 2l$. From equation (6), it follows that:

$$\begin{aligned} a_k x_{2l+(k-1)-n} &= D(a_k x_{2l+k-n}) \\ &= D \frac{\partial}{\partial x_n} U_n^k\, \theta(x_{2l}) \\ &= \frac{\partial}{\partial x_n} D U_n^k\, \theta(x_{2l}) \\ &= \frac{\partial}{\partial x_n} c_k U_n^{k-1}\, \theta(x_{2l}) \\ &= c_k a_{k-1} x_{2l+(k-1)-n} \end{aligned}$$

Therefore, $a_k = c_k a_{k-1}$. By induction, for all k with $n - 2l \leq k \leq 2(n - 2l)$:

$$a_k = c_k c_{k-1} \cdots c_{n-2l+1} a_{n-2l} = \frac{k!}{(2n - 4l - k)!} \frac{2n!(n - 2l)!}{(2l)!}$$

Moreover, if $k = n$, then $n \geq 4l$ and:

$$c_1 \cdots c_n = \frac{n!(2n - 4l)!}{(n - 4l)!}$$

This proves part (b).

For part (c), note first that $\theta(x_{2l+k-n}) = 0$ when $n-k$ is odd. In addition, it is easy to check that $a_n + (-1)^n b = 0$ when $n = k = 4l+1$. Therefore, part (c) follows from parts (a) and (b). $\square$

Theorem 5.1. *Let $s \geq 0$ be given, and let t be such that $0 \leq s - 6t \leq 5$.*

(a) *If s is even, a basis of $W_{(3,s)}$ is given by:*

$$\theta U^{4i}_{s-2i}\theta(x_{s-4i}) \ , \quad 0 \leq i \leq t$$

(b) *If $s = 6t+3$ or $s = 6t+5$, a basis of $W_{(3,s)}$ is given by:*

$$\theta U^{4i+1}_{s-2i}\theta(x_{s-(4i+1)}) \ , \quad 0 \leq i \leq t$$

(c) *If $s = 6t+1$, a basis of $W_{(3,s)}$ is given by:*

$$\theta U^{4i+1}_{s-2i}\theta(x_{s-(4i+1)}) \ , \quad 0 \leq i \leq t-1$$

Proof. By *Cor. 3.2(b)*, it suffices to show that each set of elements is linearly independent. Set $n = s - 2i$ for $0 \leq i \leq t$ ($s \neq 6t+1$) or $0 \leq i \leq t-1$ ($s = 6t+1$). Likewise, set $k = 4i$ if s is even, or $k = 4i+1$ if s is odd. Then *Lemma 5.1* implies $\deg_D U^k_n\theta(x_{2l}) = k$. Therefore:

$$U^k_n\theta(x_{s-k}) \in V_{(2,s)} \cap R^{(k)}_n \quad \Rightarrow \quad \theta U^k_n\theta(x_{s-k}) \in W_{(3,s)} \cap A_n$$

In each case, *Prop. 5.1(b)* implies that:

$$\deg_{x_n} \theta U^k_n\theta(x_{s-k}) = 1$$

Therefore, $\theta U^k_n\theta(x_{s-k}) \in A_n - A_{n-1}$ when $s \neq 0$. In each case, this suffices to conclude that the given set is linearly independent. $\square$

Corollary 5.1. *Given $n \geq 0$, let $m \geq 0$ be such that $0 \leq n - 4m \leq 3$.*

(a) $\dim\left(W_3 \cap (A_n - A_{n-1})\right) = \begin{cases} m+1 & n \equiv 0,2,3 \pmod 4 \\ m & n \equiv 1 \pmod 4 \end{cases}$

(b) $\dim(W_3 \cap A_n) = \begin{cases} 2m^2+2m+1 & n \equiv 0 \pmod 4 \\ 2m^2+3m+1 & n \equiv 1 \pmod 4 \\ 2m^2+4m+2 & n \equiv 2 \pmod 4 \\ 2m^2+5m+3 & n \equiv 3 \pmod 4 \end{cases}$

Proof. It suffices to prove part (a), since part (b) follows easily from part (a).

Consider the array $\mathcal{T}$ of integer triples (t,u,i) such that:

$$t \geq 0 \ , \ 0 \leq u \leq 5 \ , \ 0 \leq i \leq t-1 \text{ if } u = 1 \ , \ 0 \leq i \leq t \text{ if } u \neq 1$$

Order $\mathcal{T}$ lexicographically, and set

$$\lambda_{(t,u,i)} = 6t + u - 2i \quad \text{for} \quad (t,u,i) \in \mathcal{T} \ .$$

Elements of $\mathcal{T}$ are in bijective correspondence to the basis of W_3 described in *Thm. 5.1*, where (t,u,i) corresponds to $\theta U^{4i}_{s-2i}\theta(x_{s-4i})$ if $s = 6t+u$ for even u, or to $\theta U^{4i+1}_{s-2i}\theta(x_{s-(4i+1)})$ if $s = 6t+u$ for odd u. Since

$$\theta U^k_n\theta(x_{s-k}) \in A_n - A_{n-1}$$

for $n = \lambda_{(t,u,i)} = s - 2i$ and corresponding k, we have:

$$d(n) := \dim W_3 \cap (A_n - A_{n-1}) = \#\{(t,u,i) \in \mathcal{T} \mid n = \lambda_{(t,u,i)}\}$$

The first triple in $\mathcal{T}$ which gives n has the form $(t,u,0)$, i.e.,

$$n = \lambda_{(t,u,0)} = 6t + u \ .$$

The last triple in $\mathcal{T}$ giving n has the form $(t+a, v, t+a)$ for some $a \geq 0$ and $v \in \{0,2,3,5\}$, i.e.,

$$n = \lambda_{(t+a,v,t+a)} = 6(t+a) + v - 2(t+a) = 4(t+a) + v \tag{12}$$

where $d(n) = t + a + 1$. Note that $v \neq 4$, since:

$$n = \lambda_{(t+a,4,t+a)} \quad \Rightarrow \quad n = \lambda_{(t+a+1,0,t+a+1)}$$

From equation (12), we conclude that:

$$m = \begin{cases} t+a & v \in \{0,2,3\} \\ t+a+1 & v = 5 \end{cases}$$

Since $v \equiv n \pmod 4$, it follows that:

$$d(n) = \begin{cases} m+1 & n \equiv 0,2,3 \pmod 4 \\ m & n \equiv 1 \pmod 4 \end{cases}$$

This completes the proof of part (a). □

Example 5.1. For the case $n = 10$, *Cor. 5.1* implies that $\dim(W_3 \cap A_{10}) = 18$. This confirms the calculation of Cerezo ([7], Chap.I, p.10), in which the author gives an explicit list of 18 basis elements.

6. The Core Cubic Invariants

In this section, we determine, for each $n \geq 3$, a homogeneous basis for a space which is complementary to the space of reducible cubic elements of A_n, as described in *Thm. 6.2.*

6.1. **Compound and Core Invariants.** Note that A_3 admits a homogeneous generator h of standard degree 4, whereas A_4 is generated in degree 3. Therefore, h can be expressed as a polynomial in elements of strictly smaller degree, although doing so requires more variables. Specifically:

$$h = x_0\theta(4x_2x_4 - 3x_3^2) - 3\theta(x_2)\theta(x_4)$$

In classical terminology, h is a groundform of A_3, but is not a groundform of A_4. We want to identify groundforms $f \in A_n$ which remain groundforms in A_N for every $N \geq n$.

Recall that $f \in A$ is a compound invariant (relative to standard degrees) if there exist $g_1, ..., g_m \in A$ of strictly smaller degree ($m \geq 1$) and $P \in \mathbf{k}^{[m]}$ such that $f = P(g_1, ..., g_m)$. Otherwise, f is a core invariant. Given $r \geq 0$, define the vector space of compound r-forms:

$$H_r = W_r \cap \mathbf{k}[W_1, ..., W_{r-1}] = \sum_{1 \leq i \leq r/2} W_i W_{r-i}$$

Note that any element $f \in W_r$ which is not in H_r is necessarily a core invariant.

Given $s \geq 0$, set

$$H_{(r,s)} = H_r \cap W_{(r,s)}$$

and let $K_{(r,s)}$ be a complementary subspace of $H_{(r,s)}$:

$$W_{(r,s)} = H_{(r,s)} \oplus K_{(r,s)}$$

Given subspaces $K_{(r,s)}$, define:

$$K_r = \oplus_{s \geq 0} K_{(r,s)} \quad \text{and} \quad K = \oplus_{r \geq 0} K_r$$

Then $A = \mathbf{k}[K]$. Note that $A_n = \mathbf{k}[K \cap A_n]$ for $n = 1, 2, 4$, whereas $A_n \neq \mathbf{k}[K \cap A_n]$ for $n = 3, 5, 6, 7$. From this, it is easy to verify that A_1, A_2, and A_4 are degree closed subalgebras of A.

Clearly, $H_1 = \{0\}$ and $W_1 = K_1 = \mathbf{k} \cdot x_0$. Similarly, $H_2 = W_1^2 = \mathbf{k} \cdot x_0^2$, and by *Cor. 3.3*, we may take:

$$K_2 := \oplus_{k \geq 1} \mathbf{k} \cdot \theta(x_{2k})$$

Remark 6.1. In the language of classical invariant theory, the core invariants of the down operator were termed *perpetuants*. They were introduced by Sylvester in 1882 [36], and were viewed as invariants of infinite order. The generating function for the dimension of $K_{(r,s)}$ given by

$$\frac{x^{2^{r-1}-1}}{(1-x^2)(1-x^3)\cdots(1-x^r)} \quad (r \geq 3)$$

was formulated by MacMahon [28] and proved by Stroh [35]. In particular, $\dim K_{(r,s)}$ equals the coefficient of x^s in the corresponding power series. In more recent times, Kung and Rota [26] lamented that the theory of perpetuants remains in a "particularly sorry state" (p.82).

6.2. **A System of Core Cubic Invariants.** As for cubics, we have:

$$H_3 = W_1W_2 = x_0W_2$$

Therefore, given $f \in W_3$, f is a core invariant if and only if f is irreducible. When s is odd, this means $H_{(3,s)} = \{0\}$.

In general, there are many choices for a complementary subspace of H_3. *Theorem 5.1* above gives a basis of $W_{(3,s)}$ for each $s \geq 0$, thus giving a homogeneous basis $\mathcal{B}$ for W_3. The reducible elements of $\mathcal{B}$ form a basis of H_3, namely, $\{x_0\theta(x_s) \mid s \geq 0\}$. Let $\mathcal{B}'$ denote the the set of irreducible elements of $\mathcal{B}$.

Definition 6.1. K_3 is the complementary subspace of H_3 having basis $\mathcal{B}'$.

Theorem 6.1 below gives the basis for $K_{(3,s)} = K_3 \cap W_{(3,s)}$ obtained by reducing the basis for $W_{(3,s)}$. If $K_{(3,s)} \cap (A_n - A_{n-1}) \neq \{0\}$ for some n, s, then there is a unique element $C_{(n,s)} \in \mathcal{B}'$ belonging to $K_{(3,s)} \cap (A_n - A_{n-1})$. This allows us to place a total order on $\mathcal{B}'$ by using lexicographical order on the pairs (n, s). Details of this construction are spelled out in *Thm. 6.2* below.

Let $s \geq 0$ be given, and let $t \geq 0$ be such that $0 \leq s - 6t \leq 5$. Then *Cor. 3.2* implies:

$$\dim K_{(3,s)} = \begin{cases} t+1 & s \equiv 3, 5 \pmod 6 \\ t & s \equiv 0, 1, 2, 4 \pmod 6 \end{cases}$$

The reader can check that these values agree with those found via the generating function of MacMahon and Stroh for $r = 3$, which is given by:

$$\frac{x^3}{(1-x^2)(1-x^3)}$$

Theorem 6.1. *Let $s \geq 0$ be given, and let t be such that $0 \leq s - 6t \leq 5$.*

(a) *If s is even, a basis of $K_{(3,s)}$ is given by:*

$$\theta U^{4i}_{s-2i}\theta(x_{s-4i}) \ , \quad 1 \leq i \leq t$$

(b) *If $s = 6t + 3$ or $s = 6t + 5$, a basis of $K_{(3,s)}$ is given by:*

$$\theta U^{4i+1}_{s-2i}\theta(x_{s-(4i+1)}) \ , \quad 0 \leq i \leq t$$

(c) *If $s = 6t + 1$, a basis of $K_{(3,s)}$ is given by:*

$$\theta U^{4i+1}_{s-2i}\theta(x_{s-(4i+1)}) \ , \quad 0 \leq i \leq t-1$$

Since $\dim H_3 \cap (A_n - A_{n-1})$ equals 0 if n is odd, or 1 if n is even, *Cor. 5.1* implies the following.

Corollary 6.1. *Given $n \geq 0$, let $m \geq 0$ be such that $0 \leq n - 4m \leq 3$.*

(a) $\dim(K_3 \cap (A_n - A_{n-1})) = \begin{cases} m & n \equiv 0, 1, 2 \pmod 4 \\ m+1 & n \equiv 3 \pmod 4 \end{cases}$

(b) $\dim(K_3 \cap A_n) = \begin{cases} 2m^2 & n \equiv 0 \,(\mathrm{mod}\, 4) \\ 2m^2 + m & n \equiv 1 \,(\mathrm{mod}\, 4) \\ 2m^2 + 2m & n \equiv 2 \,(\mathrm{mod}\, 4) \\ 2m^2 + 3m + 1 & n \equiv 3 \,(\mathrm{mod}\, 4) \end{cases}$

In particular, the values $\dim(K_3 \cap A_n)$ for $n = 2, ..., 12$ are given respectively by:

$$0, 1, 2, 3, 4, 6, 8, 10, 12, 15, 18$$

These values confirm those found in the tables of Brouwer [4], apart from $n = 11$ for which no table is given.

Theorem 6.2. *Given $n \geq 3$, let $m \geq 0$ be such that $0 \leq n - 4m \leq 3$. Define $I_n \subset \mathbb{Z}^2$ by:*

$$I_n = \begin{cases} \{(n, n+2i) \,|\, 1 \leq i \leq m\} & n \equiv 0, 2 \,(\mathrm{mod}\, 4) \\ \{(n, n+2i) \,|\, 0 \leq i \leq m-1\} & n \equiv 1 \,(\mathrm{mod}\, 4) \\ \{(n, n+2i) \,|\, 0 \leq i \leq m\} & n \equiv 3 \,(\mathrm{mod}\, 4) \end{cases}$$

Given $(n, s) \in I_n$, define the polynomial:

$$C_{(n,s)} = \begin{cases} \theta U_n^{2(s-n)} \theta(x_{2n-s}) & n \text{ even} \\ \theta U_n^{2(s-n)+1} \theta(x_{2n-s-1}) & n \text{ odd} \end{cases}$$

(a) *A basis of $K_3 \cap (A_n - A_{n-1})$ is given by:*

$$\{C_{(n,s)} \,|\, (n, s) \in I_n\}$$

(b) *Given $(n, s) \in I_n$, there exists non-zero $a \in \mathbb{Z}$ such that:*

$$\frac{\partial}{\partial x_n} C_{(n,s)} = a\, \theta(x_{s-n})$$

Proof. Define the array $\mathcal{L}$ of integer triples (t, u, i) as follows:

$$(t, u, i) \in \mathcal{L} \quad \Leftrightarrow \quad t \geq 0 \ , \ 0 \leq u \leq 5 \ , \text{ and } \begin{cases} 1 \leq i \leq t & u = 0, 2 \\ 0 \leq i \leq t & u = 3, 5 \\ 0 \leq i \leq t-1 & u = 1 \end{cases}$$

The $\mathbb{Z}$-linear map $\gamma : \mathbb{Z}^3 \to \mathbb{Z}^2$ defined by

$$\gamma(t, u, i) = (6t + u - 2i, 6t + u)$$

is injective on $\mathcal{L}$. Define a total order on $\mathcal{L}$ as the pullback of lexicographical order on $\gamma(\mathcal{L})$. Given $n \geq 0$, define the planar subarray:

$$\mathcal{L}_n = \{(t, u, i) \in \mathcal{L} \,|\, n = 6t + u - 2i\}$$

Then:

$$\mathcal{L}_n = \begin{cases} [(t_0, u_0, 1), (m, n - 4m, m)] & n \equiv 0, 2 \,(\mathrm{mod}\, 4) \\ [(t_0, u_0, 0), (m-1, 5, m-1)] & n \equiv 1 \,(\mathrm{mod}\, 4) \\ [(t_0, u_0, 0), (m, 3, m)] & n \equiv 3 \,(\mathrm{mod}\, 4) \end{cases}$$

where t_0, u_0 are determined by $n = 6t_0 + u_0 - 2$ for n even, or $n = 6t_0 + u_0$ for n odd.

Given $(t, u, i) \in \mathcal{L}$, set $s = 6t + u$, and define the polynomial:

$$P_{(t,u,i)} = \begin{cases} \theta U_{s-2i}^{4i} \theta(x_{s-4i}) & u \text{ even} \\ \theta U_{s-2i}^{4i+1} \theta(x_{s-(4i+1)}) & u \text{ odd} \end{cases}$$

From *Thm. 6.1* and *Cor. 6.1*, we see that the set

$$\{P_{(t,u,i)} \,|\, (t, u, i) \in \mathcal{L}_n\}$$

is an ordered basis of $K_3 \cap (A_n - A_{n-1})$. Part (a) now follows from the fact that:

$$\gamma(\mathcal{L}_n) = I_n \quad \text{and} \quad C_{\gamma(t,u,i)} = P_{(t,u,i)}$$

Part (b) is implied by *Prop. 5.1(b)*. $\square$

Corollary 6.2. *Let the integers $n, s \geq 0$ be given.*

(a) *If n and s differ in parity, then:*

$$K_{(3,s)} \cap (A_n - A_{n-1}) = \{0\}$$

(b) *If n is even and s is odd, then:*

$$W_{(3,s)} \cap (A_n - A_{n-1}) = \{0\}$$

Proof. Part (a) follows from *Thm. 6.2(a)*, since the components of any element of I_n have the same parity. Part (b) follows from part (a) and the fact that $W_{(3,s)} = K_{(3,s)}$ when s is odd. $\square$

Example 6.1. The entries of *Table 3* comprise a basis of $K_3 \cap A_{32}$, which is of dimension 128. The classical order of $C_{(n,s)}$ in A_{32} equals $96 - 2s$, so these orders range in value from 0 to 90.

TABLE 3. Pairs (n, s) for the ordered basis of core cubics $C_{(n,s)}$ in A_{32}

I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}	I_{11}	I_{12}
(3,3)	(4,6)	(5,5)	(6,8)	(7,7)	(8,10)	(9,9)	(10,12)	(11,11)	(12,14)
				(7,9)	(8,12)	(9,11)	(10,14)	(11,13)	(12,16)
								(11,15)	(12,18)

I_{13}	I_{14}	I_{15}	I_{16}	I_{17}	I_{18}	I_{19}	I_{20}	I_{21}	I_{22}
(13,13)	(14,16)	(15,15)	(16,18)	(17,17)	(18,20)	(19,19)	(20,22)	(21,21)	(22,24)
(13,15)	(14,18)	(15,17)	(16,20)	(17,19)	(18,22)	(19,21)	(20,24)	(21,23)	(22,26)
(13,17)	(14,20)	(15,19)	(16,22)	(17,21)	(18,24)	(19,23)	(20,26)	(21,25)	(22,28)
		(15,21)	(16,24)	(17,23)	(18,26)	(19,25)	(20,28)	(21,27)	(22,30)
						(19,27)	(20,30)	(21,29)	(22,32)

I_{23}	I_{24}	I_{25}	I_{26}	I_{27}	I_{28}	I_{29}	I_{30}	I_{31}	I_{32}
(23,23)	(24,26)	(25,25)	(26,28)	(27,27)	(28,30)	(29,29)	(30,32)	(31,31)	(32,34)
(23,25)	(24,28)	(25,27)	(26,30)	(27,29)	(28,32)	(29,31)	(30,34)	(31,33)	(32,36)
(23,27)	(24,30)	(25,29)	(26,32)	(27,31)	(28,34)	(29,33)	(30,36)	(31,35)	(32,38)
(23,29)	(24,32)	(25,31)	(26,34)	(27,33)	(28,36)	(29,35)	(30,38)	(31,37)	(32,40)
(23,31)	(24,34)	(25,33)	(26,36)	(27,35)	(28,38)	(29,37)	(30,40)	(31,39)	(32,42)
(23,33)	(24,36)	(25,35)	(26,38)	(27,37)	(28,40)	(29,39)	(30,42)	(31,41)	(32,44)
				(27,39)	(28,42)	(29,41)	(30,44)	(31,43)	(32,46)
								(31,45)	(32,48)

The cubics listed in *Table 3* can be calculated explicitly using a computer algebra system, although the resulting integer coefficients tend to have a very large common divisor. As an illustration, we used *Maple* to find $C_{(32,48)} = \theta U_{32}^{32}\theta(x_{16})$. Note that $C_{(32,48)}$ is the unique cubic invariant of the SL_2-action on R_{32}. We find that the greatest common divisor of the coefficients of $C_{(32,48)}$ equals:

$$d = 2^{61} \cdot 3^{28} \cdot 5^{13} \cdot 7^{8} \cdot 11^{4} \cdot 13^{4} \cdot 17^{2} \cdot 19^{2} \cdot 23^{2} \cdot 29 \cdot 31$$

Dividing $C_{(32,48)}$ by d produces the following reasonable output.

$$
\begin{aligned}
&C_{(32,48)} = \\
&-39916\, x_7x_{17}x_{24} - 941732\, x_7x_{18}x_{23} + 1275204\, x_7x_{19}x_{22} - 587860\, x_7x_{20}x_{21} + 53940\, x_0x_{16}x_{32} - \\
&458490\, x_0x_{17}x_{31} + 1996650\, x_0x_{18}x_{30} - 5901210\, x_0x_{19}x_{29} + 13226850\, x_0x_{20}x_{28} - 23808330\, x_0x_{21}x_{27} + \\
&35565530\, x_0x_{22}x_{26} - 44945450\, x_0x_{23}x_{25} + 53940\, x_2x_{14}x_{32} - 350610\, x_2x_{15}x_{31} + 1133610\, x_2x_{16}x_{30} - \\
&2366400\, x_2x_{17}x_{29} + 3421080\, x_2x_{18}x_{28} - 3255840\, x_2x_{19}x_{27} + 1175720\, x_2x_{20}x_{26} + 2377280\, x_2x_{21}x_{25} - \\
&5784284\, x_2x_{22}x_{24} + 53940\, x_4x_{12}x_{32} - 242730\, x_4x_{13}x_{31} + 486330\, x_4x_{14}x_{30} - 449790\, x_4x_{15}x_{29} - \\
&178110\, x_4x_{16}x_{28} + 1219920\, x_4x_{17}x_{27} - 1914880\, x_4x_{18}x_{26} + 1472880\, x_4x_{19}x_{25} + 145996\, x_4x_{20}x_{24} - \\
&2000016\, x_4x_{21}x_{23} - 53940\, x_3x_{13}x_{32} + 296670\, x_3x_{14}x_{31} - 783000\, x_3x_{15}x_{30} + 1232790\, x_3x_{16}x_{29} - \\
&1054680\, x_3x_{17}x_{28} - 165240\, x_3x_{18}x_{27} + 2080120\, x_3x_{19}x_{26} - 3553000\, x_3x_{20}x_{25} + 3407004\, x_3x_{21}x_{24} - \\
&1406988\, x_3x_{22}x_{23} - 53940\, x_1x_{15}x_{32} + 404550\, x_1x_{16}x_{31} - 1538160\, x_1x_{17}x_{30} + 3904560\, x_1x_{18}x_{29} - \\
&7325640\, x_1x_{19}x_{28} + 10581480\, x_1x_{20}x_{27} - 11757200\, x_1x_{21}x_{26} + 9379920\, x_1x_{22}x_{25} - 3595636\, x_1x_{23}x_{24} - \\
&53940\, x_5x_{11}x_{32} + 188790\, x_5x_{12}x_{31} - 243600\, x_5x_{13}x_{30} - 36540\, x_5x_{14}x_{29} + 627900\, x_5x_{15}x_{28} - \\
&1041810\, x_5x_{16}x_{27} + 694960\, x_5x_{17}x_{26} + 442000\, x_5x_{18}x_{25} - 1618876\, x_5x_{19}x_{24} + 1854020\, x_5x_{20}x_{23} - \\
&813960\, x_5x_{21}x_{22} - 53940\, x_7x_9x_{32} + 80910\, x_7x_{10}x_{31} + 80040\, x_7x_{11}x_{30} - 334950\, x_7x_{12}x_{29} + \\
&311220\, x_7x_{13}x_{28} + 177450\, x_7x_{14}x_{27} - 760760\, x_7x_{15}x_{26} + 790110\, x_7x_{16}x_{25} + 24270543\, x_0{x_{24}}^2 + \\
&3595636\, x_2{x_{23}}^2 + 1406988\, x_4{x_{22}}^2 + 813960\, x_6{x_{21}}^2 + 26970\, {x_8}^2x_{32} + 587860\, x_8{x_{20}}^2 + 444312\, x_{12}{x_{18}}^2 + \\
&350658\, {x_{12}}^2x_{24} + 488376\, x_{10}{x_{19}}^2 + 186300\, {x_{10}}^2x_{28} + 93960\, {x_9}^2x_{30} + 278300\, {x_{11}}^2x_{26} + 427856\, x_{14}{x_{17}}^2 + \\
&418418\, {x_{14}}^2x_{20} + 396396\, {x_{13}}^2x_{22} + 424710\, {x_{15}}^2x_{18} + 141570\, {x_{16}}^3 + 53940\, x_6x_{10}x_{32} - 134850\, x_6x_{11}x_{31} + \\
&54810\, x_6x_{12}x_{30} + 280140\, x_6x_{13}x_{29} - 591360\, x_6x_{14}x_{28} + 413910\, x_6x_{15}x_{27} + 346850\, x_6x_{16}x_{26} - \\
&1136960\, x_6x_{17}x_{25} + 1176876\, x_6x_{18}x_{24} - 235144\, x_6x_{19}x_{23} - 1040060\, x_6x_{20}x_{22} - 26970\, x_8x_9x_{31} - \\
&160950\, x_8x_{10}x_{30} + 254910\, x_8x_{11}x_{29} + 23730\, x_8x_{12}x_{28} - 488670\, x_8x_{13}x_{27} + 583310\, x_8x_{14}x_{26} - \\
&29350\, x_8x_{15}x_{25} - 750194\, x_8x_{16}x_{24} + 981648\, x_8x_{17}x_{23} - 333472\, x_8x_{18}x_{22} - 687344\, x_8x_{19}x_{21} + \\
&464940\, x_{12}x_9x_{27} - 370300\, x_{12}x_{10}x_{26} - 278300\, x_{12}x_{11}x_{25} - 350658\, x_{12}x_{13}x_{23} - 442134\, x_{12}x_{14}x_{22} + \\
&838530\, x_{12}x_{15}x_{21} - 398090\, x_{12}x_{16}x_{20} - 460768\, x_{12}x_{17}x_{19} - 93960\, x_{10}x_9x_{29} - 186300\, x_{10}x_{11}x_{27} + \\
&648600\, x_{10}x_{13}x_{25} - 225584\, x_{10}x_{14}x_{24} - 548090\, x_{10}x_{15}x_{23} + 879630\, x_{10}x_{16}x_{22} - 372640\, x_{10}x_{17}x_{21} - \\
&532440\, x_{10}x_{18}x_{20} - 278640\, x_9x_{11}x_{28} - 94640\, x_9x_{13}x_{26} - 553960\, x_9x_{14}x_{25} + 779544\, x_9x_{15}x_{24} - \\
&231454\, x_9x_{16}x_{23} - 648176\, x_9x_{17}x_{22} + 1020816\, x_9x_{18}x_{21} - 488376\, x_9x_{19}x_{20} - 423016\, x_{11}x_{13}x_{24} + \\
&773674\, x_{11}x_{14}x_{23} - 331540\, x_{11}x_{15}x_{22} - 506990\, x_{11}x_{16}x_{21} + 905080\, x_{11}x_{17}x_{20} - 444312\, x_{11}x_{18}x_{19} - \\
&396396\, x_{14}x_{13}x_{21} - 418418\, x_{14}x_{15}x_{19} - 431002\, x_{14}x_{16}x_{18} - 440440\, x_{13}x_{15}x_{20} + 858858\, x_{13}x_{16}x_{19} - \\
&427856\, x_{13}x_{17}x_{18} - 424710\, x_{16}x_{15}x_{17}
\end{aligned}
$$

7. The Degree Closed Property

Proposition 7.1. *If $n \geq 3$ is odd or if $n \geq 10$, then relative to standard degrees, A_n is not degree closed in A_{n+1}.*

Proof. Given $P, Q \in A_{n+1}$ and $k \geq 0$, let $[P,Q]_k^{\partial_{n+1}}$ denote the vector product on A_{n+1} induced by ∂_{n+1} (as defined in Section 2.3). If P and Q are of degree k in x_{n+1}, then *Prop. 2.2(d)* implies:

$$[P,Q]_k^{\partial_{n+1}} \in A_n$$

We consider three cases.

Case 1: $n \geq 3$ is odd. Since $n+1$ is even, *Thm. 6.2(a)* implies that:

$$\min I_{n+1} = (n+1, n+3)$$

By *Thm. 6.2(c)*, there exists non-zero $a \in \mathbf{k}$ such that:

$$\partial_{n+1} C_{(n+1,n+3)} = a\theta(x_2)$$

Define $f \in H_{(4,n+3)} \cap A_n$ by:

$$f = [\theta(x_{n+1}), C_{(n+1,n+3)}]_1^{\partial_{n+1}} = 2x_0 C_{(n+1,n+3)} - a\theta(x_2)\theta(x_{n+1})$$

Note that $f \neq 0$, since x_0 does not divide the product $\theta(x_2)\theta(x_{n+1})$.

Suppose that f can be expressed as a polynomial in elements of A_n of degree less than 4. Given $s \geq 0$, we have:

$$H_{(4,s)} = x_0 W_{(3,s)} + \sum_{j=1}^{[s/4]} W_{(2,2j)} W_{(2,s-2j)} \tag{13}$$

Therefore, there exists $g \in W_{(3,n+3)} \cap A_n$ and $a_i \in \mathbf{k}$ such that:

$$f = x_0 g + a_1 \theta(x_4)\theta(x_{n-1}) + a_2 \theta(x_6)\theta(x_{n-3}) + \cdots$$

Therefore:

$$\partial_n f = x_0 \partial_n g$$

Since

$$\partial_n g \in W_{(2,3)} = \{0\}$$

it follows that:

$$0 = \partial_n f = 2x_0 \partial_n C_{(n+1,n+3)} + 2a x_1 \theta(x_2)$$

But this is impossible, since x_0 does not divide $x_1\theta(x_2)$. Therefore, f cannot be expressed as a polynomial in elements of A_n of degree less than 4.

Case 2: $n = 4m+2$ for $m \geq 2$. According to *Thm. 6.2(a)*, $(n+1, 6m+1), (n+1, 6m+3) \in I_{n+1}$. Define $f \in H_{(5,2n-3)} \cap A_n$ by

$$f = [C_{(n+1,6m+1)}, C_{(n+1,6m+3)}]_1^{\partial_{n+1}} = a_1 \theta(x_{2m-2}) C_{(n+1,6m+3)} - a_2 \theta(x_{2m}) C_{(n+1,6m+1)}$$

where a_1, a_2 are non-zero constants.

Suppose that f can be expressed as a polynomial in elements of A_n of degree less than 5. Since $H_5 = x_0 W_4 + W_2 W_3$, it follows that

$$f = x_0 G + \sum_{k=1}^{n/2} \theta(x_{2k}) F_{s-2k}$$

where $s = 2n - 3$, $G \in W_{(4,2n-3)} \cap A_n$, and $F_{s-2k} \in W_{(3,s-2k)} \cap A_n$. For each such k, *Cor. 6.2(b)* implies:

$$W_{(3,s-2k)} \cap (A_n - A_{n-1}) = \{0\}$$

It follows that:

$$\partial_n F_{s-2k} = 0 \quad \forall\ k \quad \Rightarrow \quad \partial_n f = x_0 \partial_n G + 2x_0 F_{s-n} \in x_0 A_n$$

Modulo x_0, it follows that:

$$a_1 \overline{\theta}(x_{2m-2}) \overline{\partial_n C}_{(n+1,6m+3)} - a_2 \overline{\theta}(x_{2m}) \overline{\partial_n C}_{(n+1,6m+1)} = 0$$

Since $\overline{\theta}(x_{2m})$ is prime in the ring $S = \mathbf{k}[x_1, ..., x_n]$, and $\overline{\theta}(x_{2m-2}) \notin \overline{\theta}(x_{2m}) \cdot S$, we conclude that

$$\overline{\partial_n C}_{(n+1,6m+3)} = \overline{\theta}(x_{2m}) \cdot h$$

for some $h \in S$. By degree considerations, $h \in V_{(0,1)} = \{0\}$. Therefore:

$$\partial_n C_{(n+1,6m+3)} \in x_0 R_n \cap V_{(2,2m+1)} = \mathbf{k} \cdot x_0 x_{2m+1}$$

In the same way we obtain:

$$\partial_n C_{(n+1,6m+1)} \in x_0 R_n \cap V_{(2,2m-1)} = \mathbf{k} \cdot x_0 x_{2m-1}$$

Therefore, there exist constants c_1, c_2 such that:

$$\partial_n f = a_1c_1x_0x_{2m+1}\theta(x_{2m-2}) - a_2c_2x_0x_{2m-1}\theta(x_{2m}) \in A_n$$

But this is clearly not possible, since $\deg_D x_{2m+1} \neq \deg_D x_{2m-1}$.

We conclude that f cannot be expressed as a polynomial in elements of A_n of degree less than 5.

Case 3: $n = 4m$ for $m \geq 3$. According to *Thm. 6.2(a)*, $(n+1, 6m-3), (n+1, 6m-1) \in I_{n+1}$. Define $f \in H_{(5,2n-5)} \cap A_n$ by

$$f = [C_{(n+1,6m-3)}, C_{(n+1,6m-1)}]_1^{\partial_{n+1}} = a_1\theta(x_{2m-4})C_{(n+1,6m-1)} - a_2\theta(x_{2m-2})C_{(n+1,6m-3)}$$

where a_1, a_2 are non-zero constants. The proof that f cannot be expressed as a polynomial in elements of A_n of degree less than 5 proceeds exactly as in Case 2. □

We next consider $H_{(4,s)} \cap A_n$ for even values of n.

Lemma 7.1. *Let $N \geq 3$ be an odd integer. Then the cubic polynomials*

$$x_1\theta(x_{N-1}) \ , \ x_3\theta(x_{N-3}) \ , \ \cdots \ , \ x_{N-2}\theta(x_2)$$

are linearly independent modulo x_0.

Proof. Consider $\bar{D} = \epsilon D \in \mathrm{LND}(\bar{R})$ as in Section 3.6, where $\bar{R} = \mathbf{k}[x_1, x_2, ...]$. Since

$$\deg_{\bar{D}} x_{2k+1}\epsilon\theta(x_{N-(2k+1)}) = 2k \ , \ 0 \leq k \leq \tfrac{N-2}{2}$$

these degrees are distinct, which implies that these polynomials are linearly independent modulo x_0. □

Proposition 7.2. *If $n \geq 0$ is even, then every element of $H_4 \cap A_n$ can be expressed as a polynomial in elements of A_n of degree less than 4.*

Proof. Let $s \geq 0$ be given. From equation (13), we see that $H_{(4,s)} = x_0W_{(3,s)}$ if s is odd. So assume that s is even. If $s \leq n$, then $W_{(r,s)} \subset A_s \subset A_n$ for each $r \geq 0$. So we may further assume that $n < s$.

Given $F \in H_{(4,s)} \cap A_n$, equation (13) implies that there exist $a_j \in \mathbf{k}$ and $G \in W_{(3,s)}$ such that:

$$F = x_0G + \sum_{j=0}^{[s/4]} a_j\theta(x_{2j})\theta(x_{s-2j})$$

Since $\deg_{U_n} F = 4n - 2s \geq 0$, it follows that $n < s \leq 2n$. Note that:

$$\partial_{n+1}\theta(x_{s-2j}) = \begin{cases} -2x_{(s-2j)-(n+1)} & n+1 \leq s-2j \\ 0 & n+1 > s-2j \end{cases}$$

Therefore:

$$0 = \partial_{n+1}F = x_0\partial_{n+1}G - \sum_{j=0}^{(s-n)/2-1} 2a_j\theta(x_{2j})x_{(s-2j)-(n+1)}$$

Note that $s \leq 2n$ insures $(s-n)/2 - 1 < [s/4]$. By *Lemma 7.1*, it follows that $a_j = 0$ when $1 \leq j \leq (s-n)/2 - 1$. Therefore:

$$x_0G = F - \sum_{j=(s-n)/2}^{[s/4]} a_j\theta(x_{2j})\theta(x_{s-2j}) \in A_n \quad \Rightarrow \quad G \in A_n$$

We conclude that, when n is even, every element of $H_{(4,s)} \cap A_n$ can be expressed as a polynomial in elements of A_n of degree less than 4. □

Remark 7.1. The results of these two propositions can be summarized as follows: (1) If $n \geq 2$ is even, then every quartic generator of A_n is a core invariant; (2) if $n \geq 3$ is odd, then A_n has a compound quartic generator; and (3) if $n \geq 10$ is even, then A_n has a compound quintic generator. This leaves open the question whether A_8 is degree closed in A.

Remark 7.2. If $A = \mathbf{k}[M]$ for any set M, then M is not bounded in degree. This was shown already by MacMahon in the Nineteenth Century; see [15]. The polynomials

$$L_n = x_0^{-1}\theta(x_1^n) \quad (n \geq 2)$$

give an easy way to see this: Each is linear and irreducible over $\mathbf{k}[x_0, x_1]$, and:

$$L_n \in W_{(n,n)} \cap (A_n - A_{n-1})$$

It follows that each L_n is a core invariant. In addition, since the x_n-coefficient of L_n is x_0^{n-1}, a unit of $\mathbb{C}(x_0)$, it follows that $L_2, ..., L_n$ forms a set of rational generators of A_n over $\mathbb{C}(x_0)$. This was known already to Weitzenböck [39], and used later in [16, 32, 37].

8. Application: Hilbert's Fourteenth Problem

Proposition 4.1 affords a surprisingly easy way to construct counterexamples to Hilbert's Fourteenth Problem. Given $n \geq 2$, define the sequence of integers k_r $(r \geq 0)$ by:

$$k_r = \begin{cases} nr/2 & nr \text{ even} \\ (nr+1)/2 & nr \text{ odd} \end{cases}$$

Define the index set $J = \{(0,0)\} \cup \{(r,s) \,|\, r \geq 1,\, k_{r-1} + 1 \leq s \leq k_r\}$.

Theorem 8.1. *There exists a sequence* $w_{(r,s)} \in R_n \cap V_{(r,s)}$ *for* $(r,s) \in J$ *such that* $w_{(0,0)} = 1$*, and for* $r \geq 1$ *:*

$$Dw_{(r,s)} = \begin{cases} w_{(r,s-1)} & k_{r-1} + 2 \leq s \leq k_r \\ x_0 w_{(r-1,k_{r-1})} & s = k_{r-1} + 1 \end{cases}$$

Proof. Given $(r,s) \in J$, set $\tilde{V}_{(r,s)} = R_n \cap V_{(r,s)}$. Using lexicographical ordering on J, assume that the sequence $w_{(i,j)} \in \tilde{V}_{(i,j)}$ has been constructed up to $(i,j) = (r-1, k_{r-1})$, where $r \geq 1$. By *Prop. 4.1*, each mapping in the following sequence of maps is surjective:

$$x_0\tilde{V}_{(r-1,k_{r-1})} \subset \tilde{V}_{(r,k_{r-1})} \xleftarrow{D} \tilde{V}_{(r,k_{r-1}+1)} \xleftarrow{D} \cdots \xleftarrow{D} \tilde{V}_{(r,k_r-1)} \xleftarrow{D} \tilde{V}_{(r,k_r)}$$

We may thus extend the sequence $w_{(i,j)}$ to $(i,j) = (r, k_r)$. □

Definition 8.1. For $k \geq 0$, the *basic* $\mathbb{G}_a$-module B_k is defined by exponentiation of the restriction of the down operator D to R_k.

Note that $B_k \cong \mathbb{A}^{k+1}$. The following result generalizes Thm. 7.13 of [18].

Theorem 8.2. *Let* n, N, λ, μ *be positive integers such that* $3 \leq n \leq N$ *and* $2\lambda = n\mu$*. Let* x_0, y_0, z_0 *denote the unique linear invariants for* B_N, B_1, B_0*, respectively, and consider the* $\mathbb{G}_a$*-module:*

$$B_N \oplus B_1 \oplus B_0$$

If X *is the* $\mathbb{G}_a$*-variety defined by* $x_0 - z_0^\lambda = y_0 - z_0^\mu = 0$*, then* $X \cong \mathbb{A}^{n+2}$ *and* $\mathbf{k}[X]^{\mathbb{G}_a}$ *is not finitely generated.*

Proof. The representation B_n is defined by the restriction of D to $R_n = \mathbf{k}[x_0, ..., x_n]$. Let $w_{(r,s)} \in R_n$ be the sequence defined in *Thm. 8.1* . Given $m \geq 1$, the theorem implies that:

$$x_0^{2i} \,|\, D^{in+j} w_{(2m,nm)} \quad (0 \leq i \leq m-1\,,\, 0 \leq j \leq n-1) \tag{14}$$

The $\mathbb{G}_a$-module $B_n \oplus B_1 \oplus B_0$ is a submodule of $B_N \oplus B_1 \oplus B_0$, and is defined by the extension of D to $R_n[y_0, y_1, z_0]$ given by:

$$\mathcal{D} = \left(\sum_{i=1}^{n} x_{i-1} \frac{\partial}{\partial x_i} \right) + y_0 \frac{\partial}{\partial y_1}$$

For each $m \geq 1$, *Prop. 2.3(a)* implies that the kernel of $\mathcal{D}$ contains the element:

$$\begin{aligned} & F_m(x_0, ..., x_n, y_0, y_1, z_0) := [w_{(2m,nm)}, y_1^{nm}]_{nm}^{\mathcal{D}} \\ = \quad & (nm)! \sum_{k=0}^{nm} \frac{(-1)^k}{k!} \mathcal{D}^k w_{(2m,nm)} (\mathcal{D} y_1)^{nm-k} y_1^k \\ = \quad & (nm)! \left(\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \frac{(-1)^{in+j}}{(in+j)!} D^{in+j} w_{(2m,nm)} y_0^{nm-(in+j)} y_1^{in+j} \right) + (-1)^{nm} x_0^{2m} y_1^{nm} \end{aligned}$$

Substitute $x_0 = z_0^\lambda$ and $y_0 = z_0^\mu$ in the term $D^{in+j} w_{(2m,nm)} y_0^{nm-(in+j)} y_1^{in+j}$. Equation (14) implies that the resulting term is divisible by:

$$z_0^{2\lambda i + \mu(nm-(in+j))} = z_0^{\mu(nm-j)}$$

In addition, substituting $x_0 = z_0^\lambda$ in the last term $x_0^{2m} y_1^{nm}$ yields $z_0^{2\lambda m} y_1^{nm} = z_0^{\mu nm} y_1^{nm}$. Since $j \leq n-1$, we have:

$$\mu(nm - j) \geq \mu(nm - n + 1)$$

Therefore, there exists $G_m \in \mathbf{k}[x_1, ..., x_n, y_1, z_0]$ such that:

$$F_m(z_0^\lambda, x_1, ..., x_n, z_0^\mu, y_1, z_0) = (-1)^{nm} z_0^{\mu(nm-n+1)} G_m$$

The coefficient of y_1^{nm} in G_m equals $z_0^{\mu(n-1)}$, which does not depend on m.

Define the triangular derivation d on $\mathbf{k}[x_1, ..., x_n, ..., x_N, z_0]$ by:

$$d = z_0^\lambda \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + \cdots + x_{n-1} \frac{\partial}{\partial x_n} + \cdots + x_{N-1} \frac{\partial}{\partial x_N}$$

The conditions $2\lambda = n\mu$ and $n \geq 3$ insure that $\lambda > \mu$, which implies that z_0^μ is not in the image of d. Extend d to $\hat{d}$ on $\mathbf{k}[x_1, ..., x_N, z_0, y_1]$ by setting $\hat{d} y_1 = z_0^\mu$. Then each polynomial G_m $(m \geq 1)$ is in the kernel of $\hat{d}$. By the Non-Finiteness Criterion (Lemma 7.4 of [18]), it follows that the kernel of $\hat{d}$ is not finitely generated. ☐

Remark 8.1. In general, the counterexamples to Hilbert's Fourteenth Problem given in *Thm. 8.2* are new, though some cases were known. The case $n = 3$, $\lambda = 3$, $\mu = 2$ yields the counterexample in dimension 5 which first appeared in [12]. The case $n = 4$, $\lambda = 2$, $\mu = 1$ yields the counterexample in dimension 6 first given in [19]. This example was used to construct a linear representation of the unipotent group $\mathbb{G}_a^4 \rtimes \mathbb{G}_a$ on $\mathbb{A}^{11}$ with non-finitely generated ring of invariants.

9. Concluding Remarks

Remark 9.1. Any algorithm to construct a finite generating set for A_n must have two ingredients: It must incorporate a technique for constructing new invariants from a given set of invariants, and it must recognize whether, at any given step, the invariants so constructed generate all of A_n. The latter step uses the fact that A_n is algebraically closed in R_n.

There are two basic methods for constructing $\mathbb{G}_a$-invariants stemming from the classical techniques. The first uses the vector product (generalized transvectants) presented in Section 2.3. By considering the down operator D on the infinite polynomial ring R, this leads naturally to the definition of the mapping θ. By combining θ with integration of invariants, we obtain a procedure which builds invariant rings by successive degrees. In particular, choose a compatible $\mathbb{Z}$-grading $\mathfrak{g}$ of R, and let U be the associated up operator. Given $f \in W_{(r,s)}$ and $k \geq 0$, the element $\theta U^k(f)$ belongs to $W_{(r+1,s+k)}$. We call this the *vertical procedure.* It is a version of Cayley's omega process. Note that the vertical procedure restricts to A_n if the grading $\mathfrak{g}$ is n-compatible.

The second standard method exploits the fact that A is factorially closed in R. In particular, if $f_1, ..., f_k \in A$ and $P(f_1, ..., f_k) = x_0 h$ for some polynomial relation P and $h \in R$, then $h \in A$. Thus, one gets new invariants from a given set of invariants by considering their ideal of relations modulo x_0. In order to capture all such relations, one typically needs Buchberger's algorithm, but this procedure was understood and used in the Nineteenth Century; see [33], §192, and [32], §15.2.

In the modern era, algorithms to compute invariant rings were given by Cerezo in 1988 for any linear $\mathbb{G}_a$-action in characteristic zero [9]; by Tan in 1989 for the basic linear $\mathbb{G}_a$-actions in any characteristic [37]; and by Bedratyuk in 2010 for the basic $\mathbb{G}_a$-actions in characteristic zero [3]. Despite their merits, these algorithms, in their current forms, lack the efficiency needed to be computationally feasible and effective in higher dimensions.

Remark 9.2. In order to create an efficient algorithm using the vertical procedure, it is necessary to gain a more refined understanding of the kernel of θ. Given $r, s \geq 0$, define:

$$T_{(r,s)} = V_{(r,s)} \cap \ker\theta \quad \text{and} \quad T_r = V_r \cap \ker\theta$$

Then the sequence

$$0 \to T_r \hookrightarrow V_r \xrightarrow{\theta} W_{r+1} \to 0$$

is split exact. Note that, by *Lemma 3.1*, $\frac{1}{r+1}\frac{\partial}{\partial x_0}$ is a section for θ. We observe two distinct types of kernel elements for θ:

1. The A-module $\sum_{k\geq 0} x_{2k+1}A$
2. Elements of the form $f\theta(g) - g\theta(f)$ $(f, g \in R)$

In particular, for $r \geq 1$ define the linear map $\psi : V_r \to T_{r+1}$ by $\psi(f) = x_0 f - \theta(f)$. Then the sequence

$$0 \to W_r \hookrightarrow V_r \xrightarrow{\psi} T_{r+1}$$

is exact.

Remark 9.3. A third method for constructing invariants is based on *Thm. 3.2*, which asserts:

$$W_{(r,s)} = x_0 W_{(r-1,s)} \oplus \tau W_{(r,s-r)}$$

Here, τ is a section of the surjective map $\sigma^{-1}\epsilon : W_{(r,s)} \to W_{(r,s-r)}$. The construction of τ described in the proof of the theorem requires choosing a basis $\{f_1, ..., f_k\}$ for $W_{(r,s-r)}$, and elements g_i such that:

$$Dg_i = \frac{1}{x_0} D\sigma(f_i) \quad (1 \leq i \leq k)$$

In this way, $W_{(r,s)}$ is built from $W_{(r-1,s)}$ and $W_{(r,s-r)}$. This is called the *horizontal procedure.*

Remark 9.4. Cerezo's work on the invariants of linear $\mathbb{G}_a$-actions is not recognized as widely as it deserves to be, perhaps because the three papers [7, 8, 9] are unpublished. The first of these is a lengthy and detailed hand-written treatise on the invariant rings A_n based on the geometric theory, and containing numerous examples. In it, Cerezo calculates explicitly the 23 generators of A_5. The generator of degree 18 involves more than eight hundred monomials with relatively prime integer coefficients on the order of 10^{10}, and requires eight pages to write. This is the SL_2-invariant which was famously discovered by Cayley and Faà di Bruno; see [13].

Remark 9.5. The idea to study all invariants of a fixed degree is in keeping with the approach laid out by Howe in [24, 25], who classified the invariants of degree $d \leq 6$ for the action of $SL_n(\mathbb{C})$ on the space of m-forms in n variables.

Remark 9.6. The paper of Olver and Sanders [30] (2000) formulates a duality between between the invariant theory of binary forms and the theory of modular forms in one variable. In this approach, the degree n of the binary form corresponds to the negative of the weight w of the modular form, and transvection corresponds to the Rankin-Cohen bracket operator. The authors write:

The key result is that the two theories of modular and binary forms have a common limiting theory as $n = -w \to \infty$. The underlying transformation group of the limiting theory is a three-dimensional Heisenberg group. This limiting procedure is made precise on the Lie algebra (infinitesimal) level, realizing the solvable Heisenberg algebra as a contraction of the semisimple unimodular algebra $\mathfrak{sl}(2, \mathbb{C})$. Complicated identities in the transvectant and Rankin-Cohen bracket algebras reduce to much simpler identities in the Heisenberg limit. (p 253)

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