Uncrossing

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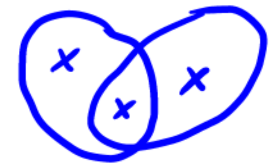
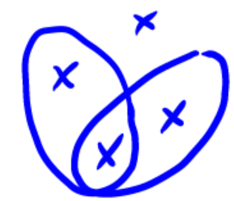
Topics

- Minimally k-edge-connected graphs
- Odd cuts, cut tree
- r-arborescence polytope
- Matroid intersection
- Lucchesi-Younger
- Submodular flows
- Matching polytope
- TDI and unimodularity
- Augmenting connectivity (w/ or w/o weights)
- Node connectivity augmentation
- Degree restricted spanning trees
- Dual uncrossing
- Primal uncrossing
- Termination, finiteness, efficiency
- TU and TDI
- TDI and unimodularity
- Iterative rounding
- Iterative relaxation
- Uncrossing set pairs

Intersecting, Crossing Sets



- Subsets A and B of S are
  - intersecting if $A \cap B \neq \emptyset$, $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$
  - crossing if intersecting and $S \setminus (A \cup B) = \overline{A \cup B} \neq \emptyset$



- Family $\mathcal{F} \subseteq 2^S$ is
  - laminar (or nested) if no two sets $A, B \in \mathcal{F}$ are intersecting (intersecting-free)
    i.e. for $A, B \in \mathcal{F}$: $A \subseteq B$ or $B \subseteq A$ or $A \cap B = \emptyset$
  - cross-free if no two sets of $\mathcal{F}$ are crossing
  - a chain if, for any two sets $A, B \in B$, either $A \subseteq B$ or $B \subseteq A$
- Uncrossing: Make a family of sets cross-free, laminar or a chain

Laminar vs. Cross-free

- If add complements to cross-free family, family remains still cross-free
- If $\mathcal{F}$ is cross-free then

$$\{S \in \mathcal{F} : v \in S\} \cup \{\bar{S} \in \mathcal{F} : v \notin S\}$$

is laminar

Tree Representation for Laminar and Cross-Free

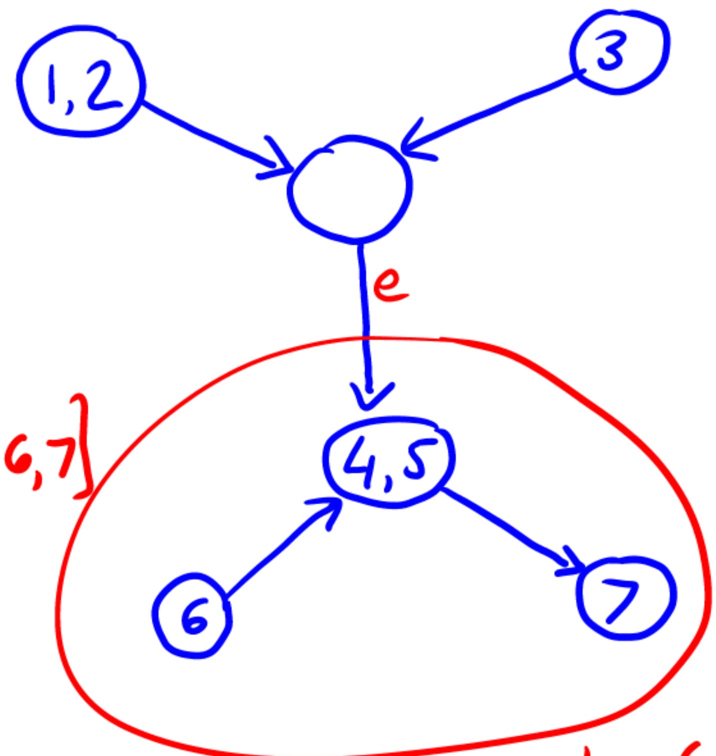
Family $\mathcal{F} \subseteq 2^V$

Tree (U, T)

$V \longrightarrow U$

$V = \{1,2,3,4,5,6,7\}$

$S = \{4,5,6,7\}$



Connected component of $T \setminus \{e\}$ containing head of e

Cross-free $\Longleftrightarrow$ $\exists$ tree (U,T)

laminar $\Longleftrightarrow$ $\exists$ rooted directed tree

Submodularity

- $f : 2^S \rightarrow R$ is submodular if for all $A, B \subseteq S$:

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

- Basic example: cut function of a nonnegatively weighted undirected graph $G = (V, E)$
  - $d(S) = w(\delta(S))$ for $S \subseteq V$
  - $d(A) + d(B) = d(A \cap B) + d(A \cup B) + 2w(A \setminus B : B \setminus A)$

count contribution of e on both sides

$d(S) = d(\bar{S}) \Rightarrow d(A) + d(B) = d(A \setminus B) + d(B \setminus A) + 2w(A \cap B : \overline{A \cup B})$

- Similarly for indegree function $d^-(\cdot) = w(\delta^-(\cdot))$ or outdegree function $d^+(\cdot) = w(\delta^+(\cdot))$ of a directed graph (with ≥ 0 weights).
- Minimizers of a submodular function form a lattice family, i.e. it is closed under $\cap$ and $\cup$

Minimally k-Edge-Connected Graphs

Theorem: In a minimally k-edge-connected graph $G = (V, E)$, we have $|E| \le k(|V| - 1)$

Witness family $\mathcal{F}$: $\forall e \in E,\ \exists S \in \mathcal{F}$: $e \in \delta(S)$, $d(S) = k$

(i) by complementing, can assume $1 \notin S \quad \forall S \in \mathcal{F}$

(ii) if $A, B \in \mathcal{F}$ intersecting, remove A, B, add $\left\{ \begin{array}{l} A \cap B \\ A \cup B \end{array} \right.$ to $\mathcal{F}$

. $k + k = d(A) + d(B) \underset{=}{\ge} d(A \cap B) + d(A \cup B) \underset{=}{\ge} 2k$

$\Rightarrow$ no edge in $(A \setminus B : B \setminus A)$

$\Rightarrow \delta(A) \cup \delta(B)$

$\subseteq \delta(A \cup B) + \delta(A \cap B)$

uncrossing process will terminate (see later)

$\longrightarrow$ laminar $\mathcal{F}$

(iii) If $\exists\ S, S_1, \ldots, S_\ell \in \mathcal{F}$
with $S_i \subset S$
and $S = \bigcup_{i=1}^{\ell} S_i$

$\Rightarrow$ can remove S from $\mathcal{F}$

$$\Rightarrow \forall S \in \mathcal{F} : S \setminus \bigcup_{S_i \in \mathcal{F},\, S_i \subseteq S} S_i \neq \emptyset$$

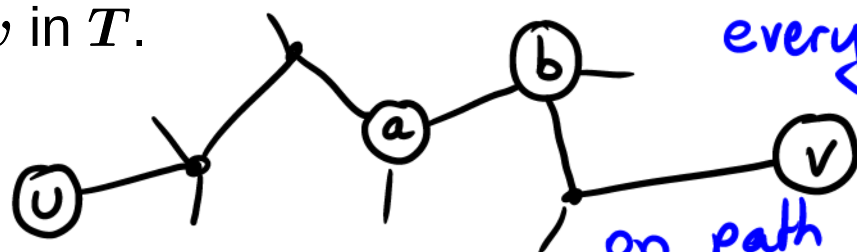
In rooted directed tree representation, every node nonempty $\Rightarrow |\mathcal{F}| \leq |V| - 1$

$$\Rightarrow |E| = \left| \bigcup_{S \in \mathcal{F}} \delta(S) \right| \leq k\,(|V| - 1)$$

$\square$

Gomory-Hu Cut Tree

- Let $G = (V, E)$ be a (nonnegatively weighted) undirected graph.
- Gomory-Hu cut tree is any tree (V, T) such that for any edge $e = (s, t) \in T$, we have that $\delta(C_e)$ is a minimum s, t-cut where C_e is any of the connected components of $T \setminus \{e\}$.
- Property of Gomory-Hu tree: For any $u, v \in V$, a min u, v-cut is given by the minimum capacity cut among $\delta(C_e)$ where e is along the path P_{uv} from u to v in T.



every u,v-cut separates two adjacent a, b on path P_{uv} $\rightarrow$ $\square$

- Gomory-Hu cut tree always exists.
- Same result holds for symmetric submodular functions [GGW]
- No need to contract if perturb

Gomory-Hu Cut Tree

Proof of existence

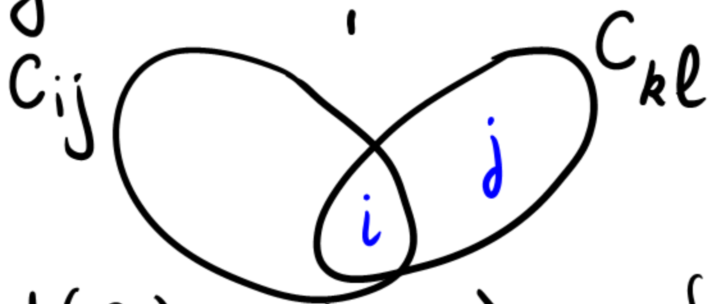
Perturb by adding ε_{ij} (lexicographically) to edge (i,j) (of complete graph) so that all cut values are distinct $\rightarrow$ mincuts are unique

$\forall\, i \neq j$, let C_{ij} : $1 \notin C_{ij}$ and $\delta(C_{ij})$ unique i,j-mincut

Let $\mathcal{F} = \{ C_{ij} \;:\; i \neq j \}$

Claim: $\mathcal{F}$ is laminar

Proof By contradiction. Assume C_{ij} & $C_{k\ell}$ cross



Assume $d(C_{ij}) > d(C_{k\ell}) \rightarrow \delta(C_{k\ell})$ does not separate i & j

Say $i \in C_{ij} \cap C_{k\ell}$ & $j \in C_{k\ell} \setminus C_{ij}$

if k or $\ell \notin C_{ij} \cup C_{k\ell}$

$$d(C_{ij}) + d(C_{k\ell}) > \underbrace{d(C_{ij} \cap C_{k\ell})}_{> d(C_{ij})} + \underbrace{d(C_{ij} \cup C_{k\ell})}_{> d(C_{k\ell})}$$

contra-diction

if k & ℓ in $C_{ij} \cup C_{k\ell}$ then k or ℓ in $C_{ij} \setminus C_{k\ell}$

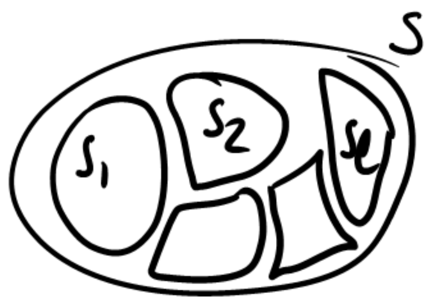
$$d(C_{ij}) + d(C_{k\ell}) > \underbrace{d(C_{ij} \setminus C_{k\ell})}_{> d(C_{k\ell})} + \underbrace{d(C_{k\ell} \setminus C_{ij})}_{> d(C_{ij})}$$

contradiction

$\rightarrow \mathcal{F}$ laminar

Claim: No $S, S_1, S_2, \ldots, S_\ell \in \mathcal{F}$ with
$S_i \subset S$
and $S = \bigcup_i S_i$

Proof:



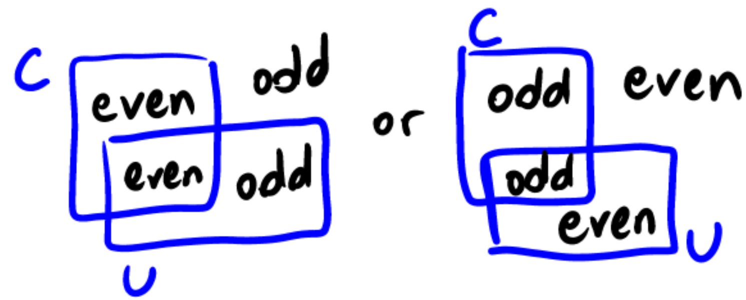
consider set maximizing
$\max\left(d(S), d(S_1), \ldots, d(S_\ell)\right)$
(max unique)

→ can remove this set since every i, j is separated by another set in family □

→ Directed tree representation ≡ tree on V
≡ Gomory-Hu cut tree

Min T-Odd Cut (Padberg and Rao '82)

- T-odd cut problem: Given (≥ 0 edge weighted) graph $G = (V, E)$ and $T \subseteq V$, find S with $|S \cap T|$ odd minimizing cut function $d(S)$
- Lemma: If $\delta(C)$ is a mincut then there exists a min T-odd cut $\delta(S)$ with either $S \subseteq C$ or $S \subseteq \overline{C}$.



if C is T-odd ✓

if C is T-even, let

$\delta(U)$ be a min T-odd cut

$$d(U) + d(C) \geq \overset{\geq d(C)}{d(U \cap C)} + d(\overline{U \cup C})$$

$$d(U) + d(C) \geq d(U \setminus C) \underset{\geq d(C)}{+} d(C \setminus U)$$

one of these min T-odd

- Lemma: If $\delta(C)$ is a mincut separating vertices of T then there exists a min T-odd cut $\delta(S)$ with either $S \subseteq C$ or $S \subseteq \overline{C}$.

Padberg-Rao's T-Odd Cut Algorithm

- Find global mincut C separating two vertices of T
- If T-odd, done.
- Else, solve subproblems
  - $G_1 = G/C$ with $T_1 = T \setminus C$
  - $G_2 = G/\overline{C}$ with $T_2 = T \setminus \overline{C}$

  and output best T-odd cut

Number of subproblems $\leq |T|$

Rizzi's Min T-Odd Cut Algorithm

$ALG(G, T)$

- Take $s, t \in T$
- Find min s, t-cut $\delta(S)$
- If S is T-odd, return $\min(d(S), ALG(G/\{s,t\}, T \setminus \{s,t\}))$
- Else return $\min(ALG(G/S, T \setminus S), ALG(G/\bar{S}, T \setminus \bar{S}))$

Min T-Cut Algorithm

Follows from Padberg-Rao: There exists $s, t \in T$ such that min T-odd cut is a min s, t-cut

Other Cut Families

- [Barahona-Conforti '87]: T-even cuts (having an even, $\geq \mathbf{2}$ vertices of T on both sides)
- [Grötschel et al. '88] (for submodular f.):
  - Lattice family $\mathcal{C}$ of sets
  - Triple subfamily $\mathcal{G}$ of $\mathcal{C}$: whenever 3 of A, B, $A \cap B$ and $A \cup B$ are in $\mathcal{C} \setminus \mathcal{G}$ then 4th is also in $\mathcal{C} \setminus \mathcal{G}$
  - Example: $\mathcal{G} = \{S \in \mathcal{C} : |S \cap T| \not\equiv q \pmod{p}\}$
    (Special case: min T-even cut separating s and t.)

More Cut Families

- Generalization: parity family (G.-Ramakrishnan) (also for submodular f.)
  - Parity subfamily $\mathcal{G}$ of a lattice family $\mathcal{C}$ if

$$A, B \in \mathcal{C} \setminus \mathcal{G} \Rightarrow (A \cap B \in \mathcal{G} \text{ iff } A \cup B \in \mathcal{G})$$

  - Example: lattice family minus a lattice family (i.e. can find second minimizer to a submodular function).
  - Need more than uncrossing. Theorem: Let S^* be a minimizer over $\mathcal{G}$. Then either $S^* \in \{\emptyset, V\}$ or there exists $a, b \in V$ such that S^* minimizer over lattice family $\mathcal{C}_{st} = \{S \in \mathcal{C} : s \in S, t \notin S\}$

Polyhedral Combinatorics

Dominant of r-Arborescence Polytope

- $X = \{$Digraphs with every vertex reachable from root $r\}$
- Minimal= r-arborescences: rooted tree at r in digraph $G = (V, A)$
- Theorem: $conv(X) = conv$(arborescences) $+ R^m_+ =$

$$\begin{array}{rcll} P & = & \{x: \; x(\delta^-(S)) \geq 1 & S \subset V \setminus \{r\} \\ & & x_a \geq 0 & a \in A\} \end{array}$$

$\subseteq$: obvious

Proof through primal uncrossing

x : vertex of polyhedron P

$A = \{a : x_a > 0\}$

$\mathcal{F} = \{S \subseteq V \setminus \{r\} : x(\delta^-(S)) = 1\}$

vertex $\Rightarrow$ $\text{span}(\chi(\delta^-(S)) : S \in \mathcal{F}) = \mathbb{R}^{|A|}$

Claim: $\exists$ laminar $\mathcal{L} \subseteq \mathcal{F}$ with $\text{span}(\mathcal{L}) = \text{span}(\mathcal{F})$

Lemma: $S, T \in \mathcal{F}$, S, T intersect $\Rightarrow$ $S \cap T,\ S \cup T \in \mathcal{F}$ and $\chi(\delta^-(S)) + \chi(\delta^-(T)) = \chi(\delta^-(S \cap T)) + \chi(\delta^-(S \cup T))$ $(*)$

Pf by submodularity

$$2 = d^-(S) + d^-(T) \geq d^-(S \cap T) + d^-(S \cup T) \geq 2$$

$=$

$\Rightarrow$ $S \cap T, S \cup T \in \mathcal{F}$ and no arc between $S \setminus T$ and $T \setminus S$

$\Rightarrow$ linear dependence $(*)$

uncrossing: $\mathcal{L} = \mathcal{F}$

while $\exists$ 2 intersecting sets S, T

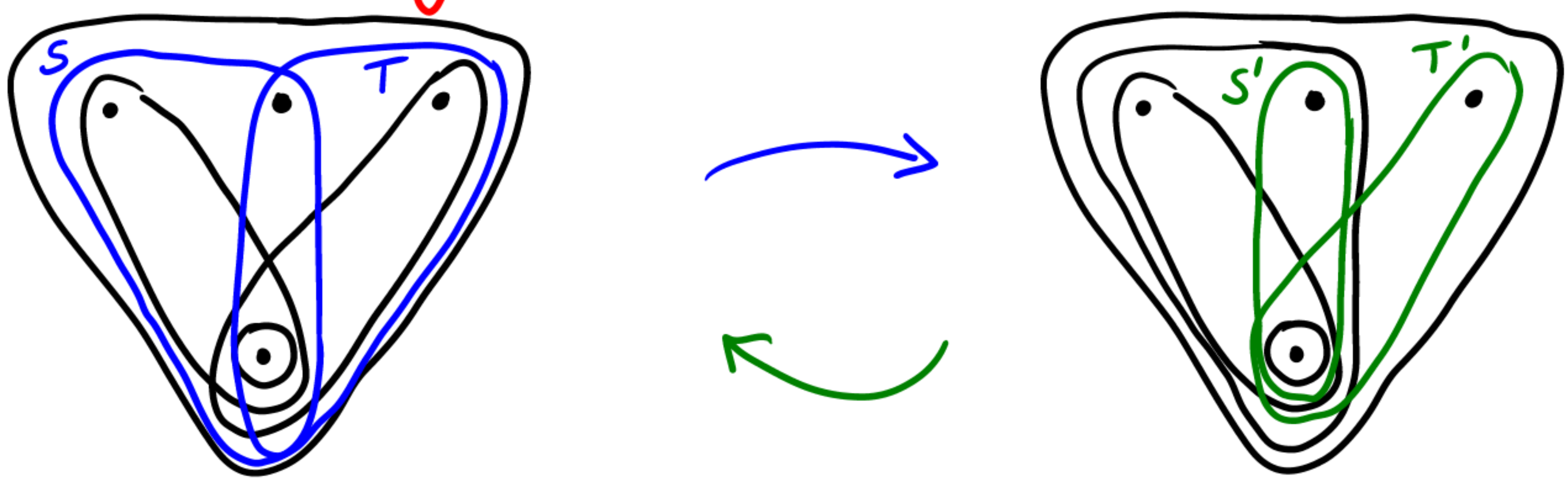
add $S \cap T$, $S \cup T$ to $\mathcal{L}$

remove either S or T

$\Rightarrow \mathrm{span}(\mathcal{L}) = \mathrm{span}(\mathcal{F})$ thanks to ⊛

Finite ?

NOT necessarily



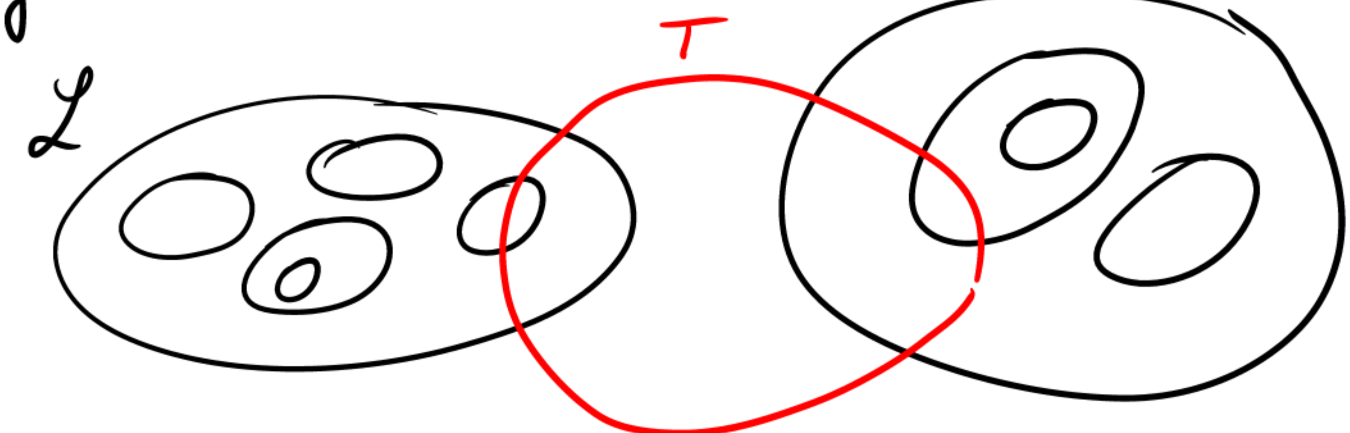
But

Lemma: For any maximal laminar $\mathcal{L} \subseteq \mathcal{F}$:

$$\operatorname{span}(\mathcal{L}) = \operatorname{span}(\mathcal{F})$$

$\rightarrow$ can construct $\mathcal{L}$ greedily

Pf by contradiction



if T cannot be added to $\mathcal{L}$ but $\chi(\delta^-(T)) \notin \operatorname{span}(\mathcal{L})$

then must intersect $S \in \mathcal{L}$

$\Rightarrow$ $S \cap T,\ S \cup T \in \mathcal{F}$ and

$$\underbrace{\chi(\delta^-(S))}_{\in \operatorname{span}(\mathcal{L})} + \underbrace{\chi(\delta^-(T))}_{\notin \operatorname{span}(\mathcal{L})} = \chi(\delta^-(S \cap T)) + \chi(\delta^-(S \cup T))$$

$\rightarrow$ either $S \cap T$ or $S \cup T$ in $\mathrm{span}(\mathcal{F})$

but both have fewer crossings with $\mathcal{L}$ than T did

$\rightarrow$ repeating get a set to add to $\mathcal{L}$ and increase span $\square$

$\Rightarrow$ $x|_A$ defined by

$$x(\delta^-(S)) = 1 \qquad S \in \mathcal{L}$$

Let A corresponding matrix (rows are $\chi(\delta^-(S))$ $S \in \mathcal{L}$)

Totally Unimodular

- A is totally unimodular (TU) if all square submatrices of A have determinant in $\{-1, 0, 1\}$
- If A is TU then for any integral b, $\{x : Ax \leq b, x \geq 0\}$ is integral.
- Ghouila-Houri: A is TU iff every subset R of rows can be partitioned into R_1 and R_2 such that

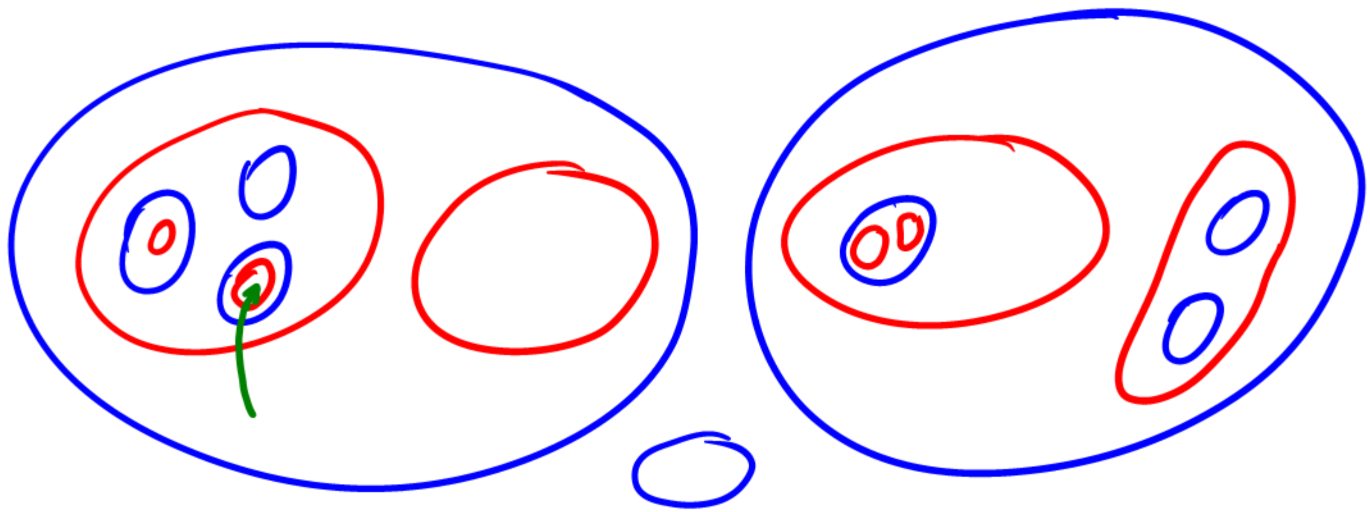
$$|\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij}| \leq 1$$

bicoloring

R_1 R_2

Claim: A is T.U.

For subset of $\mathcal{L}$, alternate between assigning sets to R_1 and R_2



For any arc, entering sets alternate $\rightarrow +1, 0, -1$ $\square$

$\Rightarrow$ Any extreme point is integral

Directed Cuts

- Digraph $D = (V, A)$
- A directed cut is $C = \delta^-(S)$ where $\delta^+(S) = \emptyset$.
- A directed cut cover is $F \subseteq A$ with $F \cap C \neq \emptyset$ for every directed cuts C
- Theorem: Polytope

$$\begin{array}{lll} \{x: & x(C) \geq 1 & C \text{ directed cut} \\ & 0 \leq x_a \leq 1 & a \in A\} \end{array}$$

integral, i.e. convex hull of directed cut covers.

- Proof: similar to arborescence with 2 differences

1. Can only uncross crossing sets
(while for arborescence intersecting $\Rightarrow$ crossing because of root)
$\rightarrow$ cross-free family $\mathcal{F}$ (rather than laminar)

2. Matrix A corresponding to directed cuts $\delta^-(S)$ for $S \in \mathcal{F}$ cross-free is T.U.
[not true if not DIRECTED cuts $\equiv \delta^+(S) = \phi$]

- For subfamily of $\mathcal{F}$, consider its tree representation T
- directed cuts $\Rightarrow$ $a \in A$ corresponds to directed path in T (no backward edge)

$\Rightarrow$ bicolor arcs of T (sets of $\mathcal{F}$) such that directed paths alternate colors

Easy: $\forall v$, use one color for incoming arcs, other for outgoing arcs
let propagate

Matroid Intersection Polytope

- Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids with rank functions r_1 and r_2

- Edmonds: The convex hull of incidence vectors of independent sets in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by:

$$\begin{array}{rclll} P & = & \{x : & x(S) \leq r_1(S) & S \subseteq E \\ & & & x(S) \leq r_2(S) & S \subseteq E \\ & & & x_i \geq 0 & i \in S\} \end{array}$$

$\subseteq$ obvious

- Proof through dual uncrossing and TDIness

TDI (Edmonds-Giles '77)

- Rational system $Ax \leq b$ is TDI if, for each $c \in Z^n$, the dual to $\min\{c^T x : Ax \leq b\}$, i.e.

$$\max\{b^T y : A^T y = c, y \geq 0\}$$

  has an integer optimum solution whenever it is finite.
- Theorem: If $Ax \leq b$ is TDI and b is integral then $Ax \leq b$ is integral (i.e. has only integral extreme points).

Dual

$$\max \quad c^T x \qquad = \min \quad \sum_S r_1(S) y_{1,S} + \sum_S r_2(S) y_{2,S}$$

$$x(S) \le r_1(S) \quad \forall S \qquad \sum_{S: i \in S} y_{1,S} + \sum_{S: i \in S} y_{2,S} \ge c_i$$

$$x(S) \le r_2(S) \quad \forall S \qquad y_{1,S}, y_{2,S} \ge 0$$

Take dual optimum y_1, y_2

Let $\mathcal{F}_i = \{S : y_{i,S} > 0\}$

Claim: Can assume that $\mathcal{F}_i$ is a chain

"Uncross" each matroid separately:

For $S, T \in \mathcal{F}_i$, $S \not\subseteq T$, $T \not\subseteq S$:

$y_{i,S} \leftarrow y_{i,S} - \varepsilon \qquad y_{i,S\cap T} \leftarrow y_{i,S\cap T} + \varepsilon$

$y_{i,T} \leftarrow y_{i,S} - \varepsilon \qquad y_{i,S\cup T} \leftarrow y_{i,S\cup T} + \varepsilon$

New y : (i) still feasible ✓

(ii) objective can only improve by subm.

$$\varepsilon\left[r_i(S\cap T)+r_i(S\cup T)-r_i(S)-r_i(T)\right]\leq 0$$

Progress towards having no $S,T\in\mathcal{F}_i$: $S\not\subseteq T, T\not\subseteq S$?

Yes. $\phi_i=\sum_S y_{i,S}\underbrace{|S|\cdot|\bar{S}|}_{d_{K_n}(S)}$

$$\text{new }\phi_i-\text{old }\phi_i=\varepsilon\underbrace{\left[d_{K_n}(S\cup T)+d_{K_n}(S\cap T)-d_{K_n}(S)-d_{K_n}(T)\right]}_{\leq-\varepsilon<0}$$

$\rightarrow$ terminate

$\Rightarrow$ can assume chain $\mathcal{C}_i=\{S: y_{i,S}>0\}$

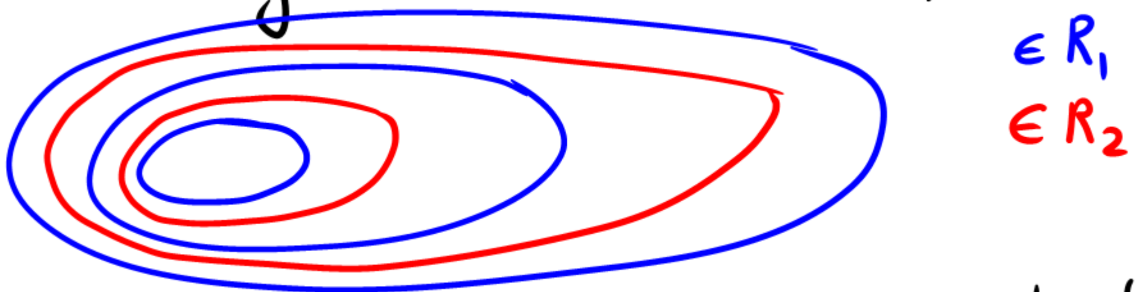
$\rightarrow$ x defined by $\begin{cases} x(S)=r_1(S) & S\in\mathcal{C}_1 \\ x(S)=r_2(S) & S\in\mathcal{C}_2\end{cases}$

Claim: Underlying matrix is T.U.

Pf. Take any subset $\mathcal{C}_1', \mathcal{C}_2'$ of $\mathcal{C}_1$ & $\mathcal{C}_2$

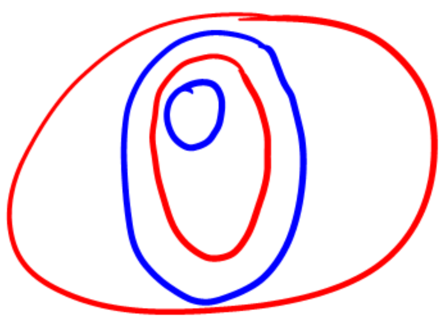
Can alternatively assign sets in $\mathcal{C}_1'$ so that any element gets contribution in $\{0, +1\}$

$\mathcal{C}_1'$:



Similarly for $\mathcal{C}_2'$ so that every element gets $\{-1, 0\}$ contribution

$\mathcal{C}_2'$



$\rightarrow$ over both $\mathcal{C}_1'$ & $\mathcal{C}_2'$ every element gets contribution in $\{-1, 0, +1\}$

$\rightarrow$ T.U $\Rightarrow$ extreme point integral $\square$

Lucchesi-Younger

- Could have done dual uncrossing and TDI proof for arborescences of directed cut covers
- Lucchesi-Younger theorem: For any digraph, min size of a directed cut cover $=$ max number of disjoint directed cuts
- If planar digraph, can take dual to get:
  Theorem: Min size of a feedback arc set (meeting all directed circuits) = max number of arc disjoint directed circuits

Perfect Matching Polytope via Uncrossing

Convex hull of perfect matchings $=$

$$\begin{aligned}
\{x: \quad & x(\delta(v)) = 1 & & v \in V \\
& x(\delta(S)) \geq 1 & & S: |S| odd \\
& 0 \leq x_e & & e \in E\}
\end{aligned}$$

Could have replaced $x(\delta(S)) \geq 1$ by $x(E(S)) \leq \frac{|S|-1}{2}$

Schrijver & Seymour: dual uncrossing $\rightarrow$ laminar
+ half dual integrality $\Rightarrow$ TDi $\Rightarrow$ integrality

Primal uncrossing

S, T odd $\Rightarrow$ either $S \cap T, S \cup T$ odd
or $S \setminus T, T \setminus S$ odd

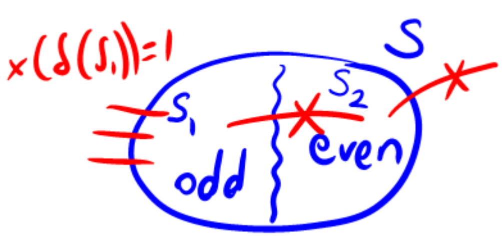
Can uncross crossing tight odd sets into $S \cap T, S \cup T$
or $S \setminus T, T \setminus S$

$\rightarrow$ vertex x defined by $x(\delta(S)) = 1 \quad S \in \mathcal{L}$
laminar

Proof adapted from Ravi & Singh

- Let E = support of $x = \{e : x_e > 0\}$ $\rightarrow |E| = |\mathcal{L}|$
- Can assume for $S \in \mathcal{L}$ that $E(S)$ connected

$\left.\begin{array}{l} x(\delta(S)) = 1 \\ x(\delta(S_1)) \geq 1 \\ x(S_1 : S_2) = 0 \end{array}\right\} \Rightarrow$



$\Rightarrow \delta(S) = \delta(S_1)$; can replace S by S_1

- Can assume E is connected and $|V|$ even
(treat separately connected comp.)

- $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$
  $\mathcal{L}_1 \hookrightarrow$ singletons

- If $\mathcal{L}_2 = \emptyset$ then extreme point is disjoint union of edges and odd cycles with $x = \frac{1}{2}$

  connected and $|V|$ even $\Rightarrow$ no odd cycles (and $|V| = 2$)

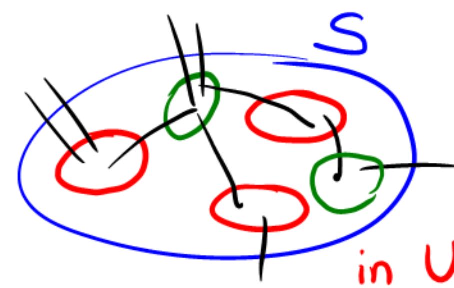
$\rightarrow$ Assume $\mathcal{L}_2 \neq \emptyset$

- <u>Claim</u>: For $S \in \mathcal{L}_2$
  Contract all children $S_i \in \mathcal{L}_2$ of S
  then G_S is not a tree

<u>Pf</u>: S odd $\Rightarrow$ after contraction, odd # k of vertices in S

if tree then bipartition U_1, U_2

odd $\Rightarrow |U_1| > |U_2|$



$$\Rightarrow 1 \leq |U_1| - |U_2| = \sum_{i \in U_1} x(\delta(i)) - \sum_{i \in U_2} x(\delta(i)) \leq x(\delta(S)) = 1$$

$\Rightarrow$ for $i \in U_2$: $\delta(i) \cap \delta(S) = \emptyset$

$$\Rightarrow x(\delta(S)) = \sum_{i \in U_1} x(\delta(i)) - \sum_{i \in U_2} x(\delta(i))$$

NOT LIN. INDEP.

□

$\Rightarrow$ Can remove one edge from $E(S)$ for each $S \in \mathcal{L}_2$ and maintain connectivity $\to$ remove $|\mathcal{L}_2|$ edges

Fix one maximal set $S \in \mathcal{L}_2$

- complement S $\to$ can also remove edge from $E(V \setminus S)$
$\to$ remove $|\mathcal{L}_2|+1$ edges

- at least 2 edges in $\delta(S)$ (otherwise $x_e = 1$ $\to$ component by itself)
$\to$ remove $|\mathcal{L}_2|+2$ edges and still connected

$$\Rightarrow \quad |E| \geq |\mathcal{L}_2| + 2 + |V| - 1 = |\mathcal{L}_2| + |V| + 1 > |\mathcal{L}_2| + |\mathcal{L}_1|$$

Contradicting $|E| = |\mathcal{L}|$

$\square$

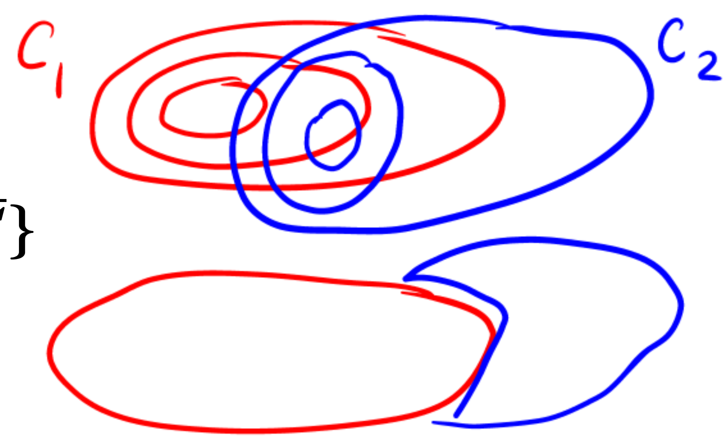
Matroid Intersection

$$\max \quad c^T x \qquad = \min \quad \sum_S r_1(S) y_{1,S} + \sum_S r_2(S) y_{2,S}$$

$$\begin{cases} x(S) \leq r_1(S) & \forall S \subseteq E \\ x(S) \leq r_2(S) & \forall S \subseteq E \\ x_i \geq 0 & i \in E \end{cases} \qquad \begin{cases} \sum_{S: i \in S} y_{1,S} + \sum_{S: i \in S} y_{2,S} \geq c_i \\ y_{1,S}, y_{2,S} \geq 0 \end{cases}$$

Min-max relation: $\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(S) + r_2(\overline{S})\}$

For $c_i = 1$, can choose y_1, y_2 integral and $\mathcal{C}_i = \{S : y_{i,S} > 0\}$ chain for $i = 1, 2$.



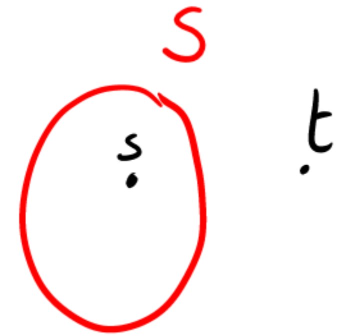
$\Rightarrow \mathcal{C}_1 = \{S\}, \mathcal{C}_2 = \{\overline{S}\}$

Connectivity Augmentation

Connectivity Augmentation

- For graph H, $\lambda_H(s,t) =$ local connectivity between s and t = max number of edge-disjoint paths between s and t
- Problem: Given graph $G = (V, E)$ and requirements $r(u,v)$ for $\forall u \neq v \in V$, add set F of (multiple) edges such that in $H=(V, E \cup F)$ $\lambda_H(u,v) \geq r(u,v)$ for all u, v
- Special case: $r_{u,v} = k$ for all u, v. Want augmentation into k-edge-connected graph
- Objective 1. Cardinality: Minimize $|F|$ [Frank]
  - Good characterization
  - Efficient algorithm
- Objective 2. Weighted: Minimize $\sum_{(i,j)\in F} w_{ij}$
  - NP-hard
  - 2-approximation algorithm [Jain]

Formulation

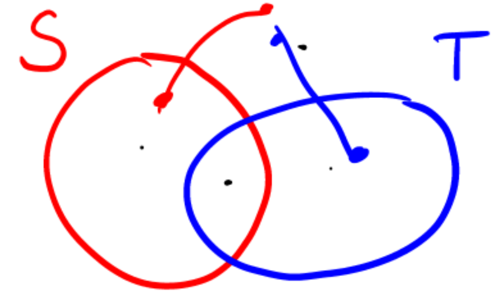


- Let $R(S) = \max_{s \in S, t \notin S} r(s,t)$
- Let $d(S) = d_E(S) = |\delta_E(S)|$
- Want integral $x \in P$:

$$P = \begin{cases} x(\delta(S)) \geq R(S) - d(S) & \forall S \\ x_{ij} \geq 0 & \forall i,j \end{cases}$$

- If relax integrality, not integral

Uncrossing



- Lemma: For crossing S and T,

  either $R(S) + R(T) \leq R(S \cup T) + R(S \cap T)$

  or $R(S) + R(T) \leq R(S \setminus T) + R(T \setminus S)$

- Uncrossing lemma: For $x \in P$, let $\mathcal{F} = \{S : x(\delta(S)) = R(S) - d(S)\}$. If $S, T \in \mathcal{F}$ and S, T crossing then

  either $S \cap T, S \cup T \in \mathcal{F}$ and $x(S \setminus T : T \setminus S) = 0$

  or $S \setminus T, T \setminus S \in \mathcal{F}$ and $x(S \cap T : \overline{S \cup T}) = 0$

$$R(S) - d(S) + R(T) - d(T) = x(\delta(S)) + x(\delta(T))$$

$$\geq x(\delta(S \cup T)) + x(\delta(S \cap T))$$

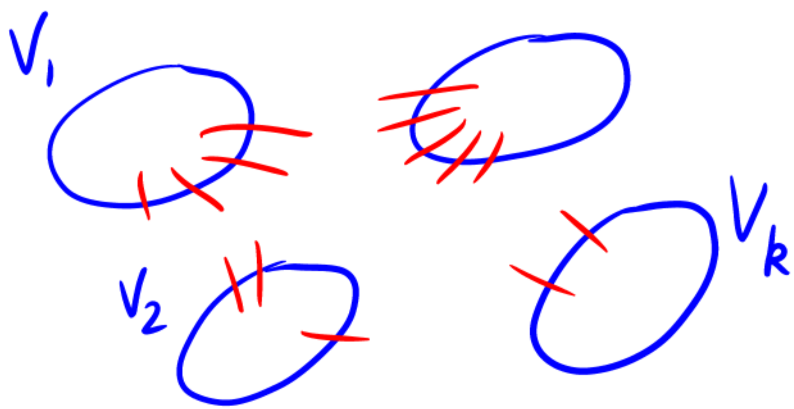
$$\geq R(S \cup T) - d(S \cup T) + R(S \cap T) - d(S \cap T)$$

$$\geq R(S) - d(S) + R(T) - d(T)$$

Lower bound

- $\gamma =$ smallest # of edges to add
- [Frank]: For any subpartition $V_1, V_2, \cdots, V_k$ of V:

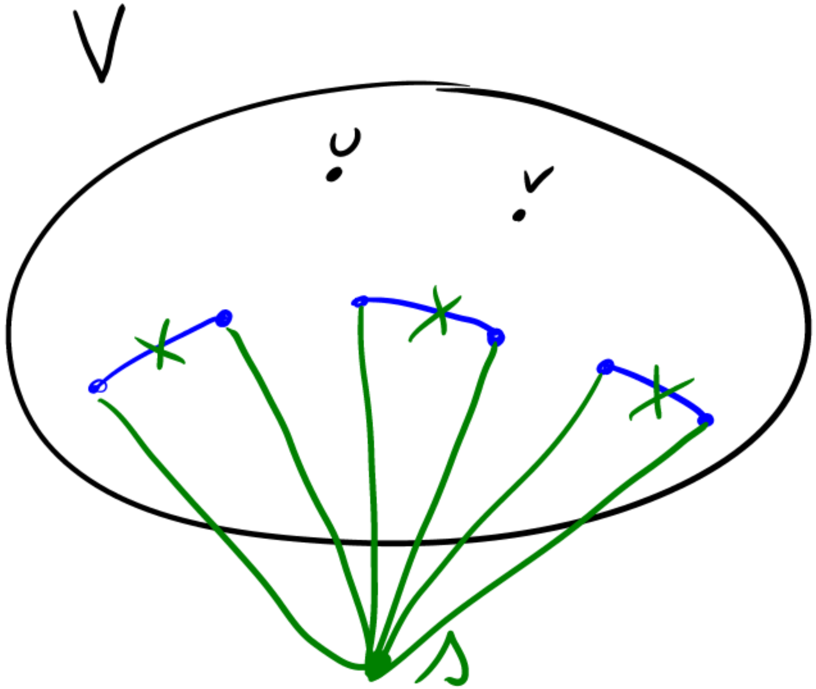
$$2\gamma \geq \sum_{i=1}^{k} [R(V_i) - d(V_i)]$$



- Hence

$$\gamma \geq \left\lceil \frac{1}{2} \max_{V_1, \cdots, V_k} [R(V_i) - d(V_i)] \right\rceil$$

Add a new vertex s



add γ new edges

pinch new edges together $\rightarrow s$

Graph on $V \cup \{s\}$: $\lambda(u,v) \geq r(u,v) \quad \forall u,v$

Frank's Algorithm

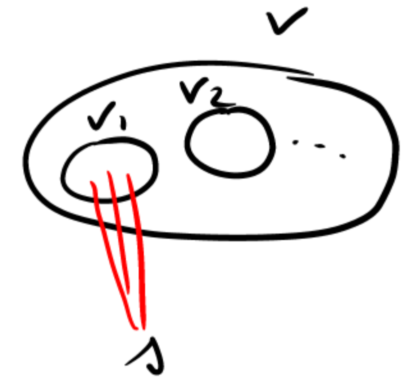
(Modulo $\cdots$)

1. Add as few edges as possible between s and V (and none within V) such that $\lambda(u, v) \geq r(u, v)$ for all u, v
2. Add one more edge if degree of s is odd
3. Use Mader's local connectivity splitting-off result to get augmenting set F (within V)

Step 1

Theorem [Frank]: Any minimal augmentation from s has

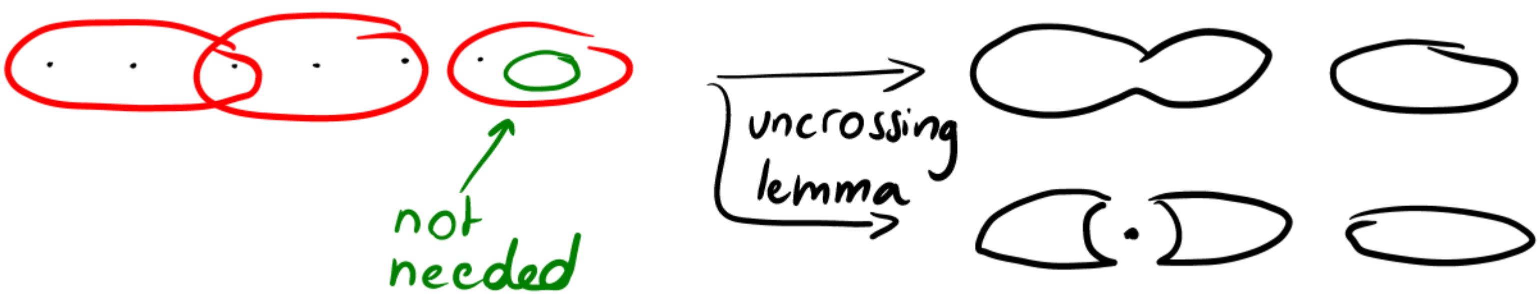
$$m = \max_{V_1,\cdots,V_k} \sum_{i=1}^{k} [R(V_i) - d(V_i)]$$



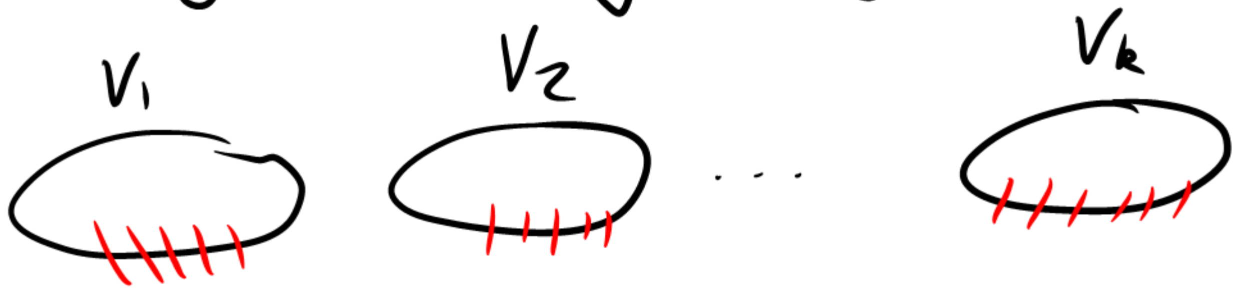
edges incident to s

Minimal solution x. Clearly has $\geq m$ edges incident to s

$x_v > 0 \Rightarrow \exists S: x(S) = R(S) - d(S)$



Get a disjoint family of tight sets



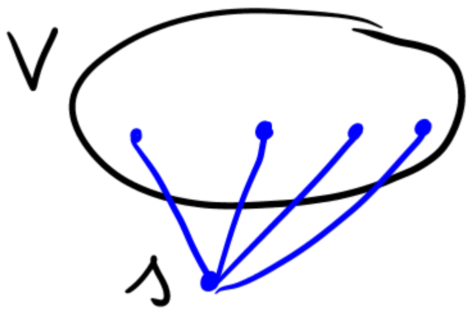
$$\bigcup V_i \supseteq \{v \in V : x_v > 0\}$$

$$\Rightarrow \sum x_v = \sum_i x(\delta(V_i)) = \sum_i R(V_i) - d(V_i) \leq m$$

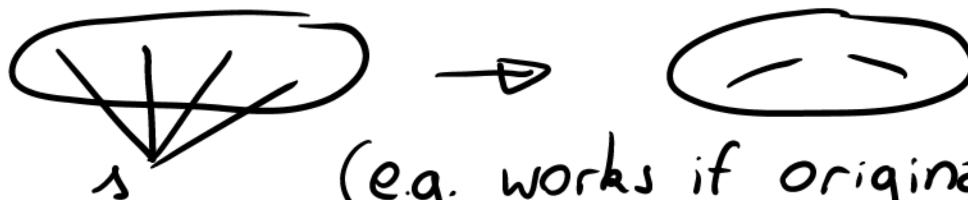
$\rightarrow$ OPTIMAL

Splitting off

- Mader: can perform splitting off and maintain local connectivity



- (Modulo $\cdots$)



(e.g. works if original graph is connected)

- Add

$$\left\lceil \frac{1}{2} \max_{V_1,\cdots,V_k} [R(V_i) - d(V_i)] \right\rceil$$

edges $\Longrightarrow$ optimal

Weighted case

$$
\begin{aligned}
LP(E) \quad &= \quad \min \sum_e w_e x_e \\
s.t. \quad & \begin{cases} x(\delta(S)) \geq R(S) - d(S) & \forall S \\ x_{ij} \geq 0 & \forall i, j \end{cases}
\end{aligned}
$$

- Extreme point x could be fractional
- Theorem [Jain]: For any extreme point x, there exists f with $x_f \geq \frac{1}{2}$
- Iterative Rounding: While connectivity reqs not met

  Solve LP(E)

  Take $f : x_f \geq \frac{1}{2}$

  add f to $E \longrightarrow F$
- 2-approximation algorithm: $w(F) \leq 2LP(E)$

Show $w(F) \leq 2\, LP(E)$

By induction, can assume $w(F \setminus \{f\}) \leq 2\, LP(E \cup \{f\})$

$$w(F) = w_f + \sum_{e \in F \setminus \{f\}} w_e \leq w_f + 2\, LP(E \cup \{f\})$$

$$\leq w_f (2 \cdot x_f) + 2\, LP(E \cup \{f\})$$

$$\leq 2\, LP(E)$$

x with edge f removed is feasible for $LP(E \cup \{f\})$

There exists f with $x_f \geq \frac{1}{2}$

Proof of Ravi, Singh, Nagarajan [2007]

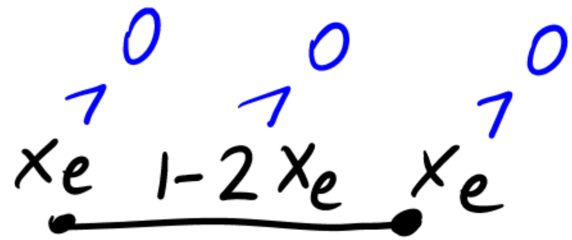
Let x: extreme point with $x_e < \frac{1}{2}$ for $e \in C = \{e : x_e > 0\}$

x is defined by $\quad x(\delta(S)) = R(S) - d(S) \quad S \in \mathcal{L}$

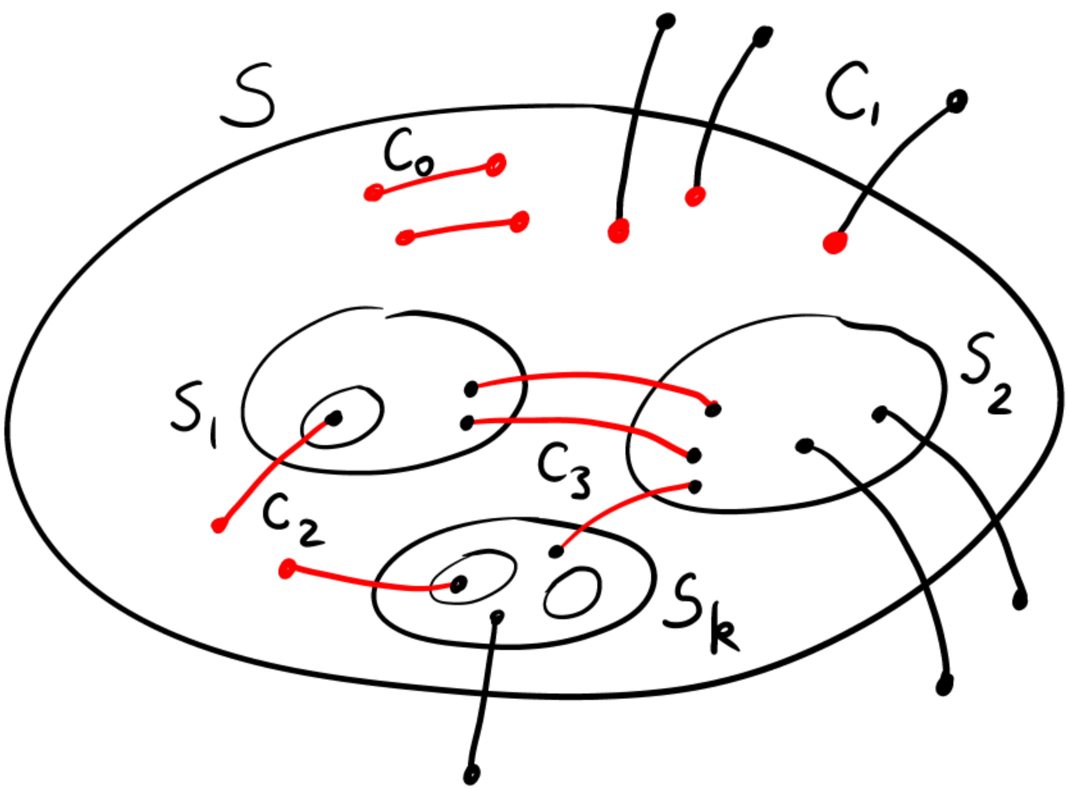
laminar

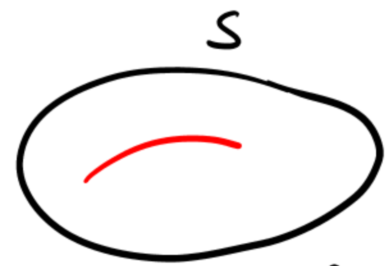
linear independence $\Rightarrow \quad |C| = |\mathcal{L}|$

Assign one unit to every edge:



Reassign to sets $S \in \mathcal{L}$





* if $e \in E(S) \setminus \bigcup_i E(S_i)$

* if $v \in S \setminus \bigcup_i S_i$

S gets

$$\begin{aligned} A_S &= \sum_{e \in C_1} x_e + \sum_{e \in C_2} (x_e + (1 - 2x_e)) + \sum_{e \in C_3} (1 - 2x_e) \\ &= \underbrace{x(C_1)} + |E_2| \underbrace{- x(C_2)} + |E_3| \underbrace{- 2x(C_3)} \end{aligned}$$

(i) $A_S > 0$ (indeed, if $C_1 = C_2 = C_3 = \phi$ then $x(\delta(S)) = \sum_i x(\delta(S_i))$)

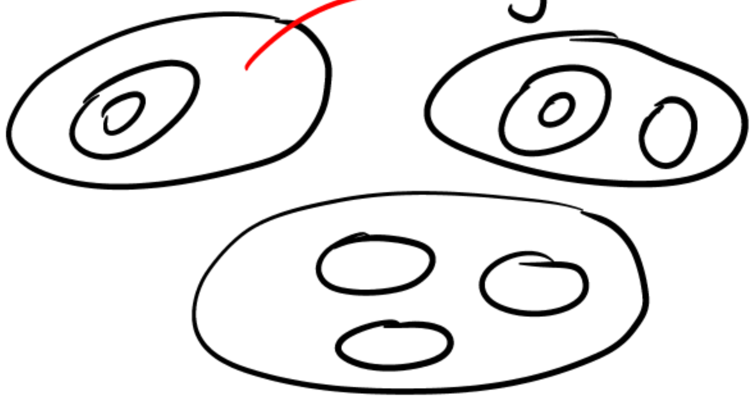
(ii) A_S integer:

$$\mathbb{Z} \ni x(\delta(S)) - \sum_i x(\delta(S_i)) = x(C_1) - x(C_2) - 2x(C_3)$$

$\Rightarrow A_S \geq 1$

Together all sets get $\geq |\mathcal{L}|$

no set gets $1-2x_e$ edge in $\delta(S)$ for S maximal



$\rightarrow |C| > |\mathcal{L}|$ Contradiction.

Degree Restricted Spanning Trees

Spanning Trees with Max Degree Bound

When does a graph have a spanning tree of maximum degree $\leq k$?

- NP-hard ($k = 2$ is Hamiltonian path...)
- S. Win [1989]: Relation to toughness
  $t(G) = \max_S \frac{|S|}{\text{\# conn. comp. of } G-S}$
  - If $t(G) \geq \frac{1}{k-2}$ then $\exists$ tree of max degree $\leq k$
  - If $\exists$ tree of max degree $\leq k$ then $t(G) \geq \frac{1}{k}$
- Algorithmically: Fürer and Raghavachari [1994], G. [unpublished, 1991]. Efficiently either show that G has no tree of maximum degree $\leq k$ or output a tree of max degree $\leq k + 1$
- Min cost version?

Bounded-Degree MST

Minimum Bounded-Degree Spanning Tree (MST) problem:

- Given $G = (V, E)$ with costs $c : E \longrightarrow R$, integer k
- find Spanning Tree T of maximum degree $\leq k$ and of minimum total cost $\sum_{e \in T} c(e)$

Even feasibility is hard.

Today

Let $OPT(k)$ be the cost of the optimum tree of maximum degree $\leq k$.

- [G. 2006]:
  Find a tree of cost $\leq OPT(k)$ and of maximum degree $\leq k+2$
  (or prove that no tree of max degree $\leq k$ exists)
- [Singh and Lau 2007]:
  Find a tree of cost $\leq OPT(k)$ and of maximum degree $\leq k+1$
  (or prove that no tree of max degree $\leq k$ exists)

Fractional Decomposition

Any convex combination of trees such that the average degree of every vertex is at most k can be viewed as a convex combination of trees each of maximum degree $k+1$

(E.g., for a $2k$-regular $2k$-edge-connected graph, there exists a convex combination of spanning trees of max degree 3 such that each edge is chosen with frequency $1/k$)

Integral decompositions?

Matroid Polytope

- [Edmonds '70] Given matroid $M = (E, \mathcal{I})$, convex hull of incidence vectors of independent sets is :

$$P(M) = \left\{ x \;\middle|\; \begin{array}{ll} x(F) \leq r_M(F) & F \subseteq E \\ x_e \geq 0 & e \in E \end{array} \right\}$$

  Convex hull $B(M)$ of bases: same with $x(E) = r_M(E)$

- For graphic matroid

$$\begin{aligned} B(M) \quad = \quad \{x : \quad & x(E(S)) \leq |S| - 1 && S \subset V \\ & x(E(V)) = |V| - 1 && \\ & x_e \geq 0 && \forall e\} \end{aligned}$$

Linear Programming Relaxation

Relaxation: $LP = \min\{c^T x : x \in Q(k)\} \leq OPT(k)$ where

$$
\begin{array}{rcll}
Q(k) & = & \{x: & x(E(S)) \leq |S| - 1 & S \subset V \\
 & & & x(E(V)) = |V| - 1 & \\
 & & & x(\delta(v)) \leq k & v \in V \\
 & & & x_e \geq 0 & e \in E\}
\end{array}
$$

Notation:

- $x(A) = \sum_{e \in A} x_e$
- $E(S) = \{e = (u, v) \in E : u, v \in S\}$
- $\delta(S) = \{(u, v) \in E : |\{u, v\} \cap S| = 1\}$

If $Q(k) = \emptyset$, no spanning tree of maximum degree $\leq k$.

Our Approach/Algorithm

- Solve LP and get an extreme point x^* of $Q(k)$ of cost LP
  E^*: support of x^*

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  $|E^*(C)| \leq 2|C| - 3$

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- Argue (polyhedrally) that cost of solution obtained $\leq LP$

Extreme points of $Q(k)$

- Recall

$$
\begin{array}{rclr}
Q(k) & = & \{x: \; x(E(S)) \leq |S| - 1 & S \subset V \\
& & x(E(V)) = |V| - 1 & \\
& & x(\delta(v)) \leq k & v \in V \\
& & x_e \geq 0 & e \in E\}
\end{array}
$$

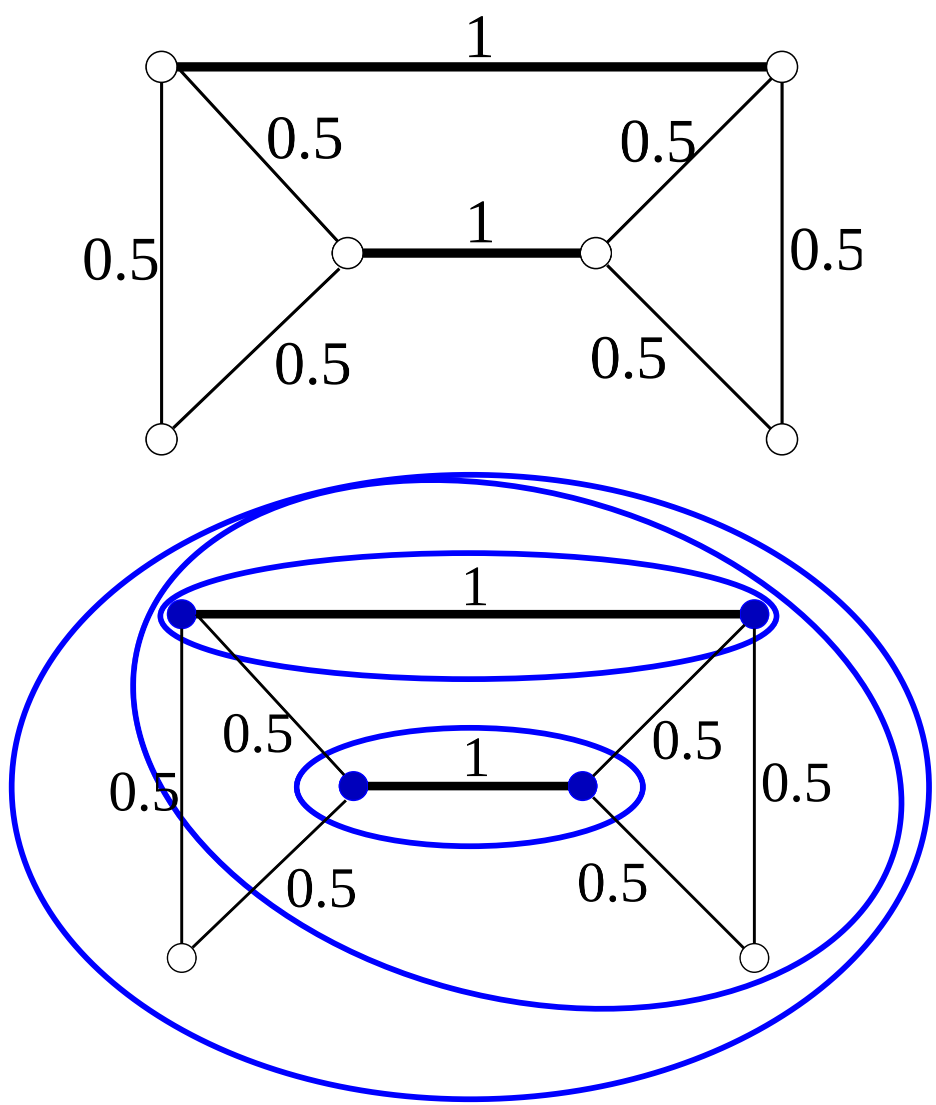
- Take an extreme point x^* of $Q(k)$
  Remove from E edges with $x^*_e = 0 \longrightarrow E^* = \{e : x^*_e > 0\}$
- x^* uniquely defined by tight inequalities:

$$
\begin{array}{ll}
x^*(E(S)) = |S| - 1 & S \in \mathcal{T} \\
x^*(\delta(v)) = k & v \in T
\end{array}
$$

or $Ax^* = b$ with $rank(A) = |E^*|$.

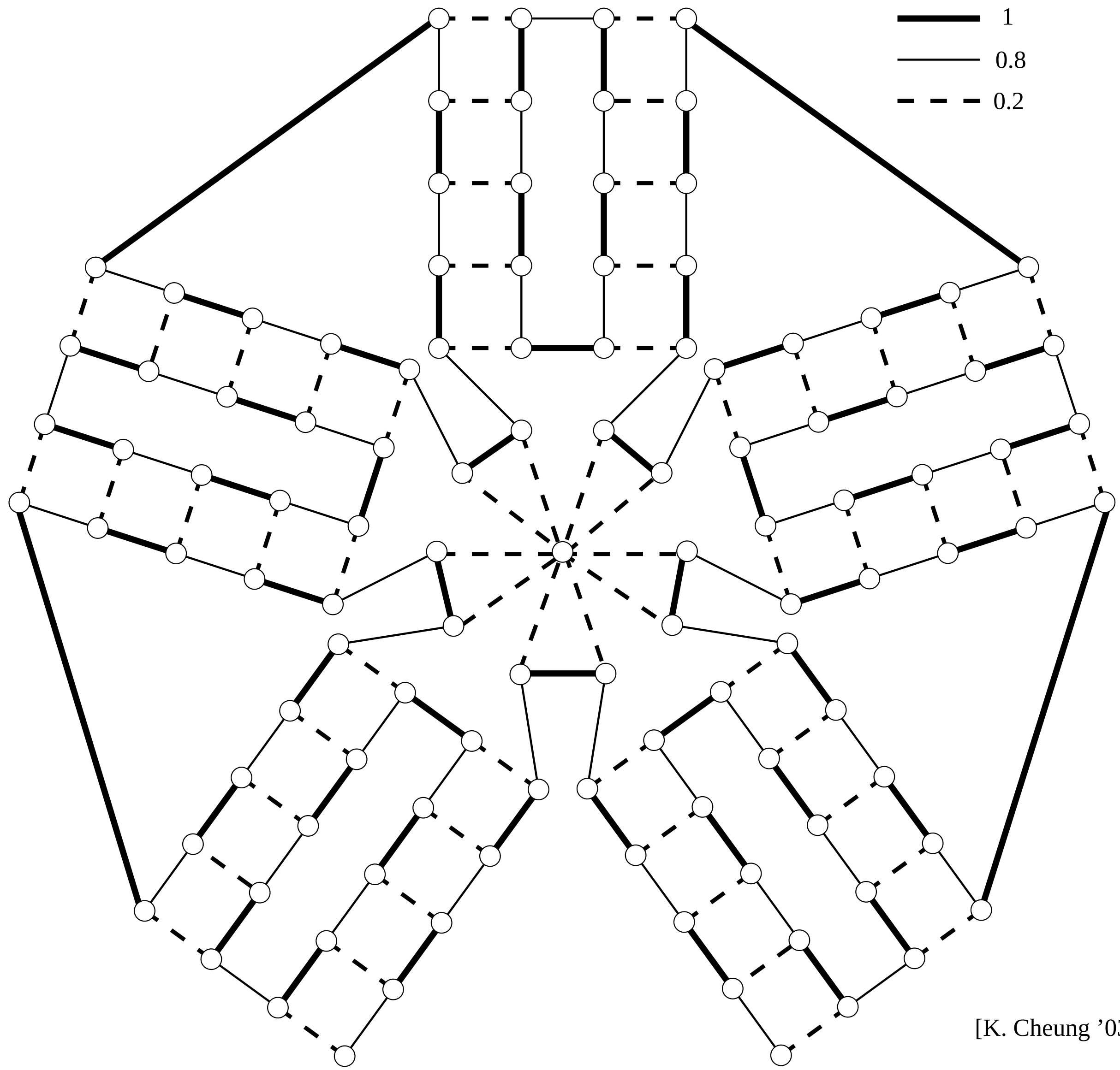
Example of Extreme Point

$k = 2$



Another Example of Extreme Point

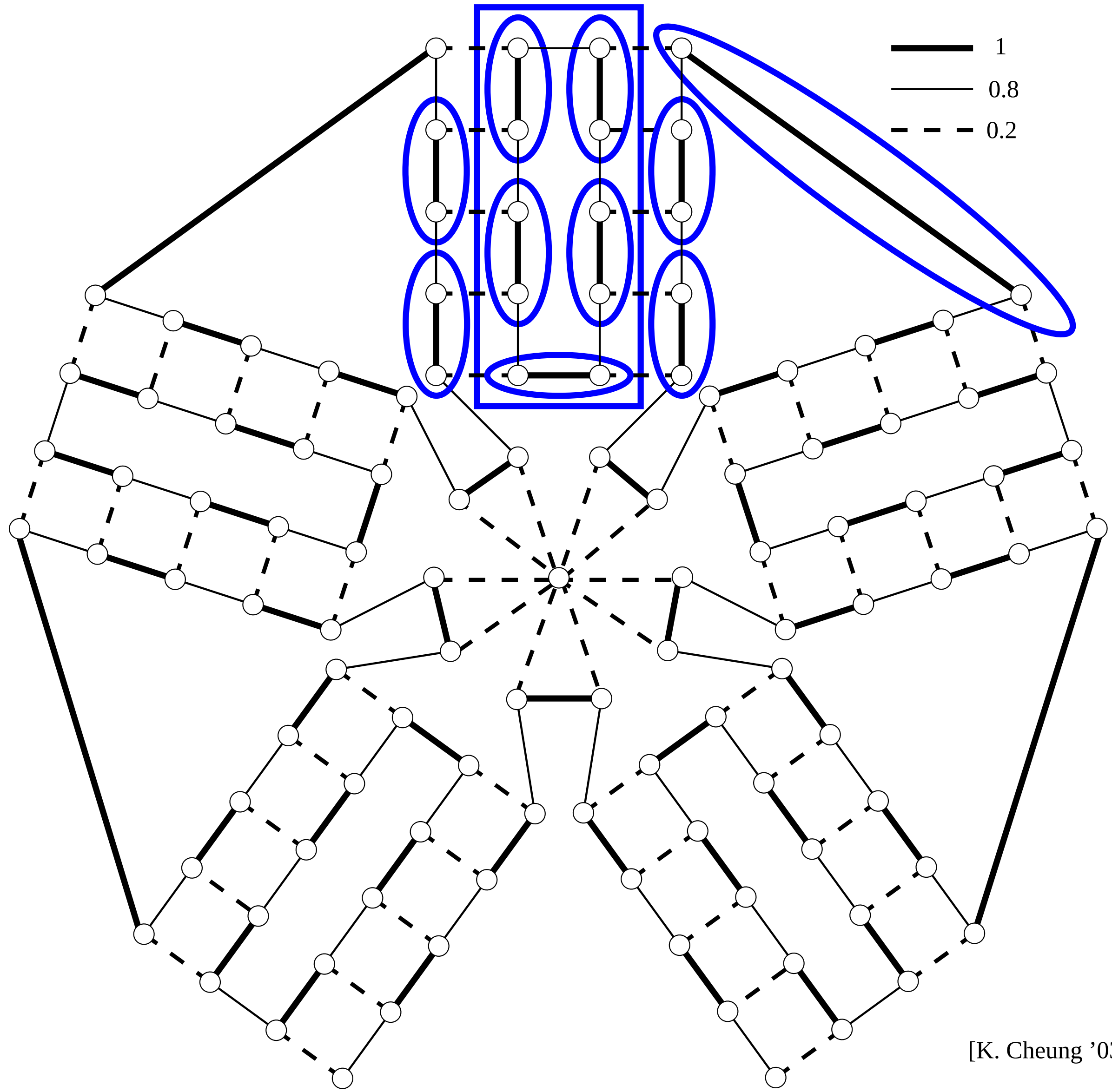
$\boldsymbol{k = 2}$



[K. Cheung ’03]

Another Example of Extreme Point

$k = 2$



[K. Cheung '03]

Extreme point

- Extreme point x^* uniquely defined by tight inequalities:

$$\begin{aligned} x^*(E(S)) &= |S| - 1 & \quad S \in \mathcal{T} \\ x^*(\delta(v)) &= k & \quad v \in T \end{aligned}$$

  or $Ax^* = b$ with $rank(A) = |E^*|$.

- Which full rank $|E^*| \times |E^*|$-submatrix of A to use?

Uncrossing

- If A, B tight ($A, B \in \mathcal{T}$) with $A \cap B \neq \emptyset$ then

$$\begin{aligned} |A| - 1 + |B| - 1 = x^*(E(A)) + x^*(E(B)) \\ \leq \quad x^*(E(A \cup B)) + x^*(E(A \cap B)) \\ \leq \quad |A \cup B| - 1 + |A \cap B| - 1. \end{aligned}$$

Thus,

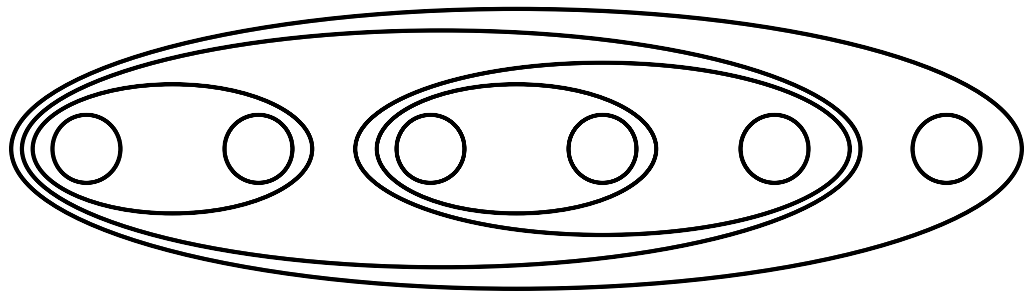
$x(E(A)) + x(E(B)) =$
$x(E(A \cup B)) + x(E(A \cap B))$

  - $A \cup B, A \cap B \in \mathcal{T}$
  - No edges between $A \setminus B$ and $B \setminus A$

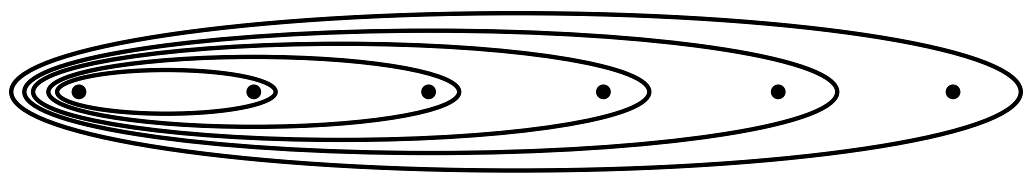
- Uncrossing argument implies: There exists laminar subfamily $\mathcal{L}$ of $\mathcal{F}$ satisfying $span(\mathcal{L}) = span(\mathcal{F})$
(Any maximal laminar subfamily works)

Size of Laminar Families

- Any laminar family on n elements contains at most $2n - 1$ sets



- If no singletons then $\leq n - 1$ sets



Small Support

- x^* defined by

$$\begin{aligned} x^*(E(S)) &= |S| - 1 & \qquad S &\in \mathcal{L} \\ x^*(\delta(v)) &= k & \qquad v &\in T \end{aligned}$$

  with $\mathcal{L}$ a laminar family of sets without singletons
- System $Ax^* = b$ with $A = |E^*| \times |E^*|$ of full rank
- $|\mathcal{L}| \leq n - 1$ implies $|E^*| = |\mathcal{L}| + |T| \leq (n-1) + n = 2n - 1$
- Similar results known in many settings. E.g. Boyd and Pulleyblank '91 for subtour polytope.

Everywhere Sparse: $|E^*(C)| \leq 2|C| - 1$ for all C

$$
A = \begin{array}{|ccccccccc|l}
\hline
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots & \leftarrow x^*(E(S)) = |S| - 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & \leftarrow x^*(E(T)) = |T| - 1 \\
\multicolumn{9}{|c|}{\dots\dots\dots\dots\dots\dots\dots\dots} & \\
\multicolumn{9}{|c|}{\dots\dots\dots\dots\dots\dots\dots\dots} & \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & \cdots & \leftarrow x^*(\delta(v)) = k \\
\multicolumn{9}{|c|}{\dots\dots\dots\dots\dots\dots\dots\dots} & \\
\multicolumn{9}{|c|}{\dots\dots\dots\dots\dots\dots\dots\dots} & \\
\hline
\end{array}
$$

Everywhere Sparse: $|E^*(C)| \leq 2|C| - 1$ for all C

$$
A = \begin{array}{|c|l}
\hline
\cdots\cdots\cdots\ \chi(E^*(S))\ \cdots\cdots\cdots & \leftarrow x^*(E(S)) = |S| - 1 \\
\cdots\cdots\cdots\ \chi(E^*(T))\ \cdots\cdots\cdots & \leftarrow x^*(E(T)) = |T| - 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\hline
\cdots\cdots\cdots\ \chi(\delta(v))\ \cdots\cdots\cdots & \leftarrow x^*(\delta(v)) = k \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
\hline
\end{array}
$$

Everywhere Sparse: $|E^*(C)| \leq 2|C| - 1$ for all C

Take any $C \subseteq V$. A has full rank

$\Longrightarrow$ columns of A corresponding to $E^*(C)$ are linearly independent

$$
A = \begin{array}{|cccccc|c|l}
\multicolumn{6}{|c|}{\leftarrow \quad E^*(C) \quad \rightarrow} & \multicolumn{1}{c}{} & \\
\hline
1 & 1 & 0 & 1 & 1 & 0 & \quad\quad\quad & \leftarrow x^*(E(S)) = |S| - 1 \\
0 & 1 & 0 & 0 & 1 & 0 & & \leftarrow x^*(E(T)) = |T| - 1 \\
\multicolumn{6}{|c|}{\dots\dots\dots\dots\dots\dots} & & \\
\multicolumn{6}{|c|}{\dots\dots\dots\dots\dots\dots} & & \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & & \leftarrow x^*(\delta(v)) = k \\
\multicolumn{6}{|c|}{\dots\dots\dots\dots\dots\dots} & & \\
\multicolumn{6}{|c|}{\dots\dots\dots\dots\dots\dots} & & \\
\hline
\end{array}
$$

Everywhere Sparse: $|E^*(C)| \leq 2|C| - 1$ for all C

Take any $C \subseteq V$. A has full rank

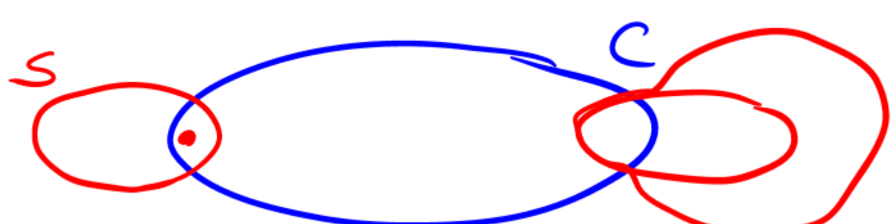
$\Longrightarrow$ columns of A corresponding to $E^*(C)$ are linearly independent

$$
A = \begin{array}{|c|c|l}
\leftarrow \quad E^*(C) \quad \rightarrow & & \\
\hline
\chi(E^*(S) \cap E^*(C)) & & \leftarrow x^*(E(S)) = |S| - 1 \\
\chi(E^*(T) \cap E^*(C)) & & \leftarrow x^*(E(T)) = |T| - 1 \\
\dots\dots\dots\dots\dots\dots & & \\
\dots\dots\dots\dots\dots\dots & & \\
\hline
.\chi(\delta(v) \cap E^*(C)). & & \leftarrow x^*(\delta(v)) = k \\
\dots\dots\dots\dots\dots\dots & & \\
\dots\dots\dots\dots\dots\dots & & \\
\hline
\end{array}
$$

Everywhere Sparse: $|E^*(C)| \leq 2|C| - 1$ for all C

Take any $C \subseteq V$. A has full rank
$\Longrightarrow$ columns of A corresponding to $E^*(C)$ are linearly independent

$$A = \begin{array}{|c|c|l}
\leftarrow \quad E^*(C) \quad \rightarrow & \multicolumn{1}{|c}{} & \\
\hline
.\ \chi(E^*(S \cap C))\ . & & \leftarrow x^*(E(S)) = |S| - 1 \\
\dots\dots\dots\dots\dots\dots & & \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 & & \leftarrow \text{if } |S \cap C| \leq 1 \\
\dots\dots\dots\dots\dots\dots & & \\
\hline
\chi(\delta(v) \cap E^*(C)) & & \leftarrow x^*(\delta(v)) = k \\
\dots\dots\dots\dots\dots\dots & & \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 & & \leftarrow \text{if } v \in (T \setminus C) \\
\hline
\end{array}$$



Everywhere Sparse: $|E^*(C)| \leq 2|C| - 1$ for all C

Take any $C \subseteq V$. A has full rank
$\Longrightarrow$ columns of A corresponding to $E^*(C)$ are linearly independent

$$
A = \begin{array}{|c|c|l}
\leftarrow \quad E^*(C) \quad \rightarrow & & \\
\hline
.\,\chi(E^*(S \cap C))\,. & & \text{distinct, non-zero rows} \\
\dots\dots\dots\dots\dots\dots & & \text{for laminar family} \\
\dots\dots\dots\dots\dots\dots & & \mathcal{L}_C = \{S \cap C : \\
\dots\dots\dots\dots\dots\dots & & S \in \mathcal{L} \text{ and } |S \cap C| \geq 2\} \\
\hline
\chi(\delta(v) \cap E^*(C)) & & \text{distinct, non-zero rows} \\
\dots\dots\dots\dots\dots\dots & & \text{for } v \in T \cap C \\
\dots\dots\dots\dots\dots\dots & & \\
\hline
\end{array}
$$

$\Longrightarrow |E^*(C)| = rank(B) \leq |C| + |C| - 1 = 2|C| - 1$ for all $C \subseteq V$

Slight Improvement

Slightly more careful rank counting argument shows $|E^*(C)| \leq 2|C| - 3$ for every $C \subseteq V$

Orientation of E^*

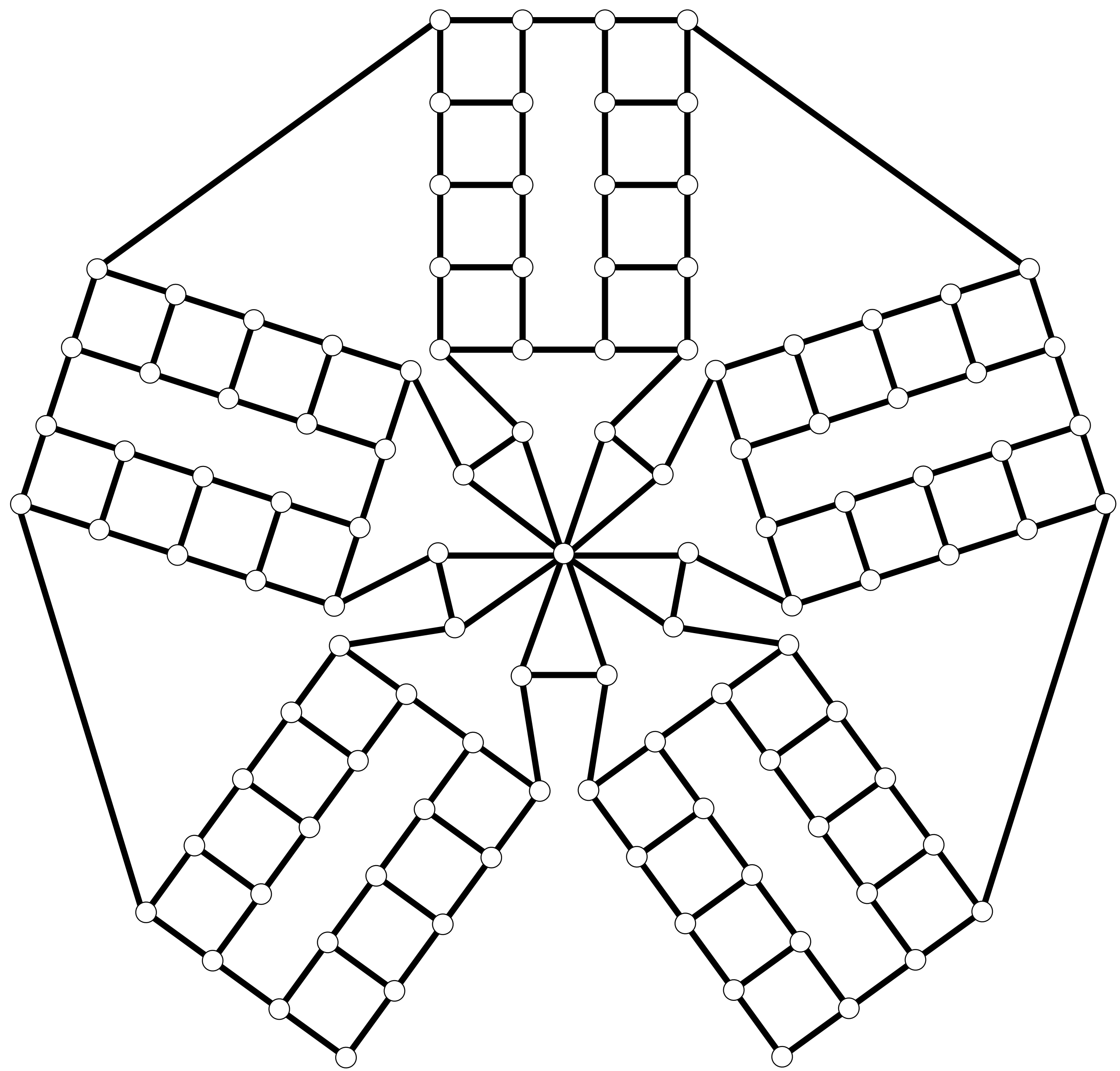
- Graph Orientation: [Hakimi '65] An undirected graph G has an orientation with indegree $d^-(v) \le u_v$ if and only if for all $C \subseteq V$:

$$|E(C)| \le \sum_{v \in C} u_v$$

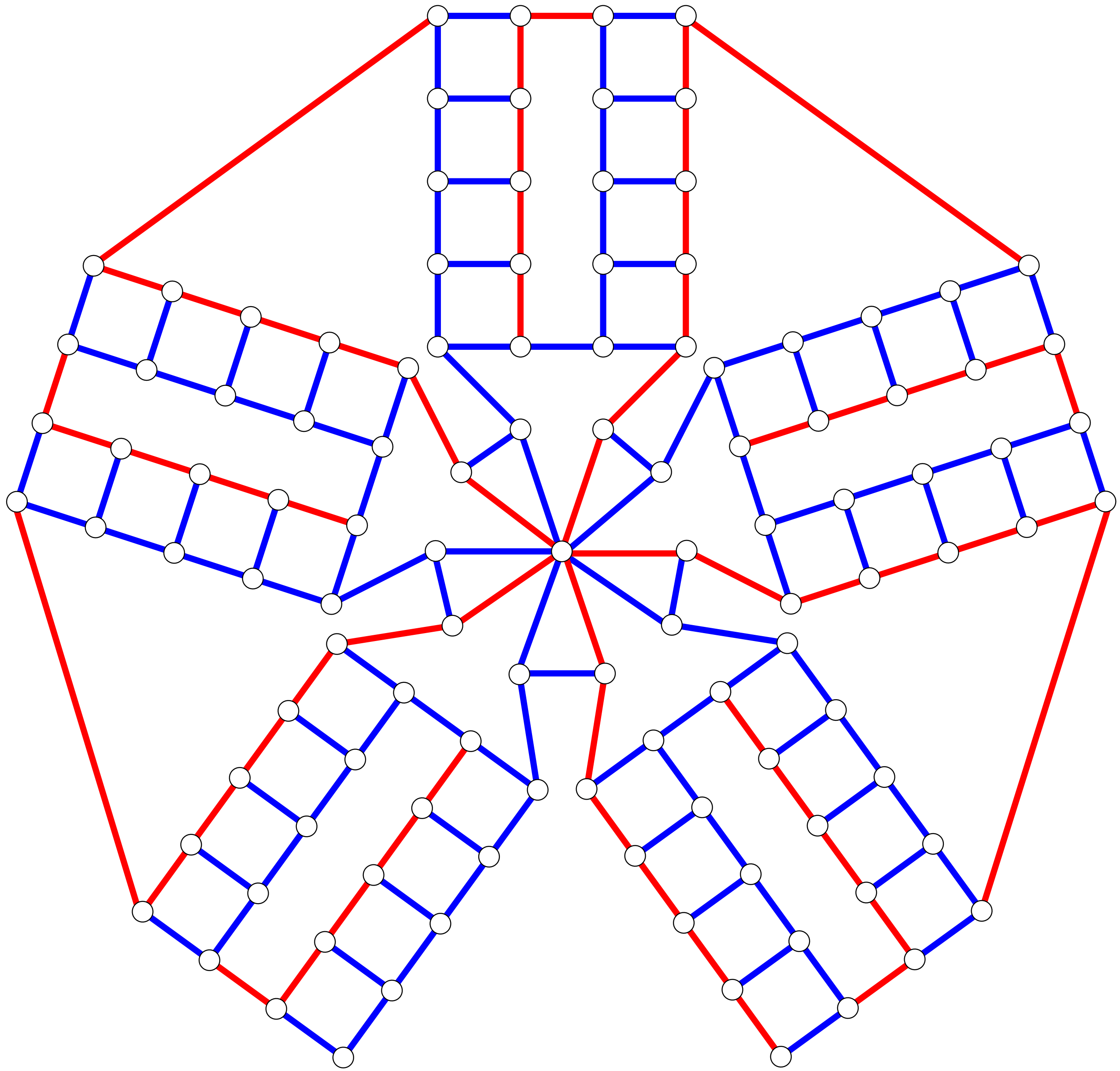
  [Easy, e.g. from max flow/min cut]

- $\Longrightarrow E^*$ can be oriented into A^* such that $d^-(v) \le 2$ for all $v \in V$
- Another way
  - [Nash-Williams' 1964] A graph can be partitioned into k forests if and only if for all $C \subseteq V$: $|E(C)| \le k(|C| - 1)$
    (Special case of Edmonds' 1965 matroid base covering theorem)
  - $\Longrightarrow E^*$ can be partitioned into 2 forests
  - Orient each forest as a branching (indegree at most 1)

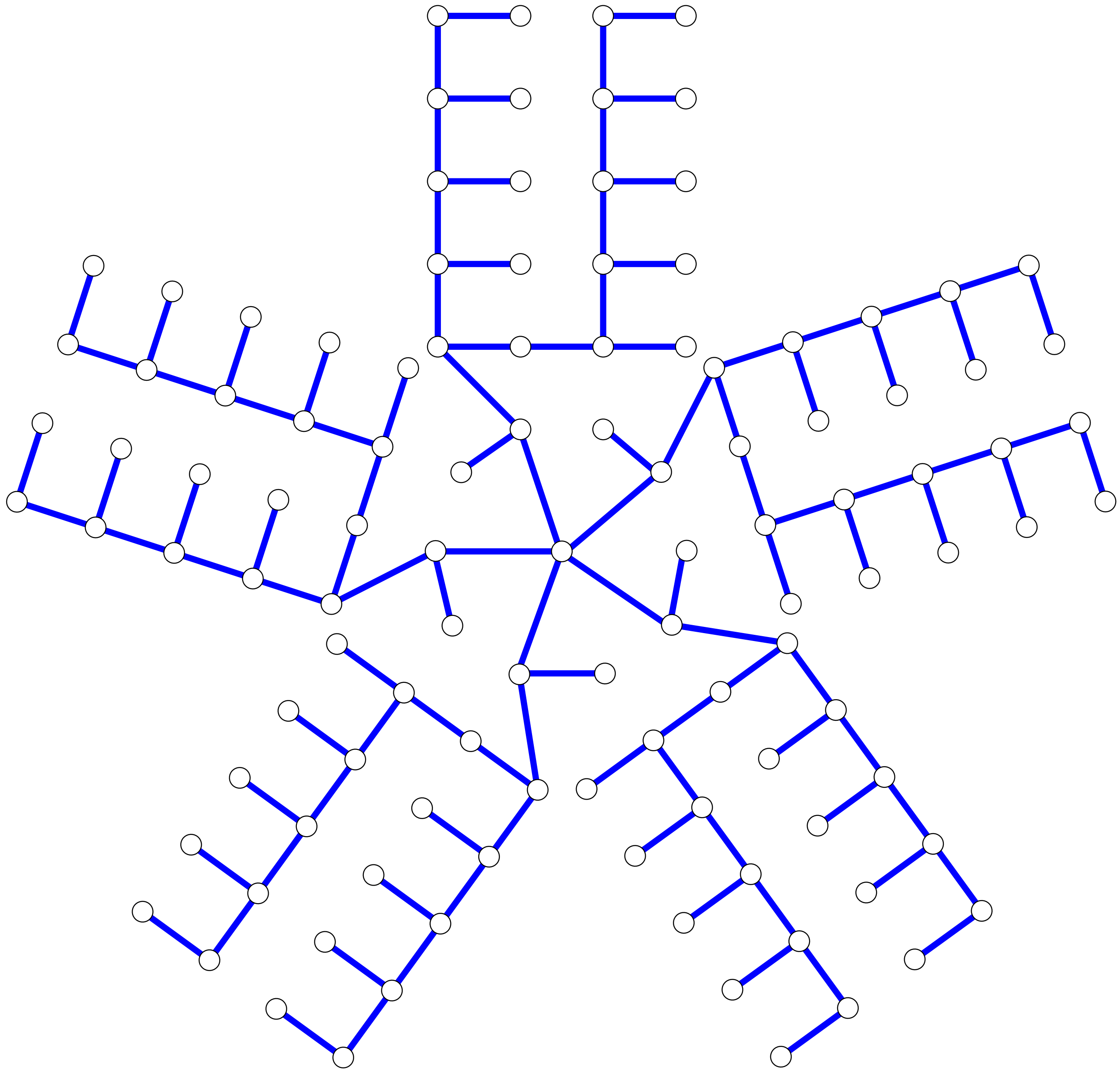
Example



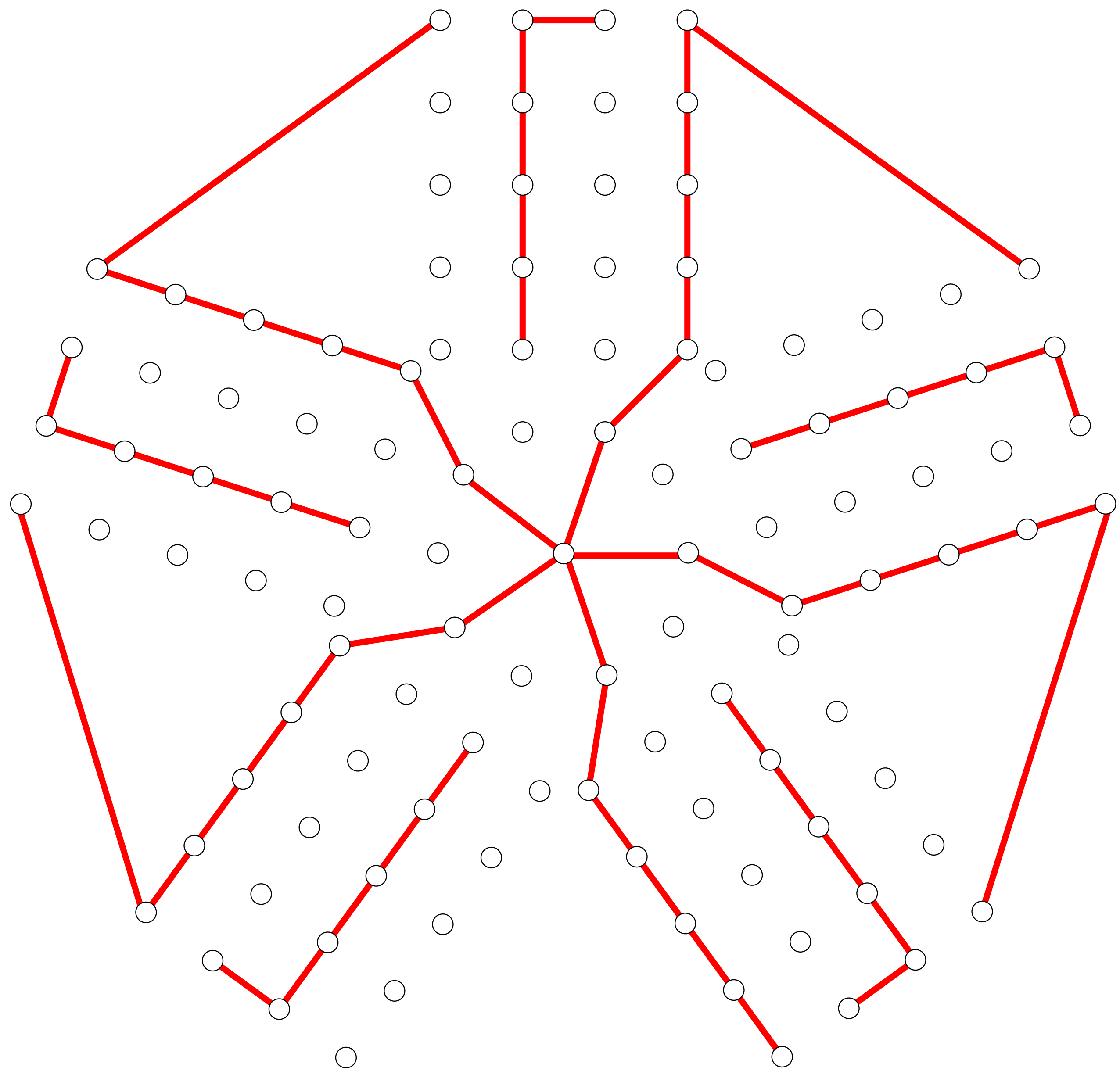
Example



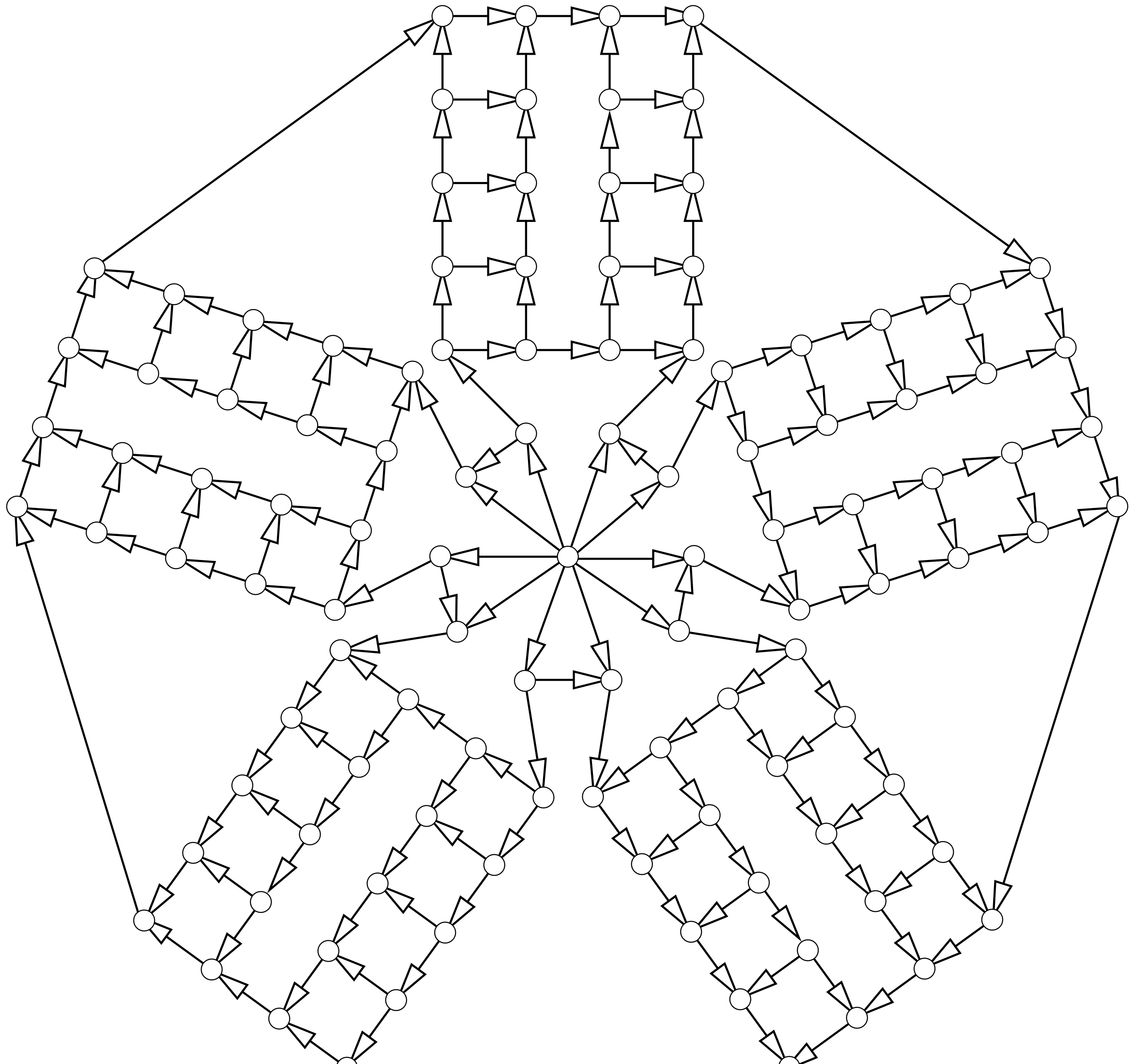
Example



Example



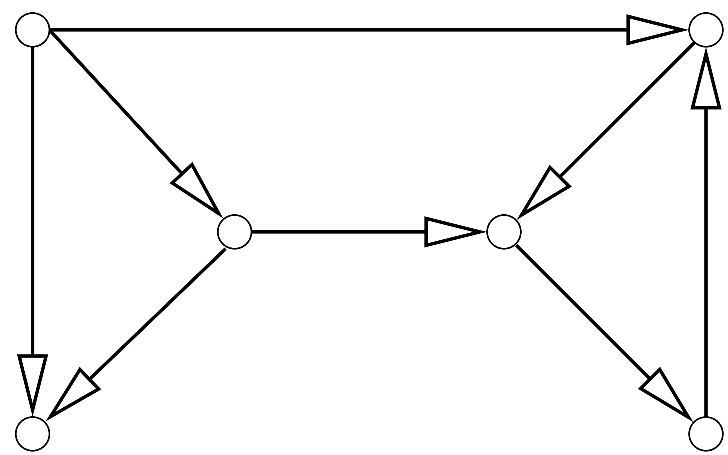
Example



Matroid M_2

- Given orientation A^* of E^* with indegree $d^-(v) \leq 2$ for $v \in V$, define partition matroid $M_2(x^*) = (E^*, \mathcal{I})$ where

$$\mathcal{I} = \{F : |F \cap \delta^+_{A^*}(v)| \leq k \text{ for all } v \in V\}$$



- Since all but at most 2 edges incident to v are outgoing in A^*, any independent set F of $M_2(x^*)$ has maximum degree $\leq k+2$
- Slack of 3 units for every $C \implies$ can assume one specific vertex of degree $\leq k$ and another of degree $\leq k+1$

Matroid Intersection Approach

- Find a minimum cost spanning tree in E^* which is also independent in $M_2(x^*)$
- M_1: graphic matroid for E^*
  $\Longrightarrow$ want a base of M_1 independent in $M_2(x^*)$
  $\Longrightarrow$ matroid intersection
- Polynomial time using matroid intersection algorithm
  - Edmonds '79 and Lawler '75
  - Brezovec, Cornuéjols and Glover '88: $O(n^3)$ algorithm for $\bigcap$ of graphic matroid and partition matroid
  - Gabow and Xu scaling algorithm for linear matroid intersection: $O(n^{2.77} \log nW)$
  - Harvey '06: $O(n^{2.38} W)$ (polynomial if weights are small)
- Bound on cost?

Matroid Polytope

- [Edmonds '70] Given matroid $M = (E, \mathcal{I})$, convex hull of incidence vectors of independent sets is :

$$P(M) = \left\{ x \;\middle|\; \begin{array}{ll} x(F) \le r_M(F) & F \subseteq E \\ x_e \ge 0 & e \in E \end{array} \right\}$$

  Convex hull $B(M)$ of bases: same with $x(E) = r_M(E)$

- For graphic matroid M_1 on E^*

$$\begin{array}{rclll} B(M_1) & = & \{x: & x(E(S)) \le |S| - 1 & S \subset V \\ & & & x(E(V)) = |V| - 1 & \\ & & & x_e \ge 0 & e \in E^*\} \end{array}$$

Matroid Polytope

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$$P(M) = \left\{ x \;\middle|\; \begin{array}{ll} x(F) \leq r_M(F) & F \subseteq E \\ x_e \geq 0 & e \in E \end{array} \right\}$$

  Convex hull $B(M)$ of bases: same with $x(E) = r_M(E)$

- For matroid $M_2(x^*)$

$$\begin{array}{rcll} P(M_2(x^*)) & = & \{x : \; x(\delta^+_{A^*}(v)) \leq k & v \in V \\ & & \quad 1 \geq x_e \geq 0 & e \in E^*\} \end{array}$$

Matroid Intersection Polytope

- [Edmonds '70] Given two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, convex hull of independent sets common to both matroids is

$$P(M_1) \cap P(M_2)$$

(Similarly, if take bases for one of them)

Cost Analysis

- Observe that $x^* \in B(M_1)$ and $x^* \in P(M_2(x^*))$
- Cost of solution returned:

$$\min\{c(x) : x \in B(M_1) \cap P(M_2(x^*))\} \leq c(x^*) = LP$$

- Thus, we get a spanning tree of maximum degree $k+2$ and of cost $\leq LP$
- Remark: We could have decomposed $x^* \in B(M_1) \cap P(M_2(x^*))$ as a convex combination of spanning trees independent for M_2 (using Cunningham '84) and take the best cost among them (enough to get at most LP)
- $x^* \in B(M_1) \cap P(M_2(x^*))$ implies that

$$Q(k) = conv(\{x^*\}) \subseteq conv(Q(k+2) \cap \mathbb{Z}^E)$$

Cost Analysis

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- $x^* \in B(M_1) \cap P(M_2(x^*))$ implies that
  Any convex combination of trees such that the average degree of every vertex is at most k can be viewed as a convex combination of trees each of maximum degree $\leq k + 2$

Without Hakimi, Nash-Williams, Edmonds, etc.

- Laplace expansion of $\det$ along column j:

$$\det(A) = \sum_i (-1)^{i+j} a_{ij} \det(M_{ij})$$

- Generalized Laplace expansion (Laplace 1772): For any I,

$$\det(A) = \sum_{J:|J|=|I|} sgn(I,J) \det(A[I,J]) \det(A[\bar{I},\bar{J}])$$

$\Longrightarrow$ If A invertible, there exists J with $A[I,J]$ and $A[\bar{I},\bar{J}]$ invertible (follows also from matroid union min-max relation)

- Algorithmically: For every $j = 1$ to n do
  - either set all entries in column j from rows in I or from rows in $\bar{I}$ to 0 so as to keep the matrix invertible

Orientation Purely Algebraically

- Take $Ax^* = b$
- Can partition E into E_1, E_2

$$A = \begin{array}{c|c|c|l} & E_1 & E_2 & \\ \hline I & B_1 & & \longleftarrow \text{ rows } x^*(E(S)) = |S| - 1 \\ \hline \bar{I} & & B_2 & \longleftarrow \text{ rows } x^*(\delta(v)) = k \\ \hline \end{array}$$

with B_1, B_2 invertible

- B_1 invertible + $\mathcal{L}$ laminar: E_1 must be a forest
- B_2 invertible: every connected component of E_2 is a tree or a tree + one edge
- $\Longrightarrow$ can trivially orient both E_1 and E_2 with indegree at most 1

Former Conjecture... Now Theorem

- Conjecture:

$$Q(k) \subseteq conv(Q(k+1) \cap \mathbb{Z}^E)$$

- Any convex combination of trees such that the average degree of every vertex is at most k can be viewed as a convex combination of trees each of maximum degree $k+1$
- Proved by Singh and Lau '07:
  - Efficient algorithm to get tree of cost $\leq OPT(k)$ and of degree $\leq k+1$
  - Uses iterative relaxation, generalizing Jain's iterative rounding

Open Questions

- Can one find E^* (combinatorially) without computing x^* (by linear programming)?
- $+1$ algorithm possible via matroid approach if, for all extreme points x^* with support E^*, there exists an orientation A^* such that for all $v \in V$:

$$\sum_{e \in \delta^-_{A^*}(v)} (1 - x^*_e) \leq 1$$

(For general (non-extreme) x^*, deciding if such orientation exists is NP-hard.)

General Lower and Upper bounds

General Degree-Bounded Spanning Trees:

- Given $l, u : V \rightarrow Z_+$, find a spanning tree T such that $l(v) \leq d_T(v) \leq u(v)$ for all $v \in V$ and of minimum cost
- Same approach gives a spanning tree of cost at most LP and of degree $l(v) - 2 \leq d_v(T) \leq u(v) + 2$ for all $v \in V$
- One step is to argue that for

$$\begin{array}{rcll} P_2 & = & \{x : \; l(v) - 2 \leq x(\delta^+_{A^*}(v)) \leq u(v) & v \in V \\ & & \quad 1 \geq x_e \geq 0 & e \in E^*\} \end{array}$$

  $B(M_1) \cap P_2$ is integral
- Singh and Lau '07: $+1$ also for general upper and lower bounds

Singh and Lau's Iterative Relaxation

- Given a forest F (initially empty) and $W \subseteq V$, consider LP relaxation for problem of augmenting F into a tree with general degree bounds $u(v)$ for $v \in W$
- Solve relaxation; remove edges of value 0 and and add edges of value 1 to F
- Theorem: If non-integral, there exists $v \in W$ with $u(v) + 1$ incident edges.
- Remove v from W and repeat

Formulation

Let E: all edges,

E_0: excluded edges,

E_1: included edges in solution,

$E' = E \setminus (E_0 \cup E_1)$

$W \subseteq V$: vertices v with degree upper bound $u(v)$

LP relaxation:

$P(E_0, E_1, W)$

$$
\begin{array}{lll}
\min & \sum_{e \in E} c_e x_e & \\
 & x(E(S)) \leq |S| - 1 & S \subset V \\
 & x(E(V)) = |V| - 1 & \\
 & x(\delta(v)) \leq u(v) & v \in W \\
 & x_e = 1 & e \in E_1 \\
 & x_e = 0 & e \in E_0 \\
 & x_e \geq 0 & e \in E'\}
\end{array}
$$

Singh and Lau's Algorithm

$E_0 = E_1 = \emptyset$, $W = V$

Repeat

Find optimum extreme point x to $LP(E_0, E_1, W)$

$E_0 = \{e : x_e = 0\}$, $E_1 = \{e : x_e = 1\}$, $E' = E \setminus (E_0 \cup E_1)$

Remove from W vertices v with $d_{E_1}(v) + d_{E'}(v) \leq u(v) + 1$

Until E_1 is a spanning tree

- Theorem [Singh and Lau '07]: Algorithm terminates $\rightarrow E_1$ satisfies the degree bounds $u(v) + 1$
- New simple proof of Bansal, Khandekar and Nagarajan '07

Tight Inequalities Can Be Uncrossed

$$\mathcal{F} = \{S : x(E'(S)) = |S| - 1 - |E_1(S)|\}$$

- $S, T \in \mathcal{F}$, $S \cap T \neq \phi$

$$\Rightarrow S \cap T, S \cup T \in \mathcal{F}$$

$$x(E'(S)) + x(E'(T)) = x(E'(S \cap T)) + x(E'(S \cup T))$$

$x|_{E'}$ uniquely defined by:

$$\begin{cases} x(E'(S)) = |S| - 1 - |E_1(S)| & S \in \mathcal{L} \\ x(\delta_{E'}(v)) = u(v) - |\delta_{E_1}(v)| & v \in T \end{cases}$$

with $\mathcal{L}$ laminar and $|E'| = |\mathcal{L}| + |T|$

W decreases

- Let
$$def(v) = \sum_{e \in \delta_{E'}(v)} (1 - x_e) = \sum_{e \in \delta_{E' \cup E_1}(v)} (1 - x_e)$$
- For $v \in T$, $def(v) = d_{E_1}(v) + d_{E'}(v) - u(v) \in \mathbb{Z}$
- Claim: There exists $v \in T$ such that $def(v) = 1$
$\rightarrow$ v can be removed from W

(i) $|\mathcal{L}| \leq x(E')$

$$\underbrace{x(E'(S))}- \sum_i \underbrace{x(E'(S_i))} \in \mathbb{Z}$$

if $=0 \Rightarrow x(E'(S)) = \sum_i x(E'(S_i))$

linear dependence

if $=$ then $E' \subseteq \bigcup_{S \in \mathcal{L}} E[S]$

(2) $$\sum_{v\in T} def(v) = \sum_{v\in T}\Big(\sum_{e\in\delta_{E'}(v)} (1-x_e)\Big)$$

if = then $E' \subseteq E(T)$ ← $$\leq 2\big(|E|' - x(E')\big)$$

$$= 2\big(|\mathcal{L}| + |T| - x(E')\big)$$

if = then $E' \subseteq \bigcup_{S\in\mathcal{L}} E(S)$ ← $$\leq 2\big(|\mathcal{L}| + |T| - |\mathcal{L}|\big) = 2|T|$$

$$\sum_{v\in T} x(\delta_{E'}(v)) = 2\,x(E') = 2 \sum_{\text{maximal sets } S \text{ in } \mathcal{L}} x(E(S))$$

→ linear dep.

→ contradiction

$$\Rightarrow \sum_{v\in T} def(v) < 2|T| \Rightarrow \exists v\in T:\ def(v) = 1$$

Iterative Relaxation

- Many more applications, see Singh and Lau '07, Lau et al. '07, Bansal et al. '07.

- Bansal et al. '07: Given a directed graph $D = (V, A)$ with root $r \in V$, and outdegree upper bounds $b(v)$ for every $v \in V$, (efficiently) either decide that D has no r-arborescence with $d^+(v) \leq b(v)$ or output an r-arborescence with $d^+(v) \leq b(v) + 2$.