Flows in graphs and matroids.

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Flows in graphs

Consider:

G : graph, $\Sigma \subseteq EG$ and $w \in \mathbb{Z}_+^{EG}$,

where

Σ : demand edges

$\overline{\Sigma}$: capacity edges

w : demand / capacity.

Let $\mathcal{C}_\Sigma = \{ C : C \text{ circuit of } G \text{ st } |C \cap \Sigma| = 1 \}$

Def: A (G, Σ, w)-flow is $y \in \mathbb{R}_+^{\mathcal{C}_\Sigma}$ st.

Capacity constraints:

$$\forall e \in \overline{\Sigma} : \quad \sum (y_C : e \in C \in \mathcal{C}_\Sigma) \leq w_e \quad (1)$$

Demand constraints:

$$\forall e \in \Sigma : \quad \sum (y_C : e \in C \in \mathcal{C}_\Sigma) = w_e \quad (2)$$

Def: Cut-condition:

$$\forall U \subseteq VG, \quad w(\delta(U) - \Sigma) \geq w(\delta(U) \cap \Sigma)$$

Qu: When is cut-condition sufficient?

Signed minors:

(G, Σ) (signed) graph

Σ — signature

$B \subseteq EG$ is odd/even if $|B \cap \Sigma|$ is odd/even.

(1) Resign: $(G, \Sigma) \longrightarrow (G, \Sigma \triangle \delta(v))$.

(2) Delete e: $(G, \Sigma) \setminus e = (G \setminus e, \Sigma - \{e\})$

(3) Contract $e \notin \Sigma$: $(G, \Sigma) / e = (G / e, \Sigma)$.

Sequence of (1), (2), (3): signed minor.

Notation: $(G, \Sigma) \succcurlyeq (H, \Gamma)$, $\widetilde{G} = (G, EG)$

Flow theorems:

Th1:

$$\left.\begin{array}{l}\text{cut-condition}\\ (G,\Sigma) \not\succeq \tilde{K}_4\end{array}\right\} \Rightarrow \exists \text{ integer } (G,\Sigma,w)\text{-flow.}$$

Th2: [G]

$$\left.\begin{array}{l}\text{cut-condition}\\ (G,\Sigma) \not\succeq \tilde{K}_5\end{array}\right\} \Rightarrow \exists\ (G,\Sigma,w)\text{-flow.}$$

Def: w is Eulerian if $\forall U \subseteq VG$, $w(\delta_G(U))$ even.

Th3: [G, Geelen]

$$\left.\begin{array}{l}\text{cut-condition}\\ (G,\Sigma) \not\succeq \tilde{K}_5\\ w \text{ Eulerian}\end{array}\right\} \Rightarrow \exists \text{ integer } (G,\Sigma,w)\text{-flow.}$$

Special cases: $(w \in \mathbb{Z}_+^{EG})$

Co: [Hu / Rothschild–Whinston]

$$\left.\begin{array}{l} |\Sigma| = 2 \\ \text{cut-condition} \end{array}\right\} \Rightarrow \exists\ (G,\Sigma,w)\text{-flow } y.$$

If w Eulerian, y can be chosen integer

Co: [Okamura, Seymour]

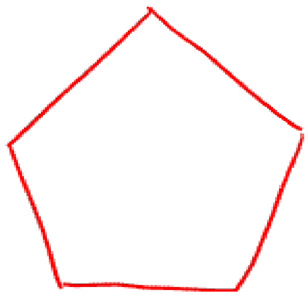
$G \setminus \Sigma$ planar.

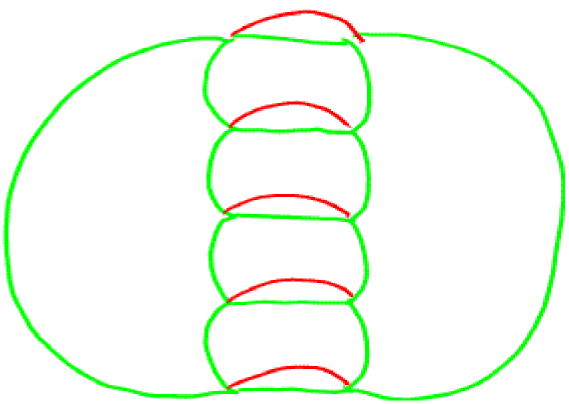
$\Sigma = \{s_1 t_1, \dots, s_k t_k\}$ where $s_1, \dots s_k, t_1, \dots t_k$ in ∞-face of $G \setminus \Sigma$.

Then cut-condition $\Rightarrow \exists\ (G,\Sigma,w)$-flow.

If w Eulerian, y can be chosen integer.

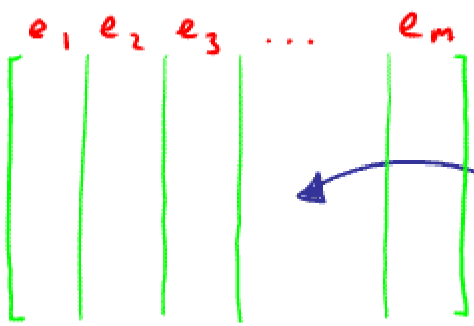
Other signed graphs without $\tilde{K}_5$ minors.

 $= \Sigma$



Even face embedding on Klein bottle.

Binary matroids:



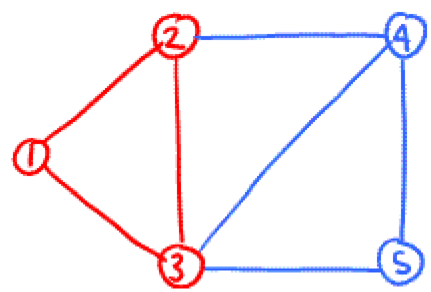
2 element field

$B \subseteq E_G$ is a **cycle** if all rows of the column submatrix of A corresponding to columns of B have an even nb of ones.

A minimal cycle is a **circuit**.

Ex:

$$
\begin{array}{c|ccccccc}
 & 12 & 13 & 23 & 24 & 45 & 34 & 35 \\
\hline
1 & 1 & 1 & & & & & \\
2 & 1 & & 1 & 1 & & & \\
3 & & 1 & 1 & & & 1 & 1 \\
4 & & & & 1 & 1 & 1 & \\
5 & & & & & 1 & & 1
\end{array}
$$



If $B \subseteq EM$ intersect all cycles of M with even parity then B is a cocycle.

The set of all cocycles of M is the set of cycles of a matroid M^* called the dual of M.

$$\text{cycles of } M \quad \perp \quad \text{cycles of } M^*$$

Flows in matroids: (M: matroid)

Let $\mathcal{C}_\Sigma = \{ C : C \text{ circuit of } M \text{ st } |C \cap \Sigma| = 1 \}$

Def: A (M, Σ, w)-flow is $y \in \mathbb{R}_+^{\mathcal{C}_\Sigma}$ st.

Capacity constraints:

$$\forall e \in \bar{\Sigma} : \quad \sum (y_C : e \in C \in \mathcal{C}_\Sigma) \leq w_e \qquad (1)$$

Demand constraints:

$$\forall e \in \Sigma : \quad \sum (y_C : e \in C \in \mathcal{C}_\Sigma) = w_e \qquad (2)$$

Def: cut-condition: $\forall$ cocycle D:

$$\underbrace{w(D - \Sigma)}_{\text{capacity}} \geq \underbrace{w(D \cap \Sigma)}_{\text{demand}}$$

Signed minors:

(M, Σ) (signed) matroid.

(1) Resign : $(M, \Sigma) \longrightarrow (M, \Sigma \triangle D)$, D cocycle.

(2) Delete e: $(M, \Sigma) \setminus e = (M \setminus e, \Sigma - \{e\})$

(3) Contract $e \notin \Sigma$: $(M, \Sigma) / e = (M / e, \Sigma)$.

Sequence of (1), (2), (3) $\Longrightarrow$ signed minor

Notation: $(M, \Sigma) \geqslant (N, \Gamma)$

Flowing cn: [Seymour]

Let (M, Σ) be signed binary matroid.

cut-condition holds and

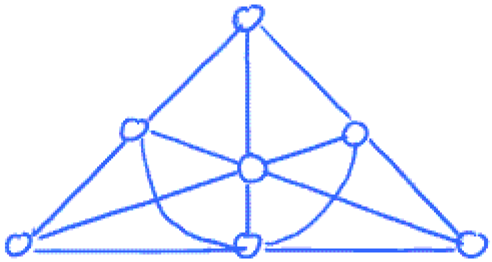
(M, Σ) has none of signed minors:

- (F_7, EF_7)
- $(M(K_5), EK_5)$
- (R_{10}, Σ_{10})

$\Rightarrow \exists\ (M, \Sigma, w)$-flow.

F_7 : representation $\begin{bmatrix} 1 & & & & 1 & 1 \\ & 1 & & 1 & & 1 \\ & & 1 & 1 & 1 & \end{bmatrix}$

Odd-circuits of (F_7, EF_7)



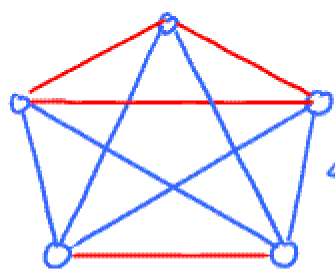
line of Fano

R_{10} : representation

$$\begin{array}{cc} & \begin{array}{cccccccccc} e & & f & & & & & & & g \end{array} \\ & \begin{bmatrix} 1 & & & & & 1 & 1 & & & 1 \\ & 1 & & & & & 1 & 1 & & 1 \\ & & 1 & & & & & 1 & 1 & 1 \\ & & & 1 & & 1 & & & 1 & 1 \\ & & & & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Σ_{10}

Odd circuits of (R_{10}, Σ_{10}) :



complements of cuts of K_5