Flows in graphs and matroids II

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Recap:

Given G, Σ, w:

Let $\mathcal{C}_\Sigma = \{C : \text{circuit of } G \text{ st } |C \cap \Sigma| = 1\}$

A (G, Σ, w)-flow is $y \in \mathbb{Z}_+^{\mathcal{C}_\Sigma}$ st

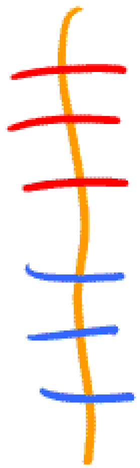
① For all $e \in \overline{\Sigma}$:

$$\sum (y_C : e \in C \in \mathcal{C}_\Sigma) \leq w_e$$

② For all $e \in \Sigma$:

$$\sum (y_C : e \in C \in \mathcal{C}_\Sigma) = w_e$$

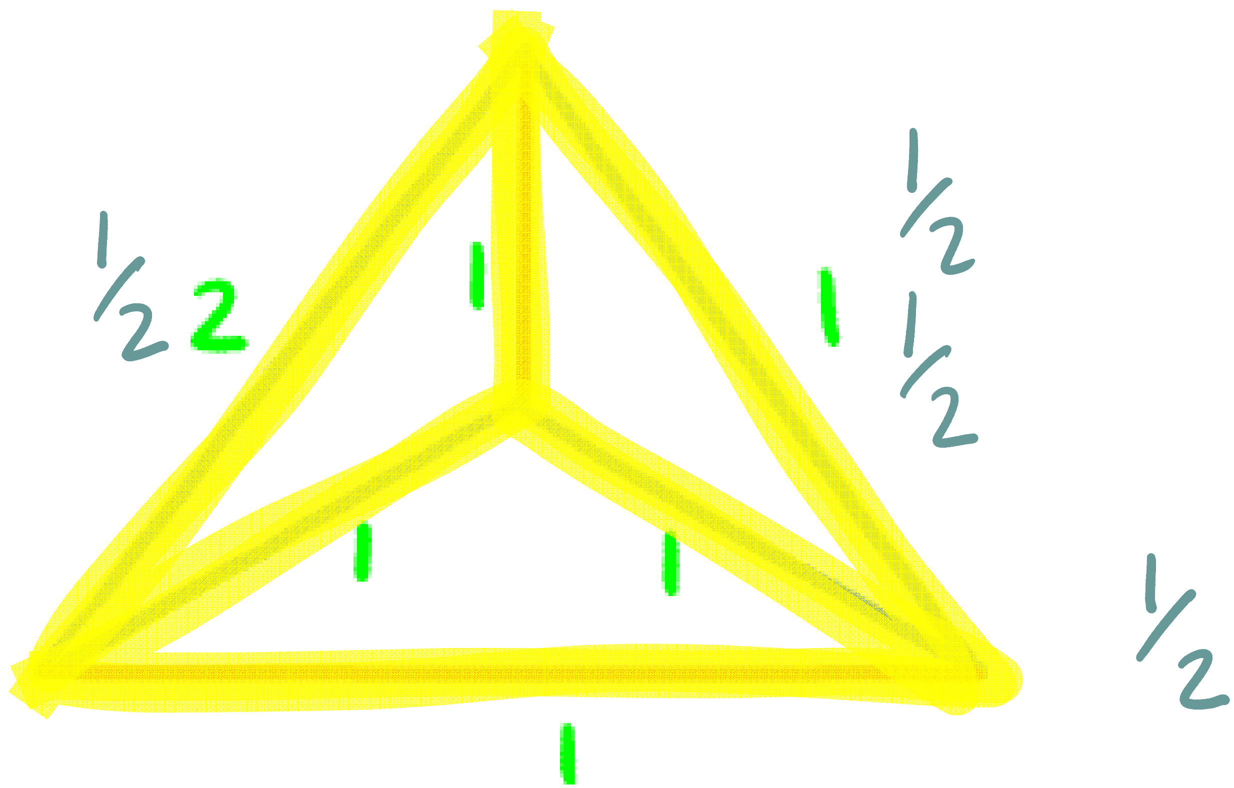
Cut condition:

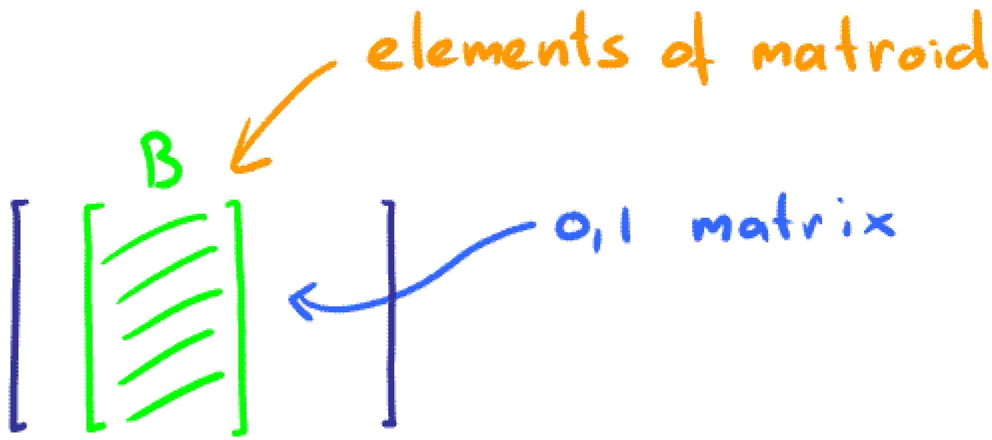


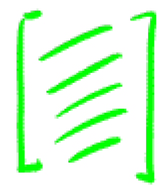
demand accross cut $\leq$ capacity accross cut

$w(D - \Sigma) \geq w(D \cap \Sigma)$ for all cuts D

Ex:





$B \subseteq E M$ cycle if all rows of  have even nb of ones

D is a cocycle if D intersects all cycles of M with even parity.

Th:

Let (G, Σ) signed graph

cut-condition
no (K_4, EK_4)-minor $\Big\} \Rightarrow$

$\exists$ integer flow

Th: [Seymour]

binary

Let (M, Σ) signed matroid

cut-condition
no $(M(K_4), EK_4)$-minor $\Big\} \Rightarrow$

$\exists$ integer flow

Th: [Guenin]

Let (G, Σ) signed graph

$$\left.\begin{array}{l}\text{cut condition} \\ \text{no } (K_5, EK_5)\text{-minor}\end{array}\right\} \Rightarrow \exists \text{ a flow.}$$

Flowing conjecture [Seymour]

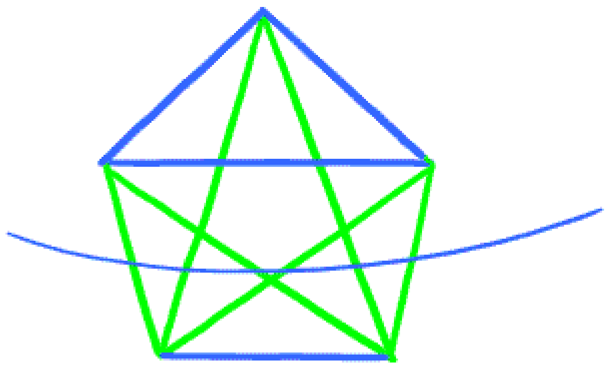
Let (M, Σ) signed matroid

$$\left.\begin{array}{l}\text{cut condition} \\ \text{no minor} \\ \quad (1)\ (M(K_5), EK_5) \\ \quad (2)\ (R_{10}, \Sigma_{10}) \\ \quad (3)\ (F_7, EF_7)\end{array}\right\} \Rightarrow$$

$\exists$ a flow.

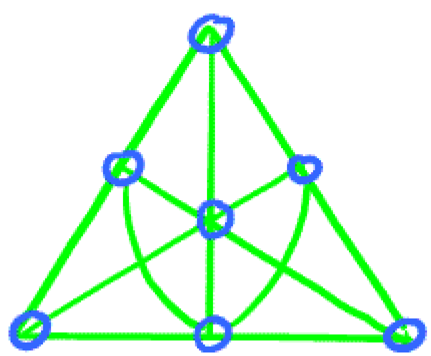
(2) odd circuits =

complements of cuts of K_5



(3) odd circuits =

lines of Fano matroid



Th: [Geelen, Guenin]

Let (G, Σ) signed graph.

$$\left.\begin{array}{l}\text{cut-condition} \\ w\text{-eulerian} \\ \text{no } (M(K_5), EK_5)\text{-minor}\end{array}\right\} \Rightarrow$$

$\exists$ an integer flow

Def: $w \in \mathbb{Z}_+^{EM}$ is eulerian if

$\forall$ cocycle $w(D)$: even

Cycling conjecture (Seymour):

Let (M, Σ) signed matroid

cut-condition

w-eulerian

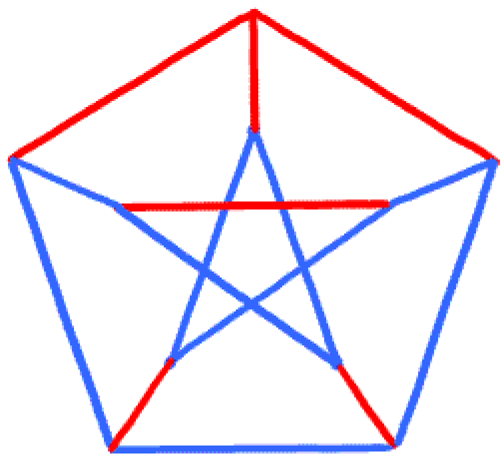
no minors (1), (2), (3) (4): (T_{15}, Σ_{15})

$\Rightarrow$

$\exists$ integer flow.

(4) odd circuits =

postman sets of Petersen graph



sets of edges B st all vertices are odd in graph induced by B

Clutters: (minimax relations)

Def:

A family H of sets with groundset EH is a **clutter** if no set is included in another.

Let $w \in \mathbb{Z}_+^{EH}$

$$\min \; w^T x$$
st
$$x(C) \geq 1 \quad (C \in H) \qquad (P)$$
$$x \geq 0$$

$$\max \; \underline{1}^T y$$
st
$$\sum (y_C : e \in C \in H) \leq w_e \quad (e \in EH) \qquad (D)$$
$$y \geq 0$$

$\tau(H,w)$ = value opt. integer sol. to (P)
$\tau^*(H,w)$ = " " fractional " " "

$\nu(H,w)$ = value opt. integer sol to (D)
$\nu^*(H,w)$ = " " fractional " " ".

$$\tau(H,w) \geq$$

$$\tau^*(H,w) =$$ LP duality

$$\nu^*(H,w) \geq$$

$$\nu(H,w)$$

Let H: clutter of odd circuits of (M, Σ).

$$\left.\begin{array}{l}\text{cut-condition} \\ \tau(H,w) = \nu(H,w)\end{array}\right\} \Longrightarrow \exists \text{ an integer flow}$$

$$\left.\begin{array}{l}\text{cut-condition} \\ \tau(H,w) = \nu^*(H,w\end{array}\right\} \Longrightarrow \exists \text{ a flow}$$

$$\left.\begin{array}{l}\text{cut-condition}\\ \tau(H,w) = \nu(H,w)\end{array}\right\} \Rightarrow \exists \text{ an integer flow}$$

$$\begin{array}{ll} \min & w^T x \\ \text{st} & \\ & x(C) \geq 1 \quad (C \in H) \\ & x \geq 0 \end{array} \qquad (P)$$

$$\begin{array}{ll} \max & \underline{1}^T y \\ \text{st} & \\ & \sum (y_C : e \in C \in H) \leq w_e \quad (e \in EH) \\ & y \geq 0 \end{array} \qquad (D)$$

Def clutter H is ideal if $\forall w \in \mathbb{Z}_+^{EH}$

$$\tau(H,w) = \nu^*(H,w)$$

$$\Updownarrow$$

$$\{x \geq 0 : x(C) \geq 1, C \in H\} \text{ integral}$$

Def: minor of clutter H

$$H \setminus e = \{S \in H : e \notin S\}$$

$$H / e = \text{minimal sets in } \{S - \{e\} : S \in H\}$$

Rem: H ideal $\Rightarrow$ minors of H ideal

Th: [Guenin]

Let (G, Σ) signed graph

$\left.\begin{array}{l} \text{cut condition} \\ \text{no } (K_5, EK_5)\text{-minor} \end{array}\right\} \Longrightarrow \exists$ a flow.

equivalent to

Let H clutter of odd circuits of (G, Σ).

H ideal $\Longleftrightarrow$

clutter of odd circuits of K_5 not a minor.

Binary clutters:

A clutter H is **binary** if

$\forall S_1, S_2, S_3 \in H, \quad S_1 \triangle S_2 \triangle S_3 \supseteq S \in H$

Ex:

- odd circuits of (G, Σ)
- st-paths

Generalizes to binary matroids M:

① odd circuits of (M, G)

② **port(M, e)** :=

$\{C - \{e\} : C \text{ circuit of } M \text{ using } e\}$

All binary clutters can be viewed as ① or ②

Flowing conjecture

Let (M,Σ) signed matroid

cut condition

no minor ①, ②, ③ $\Rightarrow$ $\exists$ a flow.

equivalent to

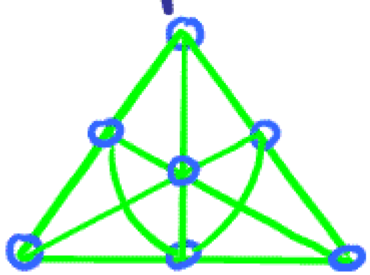
Let H be binary clutter.

H ideal $\Longleftrightarrow$

no minor :

① odd circuits of K_5

② complements of cuts of K_5

③  line of Fano

The **blocker** bH of H is the clutter of minimal sets in $\{T : T \cap S \neq \emptyset\}$

Ex: H : st-paths, bH : st-cuts

Spse H is clutter of odd circuits of (G, Σ)

$\Rightarrow$ all sets of bH of the form: $\Sigma \triangle \delta(U)$.

Let $T_1 = \Sigma \triangle \delta(U_1) \in bH$

$T_2 = \Sigma \triangle \delta(U_2) \in bH$

$\Rightarrow T_1 \triangle T_2 = \delta(U_1) \triangle \delta(U_2)$: cut

In general:

binary matroid

Spse H is clutter of odd circuits of (M, Σ)

Let $T_1, T_2 \in bH$

$\Rightarrow T_1 \triangle T_2$: cocycle.

Def: Let H be a binary clutter and $w \in \mathbb{Z}_+^{E(H)}$. Then w is eulerian if

$$\forall T_1, T_2 \in bH, \quad w(T_1 \triangle T_2) \text{ even.}$$

Def: H cycling if

$$\forall \text{eulerian } w: \ \tau(H,w) = \nu(H,w)$$

Cycling conjecture :

Let (M, Σ) signed matroid

$$\left.\begin{array}{l}\text{cut-condition}\\ \text{w-eulerian}\\ \text{no minors } (1), (2), (3)\ (4)\end{array}\right\} \Rightarrow \exists \text{ integer flow.}$$

equivalent to

Let H be binary clutter.

H cycling $\Longleftrightarrow$ no minor :

1. odd circuits of K_5
2. complements of cuts of K_5
3. line of Fano
4. postman set of Petersen

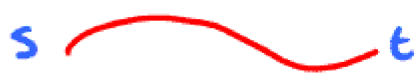
Known special cases of flowing conj:

1. Odd circuits of graphs
2. T-cuts [Edmonds]
3. blockers of 1, 2

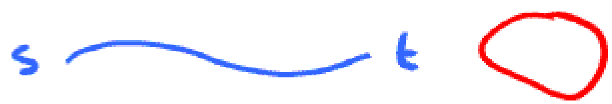
Th: [Lehman] H ideal $\Longleftrightarrow$ bH ideal

Generalizations:

1. Odd st-walks [Guenin]

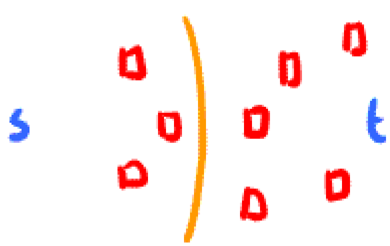


odd st-path <u>or</u>



even st-path + odd circuit

2. st-T-cuts



Cycling conjecture:

Let H odd circuits of (G, Σ)
$\Rightarrow$ bH sets of the form: $\Sigma \triangle \delta(U)$.

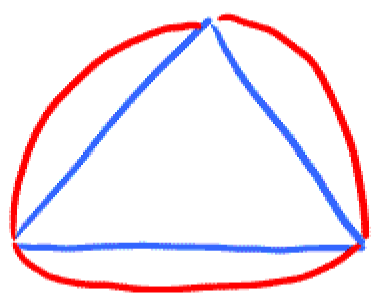
Restrict conj. to bH:

Then w eulerian if:

$w(C_1 \triangle C_2)$ even

$\forall$ odd circuits C_1, C_2

consistent $(w = 1)$



not consistent

$\tau(bH, 1) =$ odd girth of (G, Σ)

$\nu(bH, 1) =$ max nb of disjoint (odd circuit) covers.

Consistency conj:

Spse (G, Σ) consistent.
If $(G, \Sigma) \not\succeq \tilde{K}_5$ then

odd girth = max # of disjoint covers

Special cases:

Th: [Guenin]

If $\tilde{G} \not> \tilde{K}_5$ then G can be 4-coloured.

Def: r-graph is an r-regular graph G st $\forall U \subseteq VG$, $|U|$ odd, $|\delta(U)|$

Cn: [Seymour]

Planar r-graphs have chromatic index r.

Special cases:

$r = 3$: 4-colour th.

$r = 4, 5$: ✓ [Guenin]