Flows in graphs and matroids III

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We will prove :

Th: [Guenin]

Let (G, Σ) signed graph

$$\left.\begin{array}{l}\text{cut condition}\\ \text{no } \tilde{K}_5\text{-minor}\end{array}\right\} \Rightarrow \exists \text{ a flow.}$$

Short proof of Schrijver.

Recap:

Clutter H is ideal if $\forall w \in \mathbb{Z}_+^{EH}$

$$\begin{aligned} \min \quad & \sum (w_e x_e : e \in EH) \\ \text{st} \quad & x(C) \geq 1 \qquad (C \in H) \\ & x \geq 0 \quad \text{integer} \end{aligned}$$

$=$

$$\begin{aligned} \max \quad & \sum (y_C : C \in H) \\ \text{st} \quad & \sum (y_C : e \in C \in H) \leq w_e \quad (e \in EH) \\ & y \geq 0 \end{aligned}$$

$\Updownarrow$ Edmonds, Giles

The polyhedron:

$$x(C) \geq 1 \quad \forall C \in H$$
$$x \geq 0$$

is integral.

Th: [Guenin]

Let (G, Σ) signed graph

cut condition
no $\tilde{K}_5$-minor
$\Big\} \Rightarrow \exists$ a flow.

equivalent to

Let H clutter of odd circuits of (G, Σ)

H NOT ideal $\Rightarrow \exists\ \tilde{K}_5$ minor

Lehman's theorem:

Clutters as matrices:

$$
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\left[\begin{array}{cccccccccc}
1 & 1 & 1 & & & & & & & \\
1 & & & 1 & 1 & & & & & \\
1 & & & & & 1 & 1 & & & \\
 & 1 & & 1 & & & & 1 & & \\
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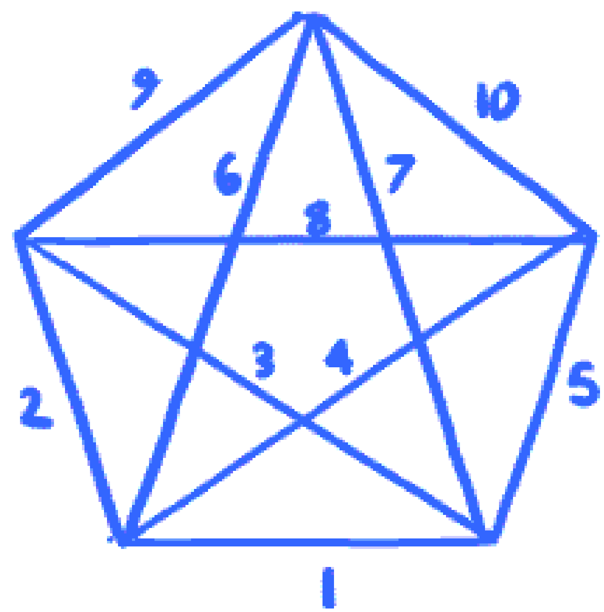
1 & 1 & & & 1 & & & & 1 & 1 \\
 & & 1 & 1 & & 1 & 1 & 1 & & \\
\ldots & & & & & & & & &
\end{array}\right]
\end{array}
$$



Let A 0,1 matrix.

$$Q(A) := \{ x \geq 0 \mid Ax \geq 1 \}$$

Minor of A :

$Q(A) \cap \{x \mid x_j = 0\}$ ← face

$Q(A) \cap \{x \mid x_j = 1\}$ ← projection

Thus A ideal ⟹ minor of A ideal.

Def: A is minimaly non ideal (mni) if :

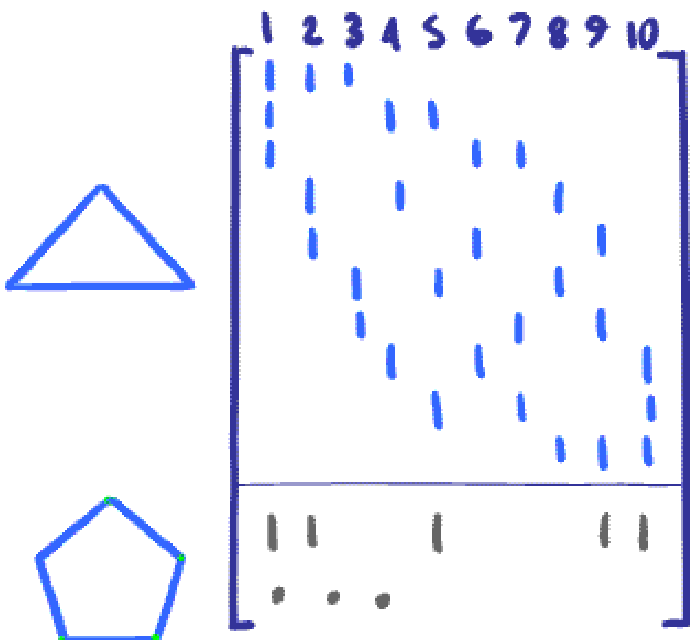
- A not ideal
- all minors of A ideal

A mni $\Longrightarrow$

$\exists$ extreme point $\hat{x}$ of $Q(A)$ and

$$0 < x_j < 1 \quad \forall j$$

Ex:



$$\hat{x} = \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \cdots\ \tfrac{1}{3}\right)$$

Let A be mni.

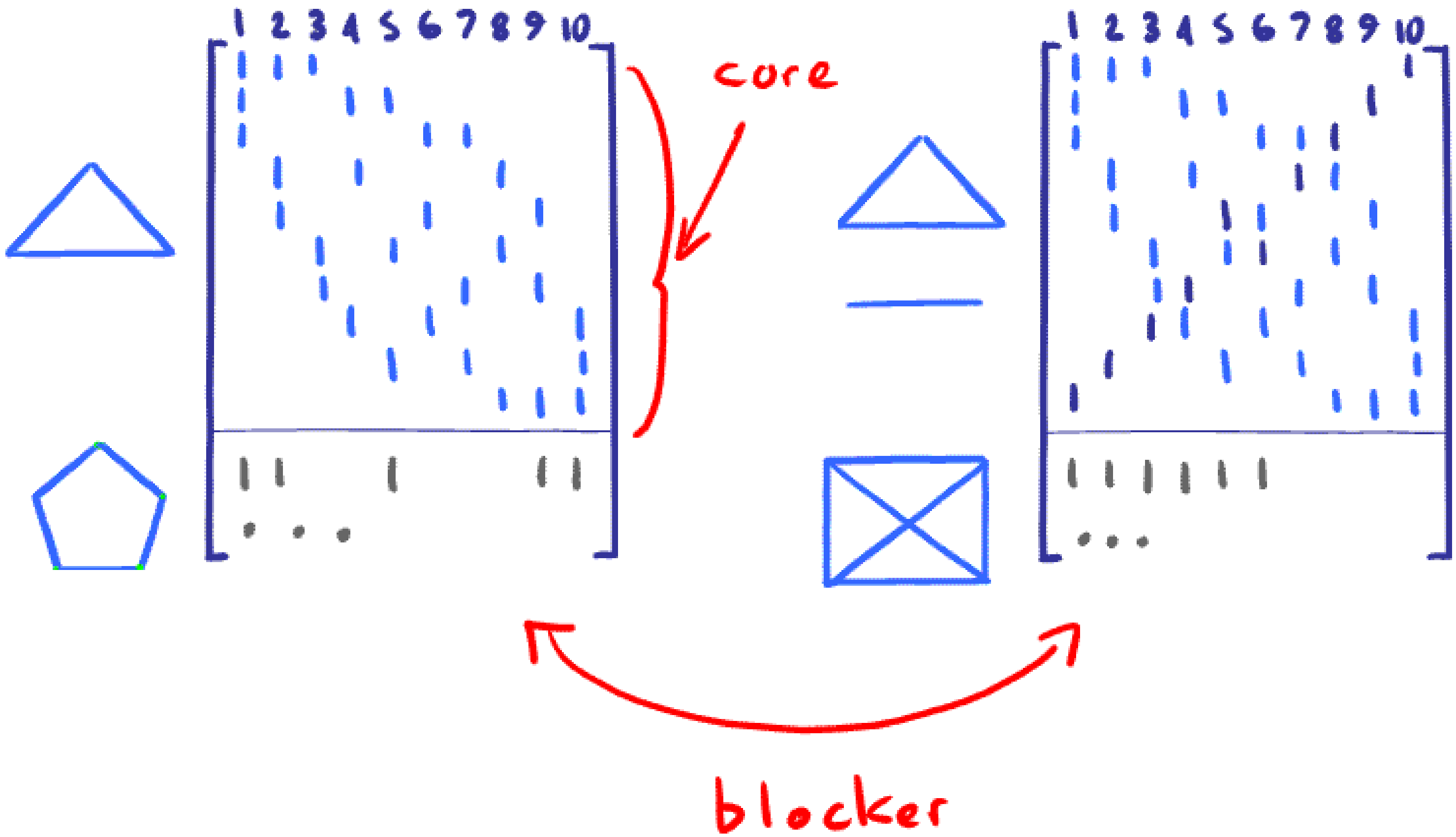
Def: A core of A is a maximal row submatrix $\bar{A}$ of A st for some fractional point $\hat{x}$:

$$\bar{A}\hat{x} = 1$$

Recall:

H clutter, the blocker

$$bH = \text{minimal sets in } \{T \mid S \cap T \neq \emptyset \ \forall S \in H\}$$


1 2 3 4 5 6 7 8 9 10
core
1 2 3 4 5 6 7 8 9 10
blocker

Th [Lehman]

blocker

Let A be mni, $B = bA$

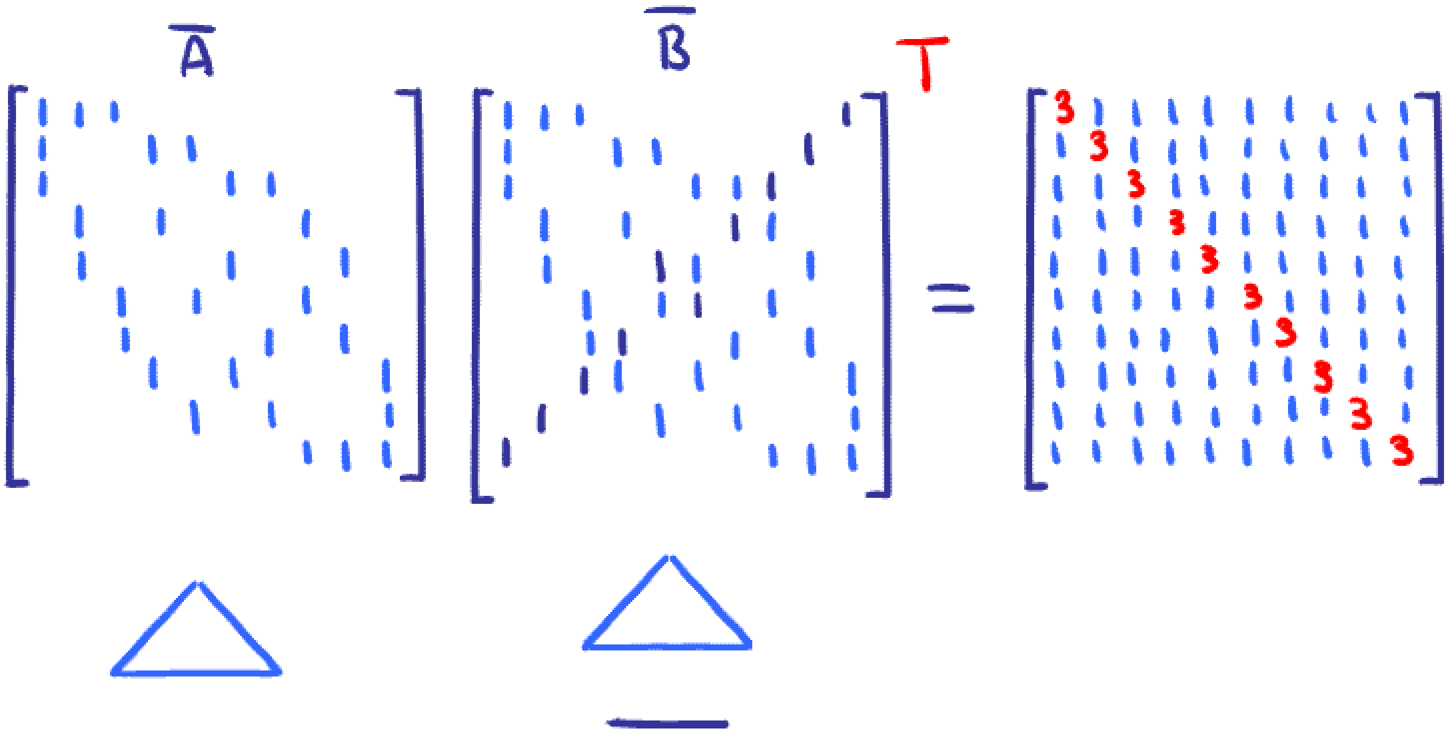
(1) A has unique core $\bar{A}$ which is square.
B " " " $\bar{B}$ " " " .

(2) Either:

not odd circuit of graph

$$A = \begin{bmatrix} 0 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & & & & \\ 1 & & 1 & & 0 & \\ \vdots & & & 1 & & \\ \vdots & & 0 & & \ddots & \\ 1 & & & & & 1 \end{bmatrix}$$

or $\bar{A}\,\bar{B}^T = E + dI \quad (d \geq 1)$


$\bar{A}$
$\bar{B}^T$
=
3 3 3 3 3 3 3 3 3 3

Th: [Bridges & Ryser]

Let $\bar{A}, \bar{B}$ be square 0,1 matrices st

$$\bar{A}\bar{B}^T = E + dI \quad (d \geq 1)$$

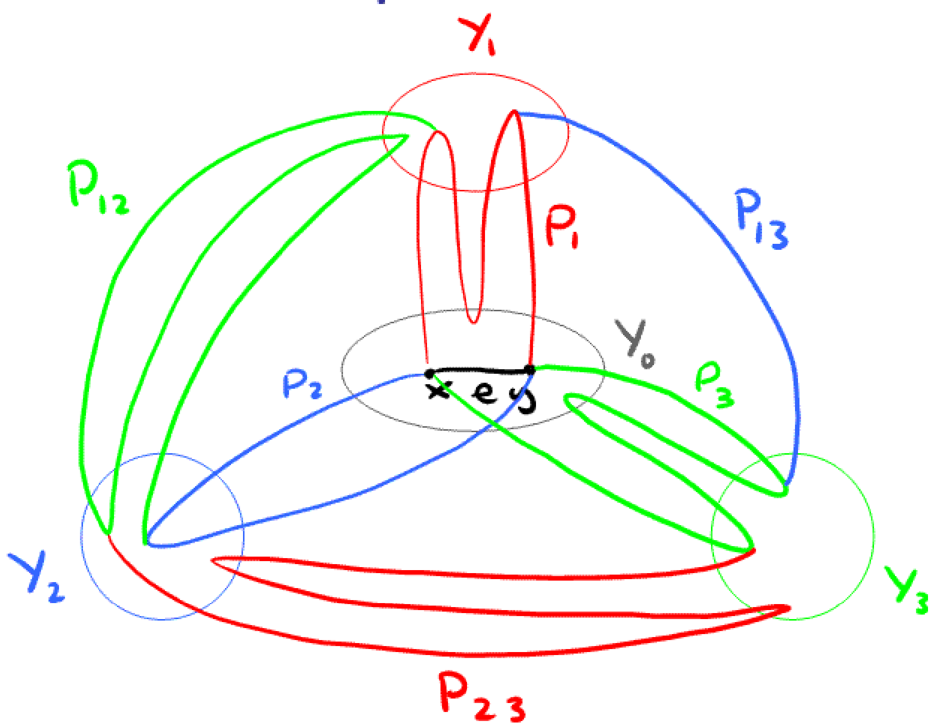
(1) A is r-regular
B is s-regular

(2) $AB^T = B^T A$

(1) $\Rightarrow$ rows in A not in $\bar{A}$ have $> r$ ones.

K_5-lemma: (Schrijver)

If $\tilde{G}$ has following properties $\tilde{G} \geq \tilde{K}_5$



(1) $Y_0 - \{e\}, Y_1, Y_2, Y_3$: stable sets

(2) $\forall i$, $\exists$ xy-path P_i in $Y_0 \cup Y_i$

(3) P_1, P_2, P_3 intersect in x, y only.

(4) $\forall v_i \in Y_i$, $v_j \in Y_j$ $(i \neq j)$

$\exists$ $v_i v_j$-path P_{ij} in $Y_i \cup Y_j$

Proof:

$\mathcal{A}$: clutter of odd circuits of (G, Σ)

$\mathcal{B}$: blocker of $\mathcal{A}$.

To show: $\mathcal{A}$ mni $\Rightarrow (G, \Sigma) \succcurlyeq \widetilde{K}_5$

Sets in $\mathcal{B}$ of the form: $\underbrace{\Sigma \triangle \delta(U)}_{\text{signature}}$

- $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \triangle B_2$: cut
- $B_1, B_2, B_3 \in \mathcal{B} \Rightarrow B_1 \triangle B_2 \triangle B_3$: signature

$\bar{A}$ core of A
$\bar{B}$ core of B

Lehman's th $\Rightarrow$ $\bar{A}\bar{B}^T = E + dI$

$$\bar{A} = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}$$

$$\Rightarrow \quad |C_i \cap B_j| = \begin{cases} \geq 3 & \text{if } i = j \\ = 1 & \text{otherwise} \end{cases}$$

Pairs C_i, B_i are mates.

Claim 1:

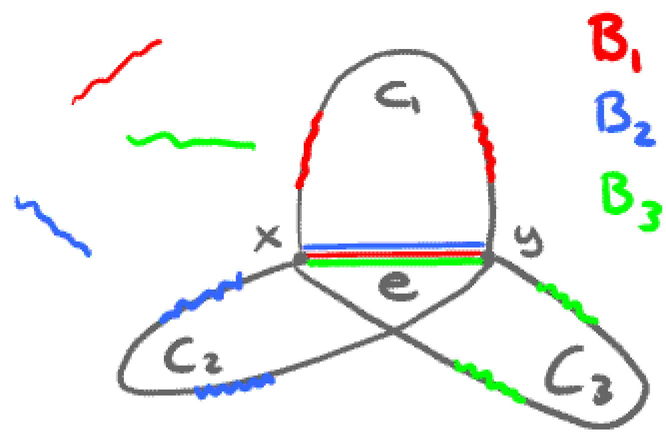
(1) $C \subseteq C_1 \cup C_2$ and $C \in \mathcal{A} \Rightarrow C = C_1$ or $C = C_2$.

(2) $B \subseteq B_1 \cup B_2$ " $B \in \mathcal{B} \Rightarrow B = B_1$ or $B = B_2$

Pick arbitrary edge $e = xy$.

Claim 2: We may assume:

$$C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = \{e\}$$
$$B_1 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_3 = \{e\}$$



Claim 3: C_1, C_2, C_3 only share vertices x, y.

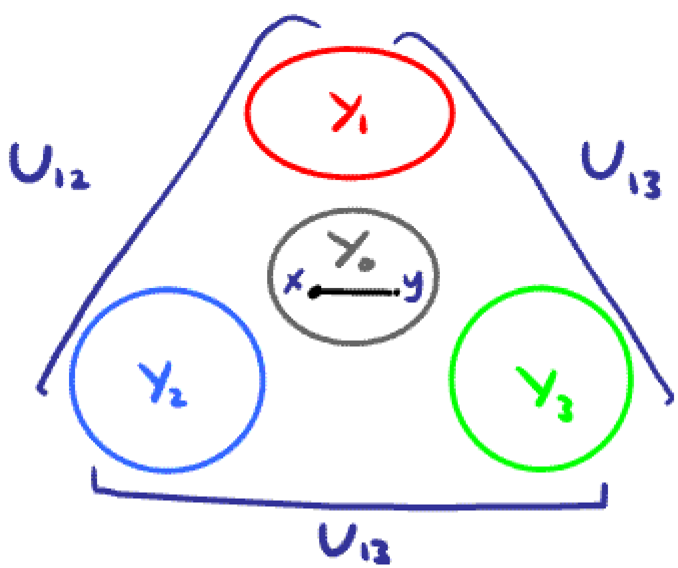
cut

$$B_1 \triangle B_2 = \delta(U_{12})$$
$$B_1 \triangle B_3 = \delta(U_{13})$$
$$B_2 \triangle B_3 = \delta(U_{23})$$
$$(x, y \notin U_{12} \cup U_{13} \cup U_{23})$$

$$Y_1 = U_{12} \cap U_{13}$$
$$Y_2 = U_{12} \cap U_{23}$$
$$Y_3 = U_{13} \cap U_{23}$$
$$Y_0 = VG - Y_1 - Y_2 - Y_3$$

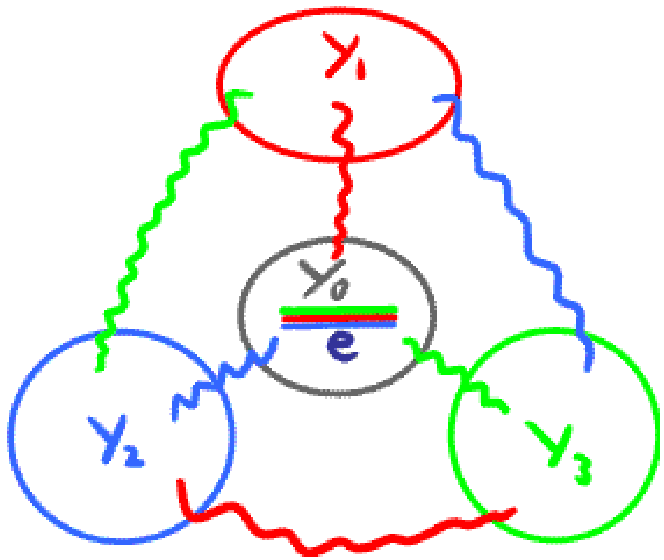
Claim 4:

$$U_{12} = Y_1 \cup Y_2$$
$$U_{13} = Y_1 \cup Y_3$$
$$U_{23} = Y_2 \cup Y_3$$



Claim 5:

Edges B_1, B_2 B_3 in following locations only



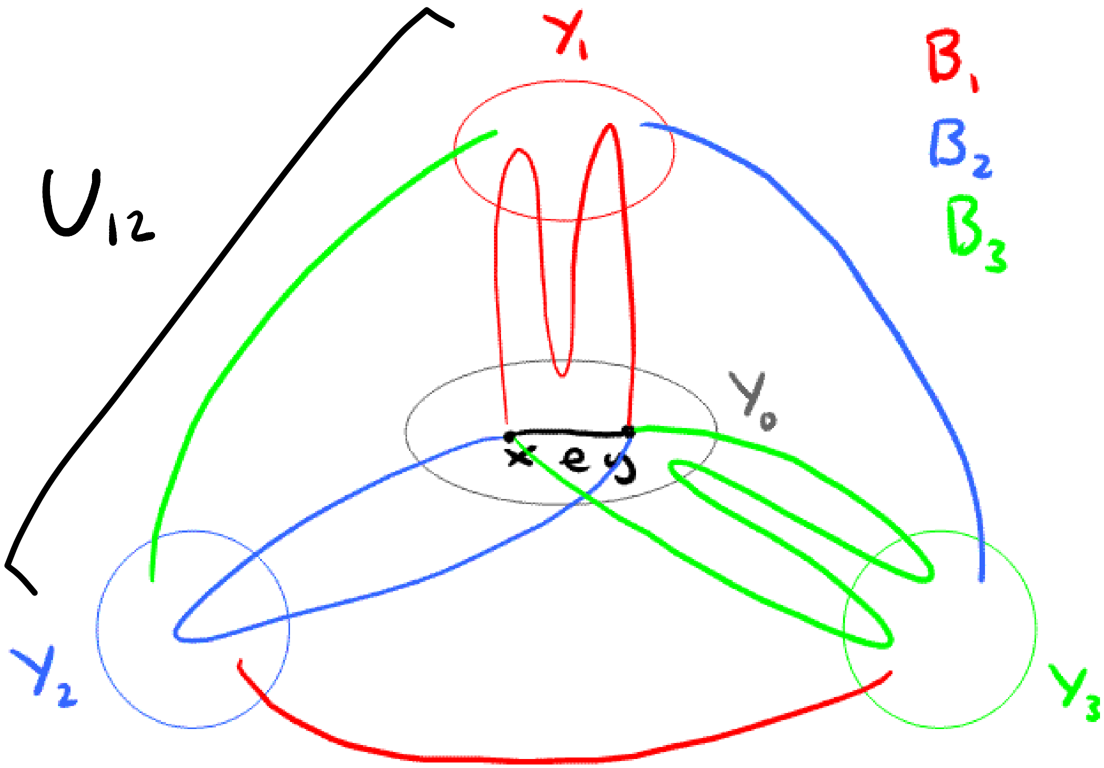
no other edges betwee Y_i's

$B_1 \triangle B_2 \triangle B_3$ signature

Contract all edges $\notin B_1 \cup B_2 \cup B_3$

$\Longrightarrow$ $Y_0 - \{e\}, Y_1, Y_2, Y_3$ disjoint stable set

$\Longrightarrow$ all edges odd



Claim 6: $G[U_{12}]$, $G[U_{13}]$, $G[U_{23}]$ connected

K_5-lemma $\Rightarrow (G, EG) \geq \tilde{K}_5$!