Flows in graphs and matroids IV

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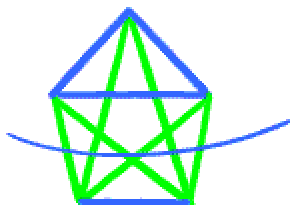
For a signed matroid (M, Σ) when is the cut-condition sufficient for existence of flow?

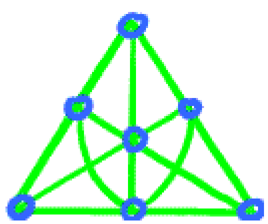
<u>Flowing conj:</u> [Seymour]

If we have none of the signed minors

- $(M(K_5), EK_5)$
- (R_{10}, Σ_{10})



- (F_7, EF_7)



Hard !!

Are single commodity flows easier? No !

cut condition always sufficient for (M, Σ)

$$\Updownarrow$$

odd circuits of (M, Σ) $\rightsquigarrow$ $\begin{bmatrix} \; A \; \end{bmatrix}$ ideal

$$\Updownarrow$$

odd circuits of $(N, \{\Omega\})$ $\rightsquigarrow$ $\begin{array}{c} \begin{array}{cc} \Omega & \end{array} \\ \left[\begin{array}{c|c} \vdots & A \end{array}\right] \end{array}$ ideal

$$\Updownarrow$$

cut condition always sufficient for $(N, \{\Omega\})$

Def: M is 1-flowing if $\forall\, \Omega \in EM$:
cut-condition sufficient for flow in $(M, \{\Omega\})$

Flowing conjecture $\Rightarrow$

1-flowing conj:

M is 1-flowing $\Leftrightarrow$
none of following minors:

(1) $AG(3,2)$
(2) T_{11}
(3) T_{11}^{*}

Conjecture holds for: [Guenin]

(1) even cycle matroids ⇜ odd st-walk
(2) even cut matroids ⇜ st-T-cuts
(3) dual of (1)
(4) dual of (2)

Working assumption:

Basic 1-flowing matroids are of type (1) – (4).

What is "outside" this class.

Generic problem:

Let $\mathcal{C}$ be a class of minor closed matroids.

Def: M is is minimaly not in $\mathcal{C}$ if

- $M \notin \mathcal{C}$
- all minors of M are in $\mathcal{C}$.

Characterize all 3c matroid minimaly not in $\mathcal{C}$.

We would like to answer question where

$\mathcal{C}$ = set of matroids in ①-④
with no $AG(3,2)$, T_{11}, T_{11}^* minor.

Today:

$\mathcal{C}$ = even cycle matroids
with no $AG(3,2)$ minor.

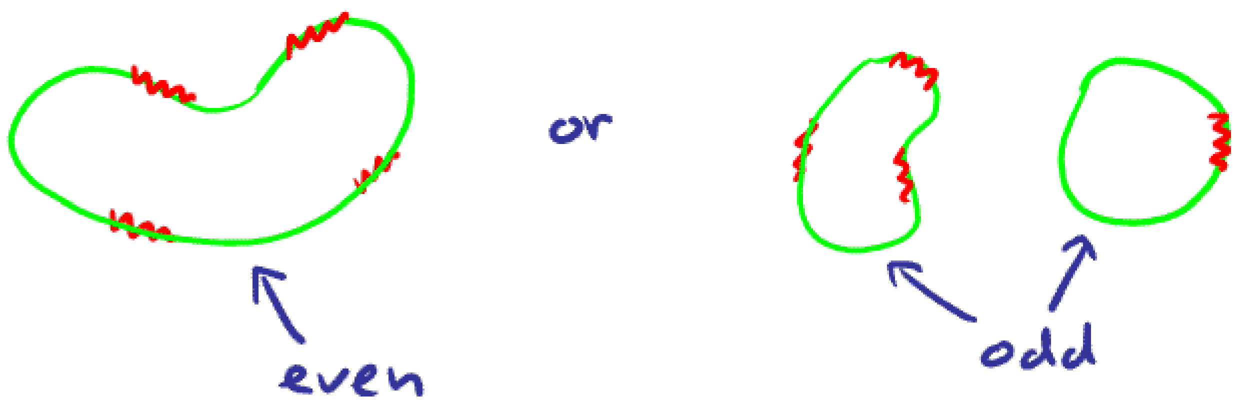
Even cycle matroids:

Representation of $\text{ecycle}(G, \Sigma)$:

$$\left[\begin{array}{c} \text{cuts of } G \\ \hline \Sigma \end{array}\right]$$

$\Rightarrow$ cycles of $\text{ecycle}(G, \Sigma)$ = even cycles of (G, Σ).

$\Rightarrow$ circuits of $\text{ecycle}(G, \Sigma)$:



Representations:

Def: (G, Σ) is a representation of even cycle matroid M if $M = \text{ecycle}(G, \Sigma)$.

Difficulty:

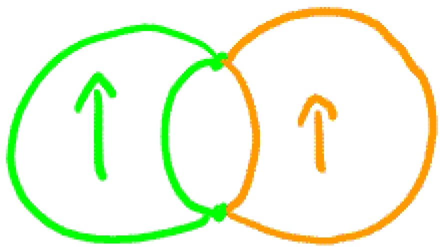
Same even cycle matroid can have different representations

Resigning:

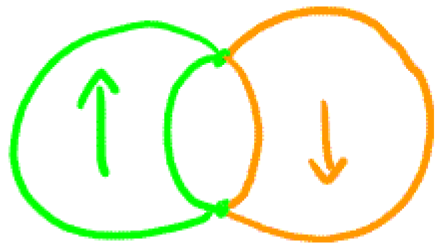
signature

(G, Σ) and $(G, \Sigma \,\Delta\, \delta(v))$

Whitney flips:

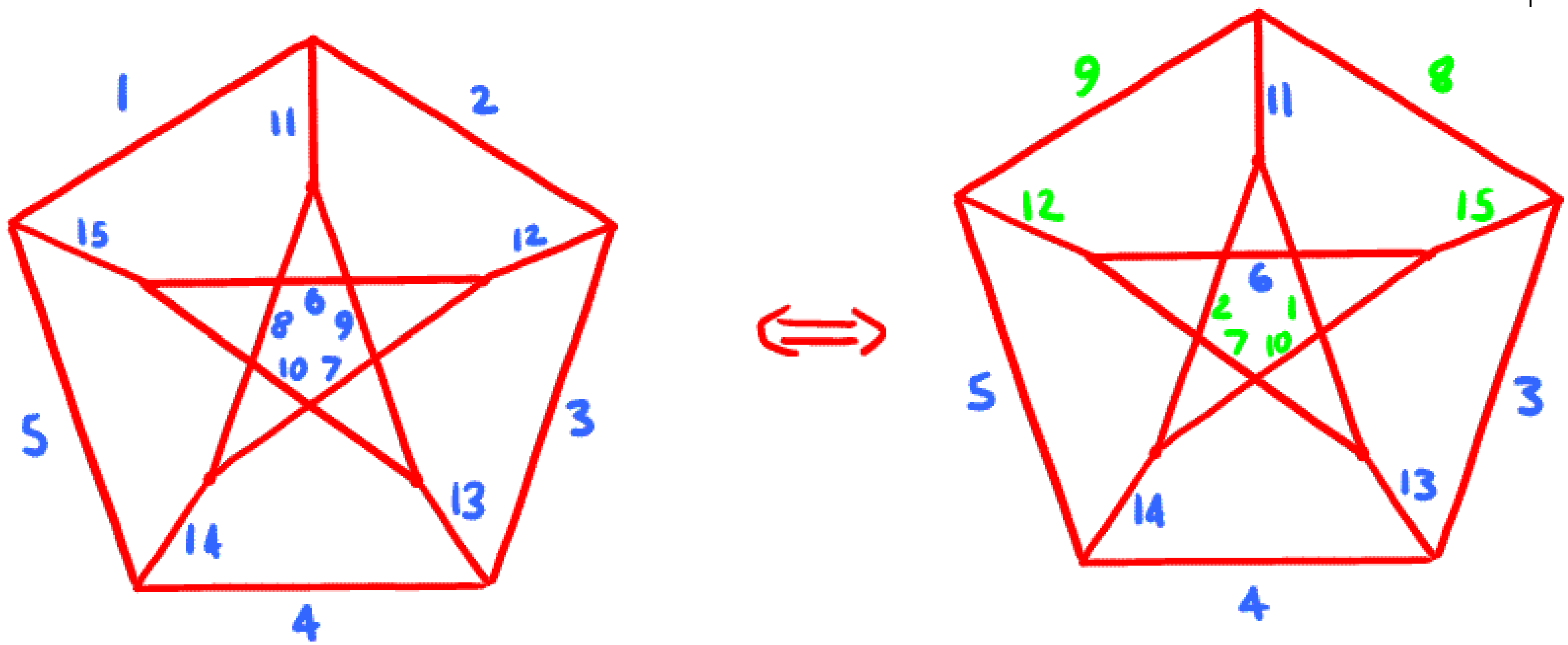


and

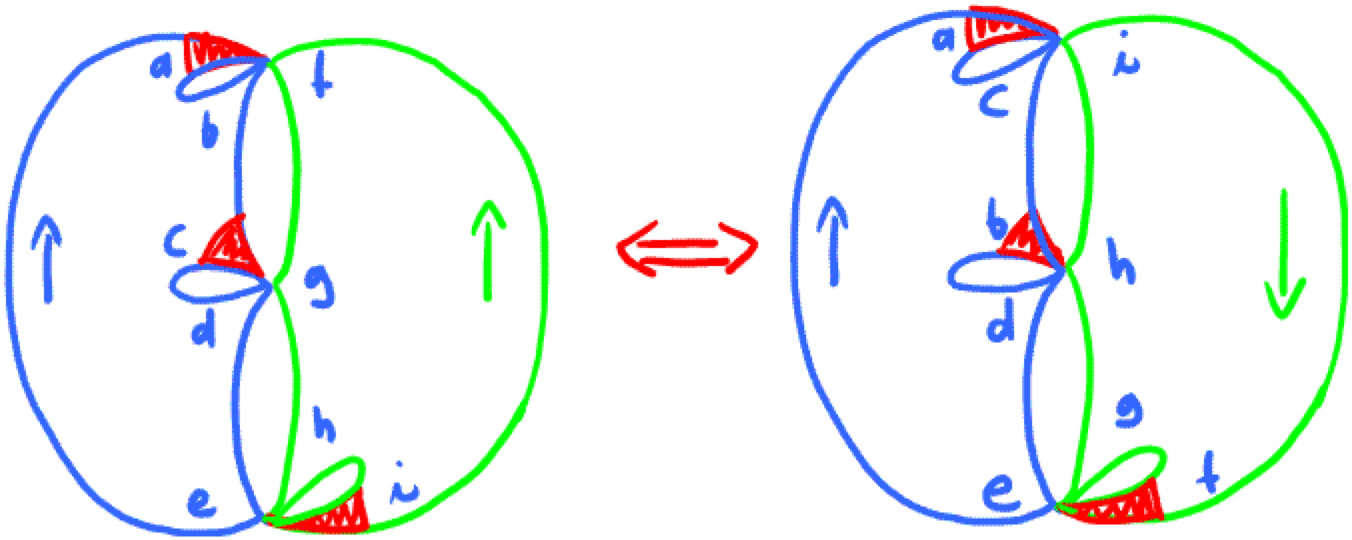


Def: (H, Γ) and (G, Σ) are equivalent if one can be obtained from the other by a sequence of Whitney-flips and resigning

Non-equivalent representations:



[Sergei Norine, Robin Thomas]



Finding all representations of a given even cycle matroid is hard.

Sol: Stabilizer !

$$
\begin{tikzcd}
M = \operatorname{ecycle}(G,\Sigma) \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, "\text{matroid minor}"'] & (G,\Sigma) \arrow[d, Rightarrow, "\text{signed graph minor}"] \\
N = \operatorname{ecycle}(H,\Gamma) \arrow[r, Leftrightarrow] & (H,\Gamma)
\end{tikzcd}
$$

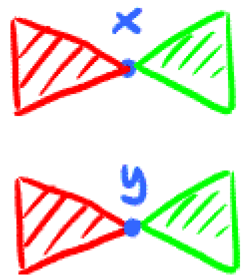
Def: (G,Σ) **extends** representation (H,Γ) of N to M

Does every representation extend **uniquely**?

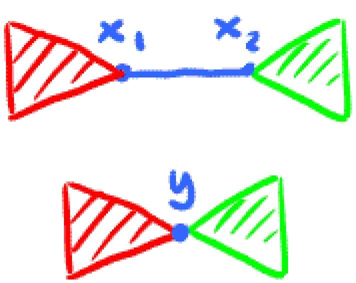
No!

Def: x, y is a **blocking pair** of (G, Σ) if no odd circuit avoids both x, y.

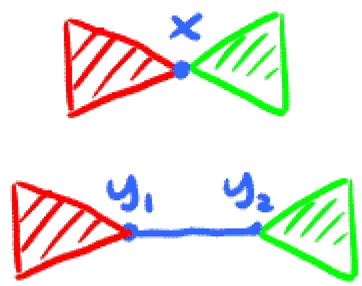
(H, r)



$N = \text{ecycle}(H, r)$



(G_1, r)

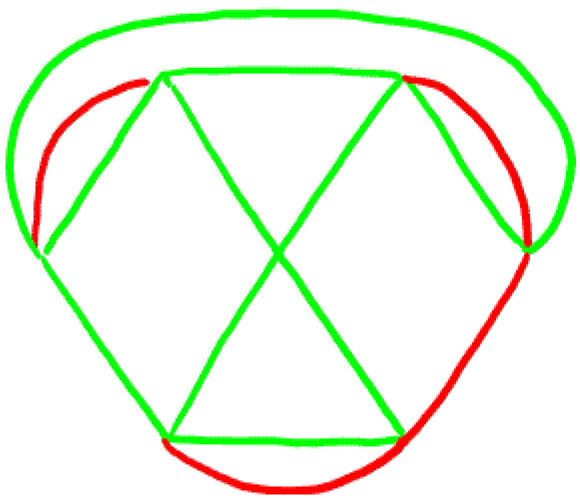


(G_2, r)

Then $\text{ecycle}(G_1, r) = \text{ecycle}(G_2, r)$

Def: an even cycle matroid is **substantial** if it is 3C and none of its representation has a blocking pair

Ex:



$\Rightarrow T_{12}$

Recall:

$\mathcal{C}$ = even cycle matroids without $AG(3,2)$ minor.

Conjecture:

$\exists$ a finite nb of 3c matroids minimaly not in $\mathcal{C}$

Step 1:

Let N be any fixed substantial matroid.

$\exists$ a finite nb of 3c matroids minimaly not in $\mathcal{C}$ that contain N as minor.

$\Rightarrow$ wma matroid do not contain any fixed substantial matroid.

$\Rightarrow$ thin class

Step 2:

Characterize thin class ??

"No blocking pair $\Rightarrow$ unique extension"

Stabilizer th: [Guenin, Pivotto, Wollan]

Let $N = \text{ecycle}(H, \Gamma)$ and $M \geq N$ where $AG(3,2) \not\leq M$.
Spse (H, Γ) has no blocking pair.
Then there is at most one representation (G, Σ) which extends (H, Γ) to M (up to equivalence)

Let N be a substantial matroid.

Let $M \geq N$ where M 3c is minimally not in $\mathcal{C}$.

To show: size$(M) \leq f($size$(N))$.

Since N, M 3c, $\exists$ sequence

$N_1 = N, N_2, N_3 \ldots \quad N_k = M$ such that

- N_i 3c
- N_i obtained from N_{i-1} by
  - uncontracting or
  - undeleting

Let $(H_1, r_1), \ldots, (H_t, r_t)$ representations of N.

Focus on one representation at the time.

Need :

"representation dies $\Rightarrow$ dies quicly"

Escape theorem: [Guenin, Pivotto, Wolan]

Let (H, r) representation of N 3c with no blocking pair. H 3c ?

If (H, r) does not extend to $M \succeq N$ where M 3c then $\exists N'$ 3c such that:

(1) $N \preceq N' \preceq M$

(2) $\mathrm{size}(N') \leq f(\mathrm{size}(N))$

(3) (H, r) does not extend to N'.

Stabilizer + escape th. without 3c assumption

$\Rightarrow$

Let N be any fixed substantial matroid.

$\exists$ a finite nb of 3c matroids minimaly not in $\mathcal{E}$ that contain N as minor.

Some ideas in proof of escape theorem

Undeleting:

$$
\begin{array}{c}
\\ a \\ b \\ c \\ d \\ e \\ f \\ \Sigma
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\left[\begin{array}{cccccccccccc}
 & & & & 1 & 1 & & & 1 & & & 1 \\
 & & & 1 & 1 & & & 1 & & & & 1 \\
 & & 1 & 1 & & & 1 & & & 1 & & \\
 & 1 & 1 & & & & & & 1 & 1 & & \\
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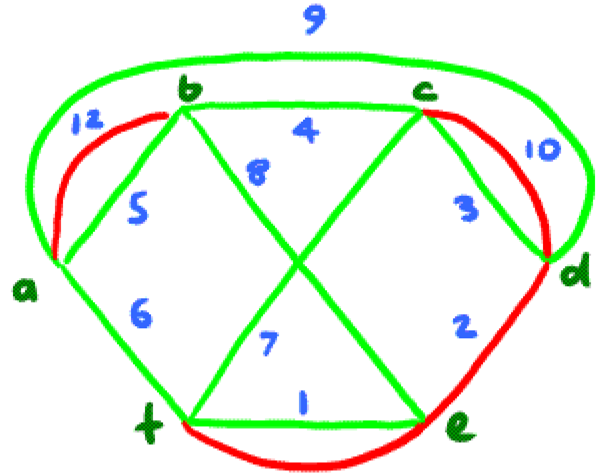
\end{array}\right]
\end{array}
$$



$\Downarrow$ undeleting

$$
\begin{array}{c}
\\ a \\ b \\ c \\ d \\ e \\ f \\ \Sigma
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \Omega \end{array} \\
\left[\begin{array}{ccccccccccccc}
 & & & & 1 & 1 & & & 1 & & & 1 & 1 \\
 & & & 1 & 1 & & & 1 & & & & 1 & 1 \\
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\end{array}\right]
\end{array}
$$

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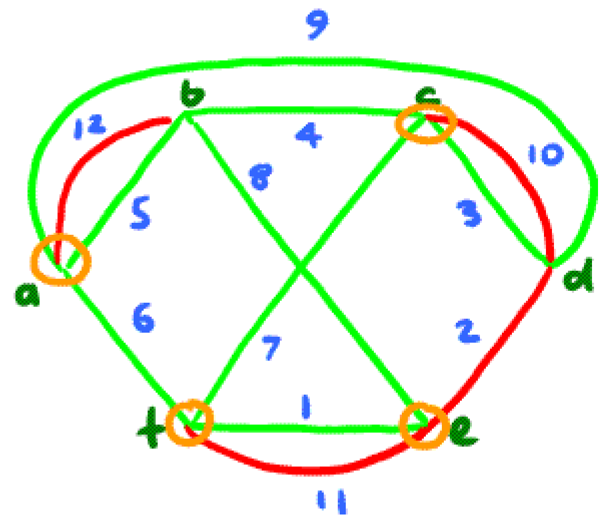


Circuits using Ω = Σ-even T-joins

Rem:

$|T| \leq 2 \Rightarrow$ representation extends

$|T| \geq 4 \Rightarrow$ " does not extend

certificate

We have

$N =$ ecycle (H, r) : fixed size 3c

$N \preceq M$: big 3c
representation (H, r) does not extend to M.

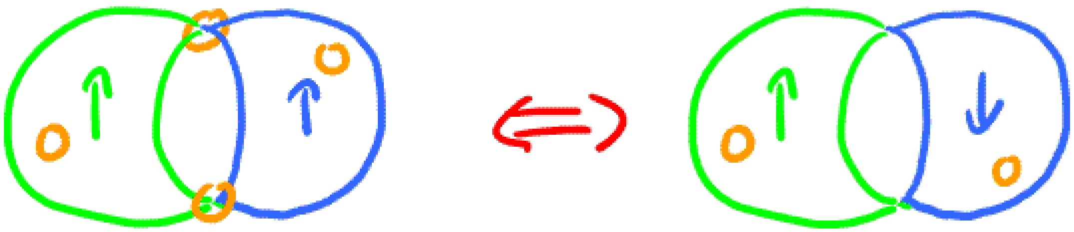
To show : $\exists M'$ st $N \preceq M' \preceq M$, M' 3c and representation (H, r) does not extend to M'.

$\Leftarrow\!\!\Rightarrow$

We can delete (contract) edge of (G, Σ) st:

- resulting matroid 3c
- $(G, \Sigma) \setminus e$ (or $(G, \Sigma)/e$) contains (H, r)
- Resulting set $|T'| \geq 4$.

Problem: 2-cuts



$|T| = 4$ $\qquad$ $|T| = 2$

We want $|T| \geq 4$ even after whitney-flips.

Splitter th:

Let $(G,\Sigma) \geq (H,\Gamma)$, G 3c, H : 2c.

Let $A \subseteq EG$ (forbidden edges).

If $|VG| \geq f(|VH|, |A|)$ there $\exists\, e \in EG \setminus A$ st either $(G,\Sigma)\setminus e$ or $(G,\Sigma)/e$ contains (H,Γ) as minor and is 3c.

Trees of 3-cuts (Oxley, Semple, Whittle / Oxley, Beavers)