Submodular Functions

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Outline

- Submodular Functions

  Examples

  Discrete Convexity

  Base Polyhedra
- Submodular Function Minimization

  Min-Max Theorem

  Combinatorial Algorithms
- Symmetric Submodular Functions

  Minimum Degree Orderings

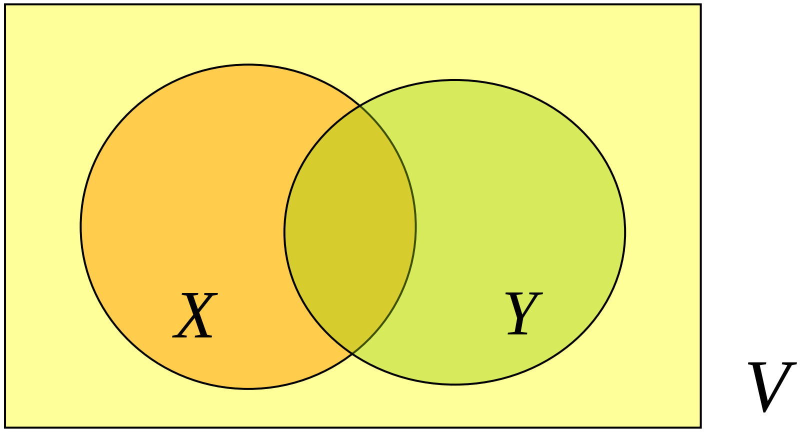
Submodular Functions

V : Finite Set

$$f : 2^V \to \mathrm{R} \qquad \forall X, Y \subseteq V$$

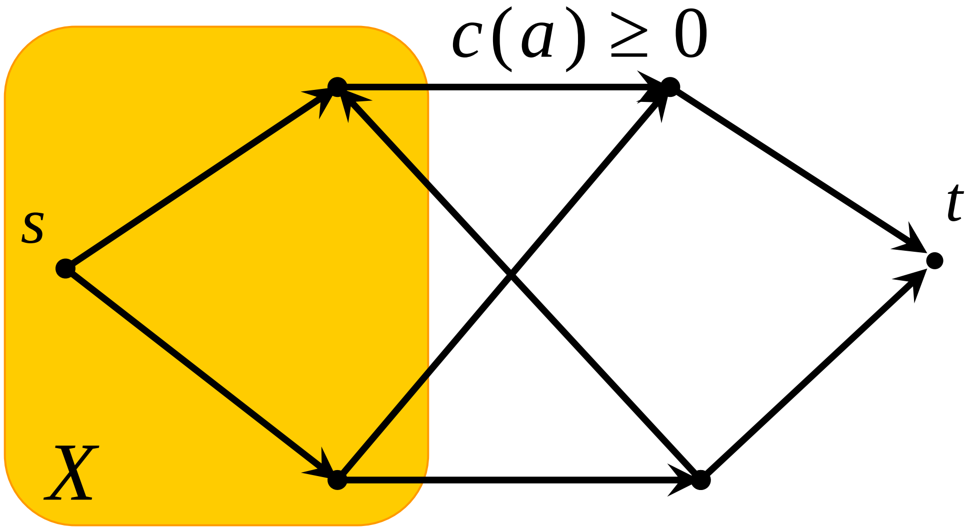
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

- Cut Capacity Functions
- Matroid Rank Functions
- Entropy Functions



Cut Capacity Function

Cut Capacity $\kappa(X) = \sum \{c(a) \mid a : \text{leaving } X\}$



Max Flow Value = Min Cut Capacity

Matroids

$\mathrm{M} = (E, \Im)$

Whitney (1935)

$$\phi \in \Im$$

$$I \subseteq J \in \Im \Rightarrow I \in \Im$$

$$\forall I, J \in \Im, |I| < |J| \Rightarrow \exists j \in J - I, I \cup \{j\} \in \Im$$

Rank Function

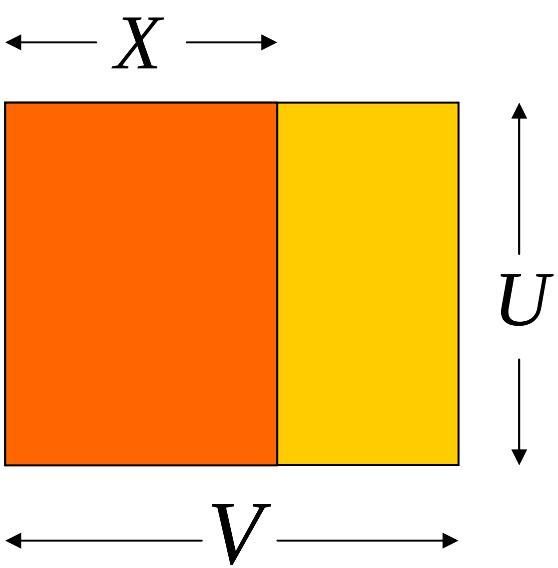
$$\rho(Y) = \max\{|J| : J \subseteq Y, J \in \Im\}$$

Matroid Rank Functions

Matrix Rank Function

$$\rho(X) = \operatorname{rank} A[U, X]$$

$A =$



$$\rho : 2^V \to \mathbf{Z}_+$$

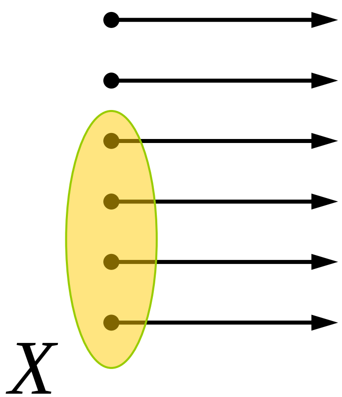
$$\forall X \subseteq V, \rho(X) \leq |X|.$$

$$X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y).$$

$\rho :$ Submodular.

Entropy Functions

Information
Sources



$$h(\phi) = 0$$

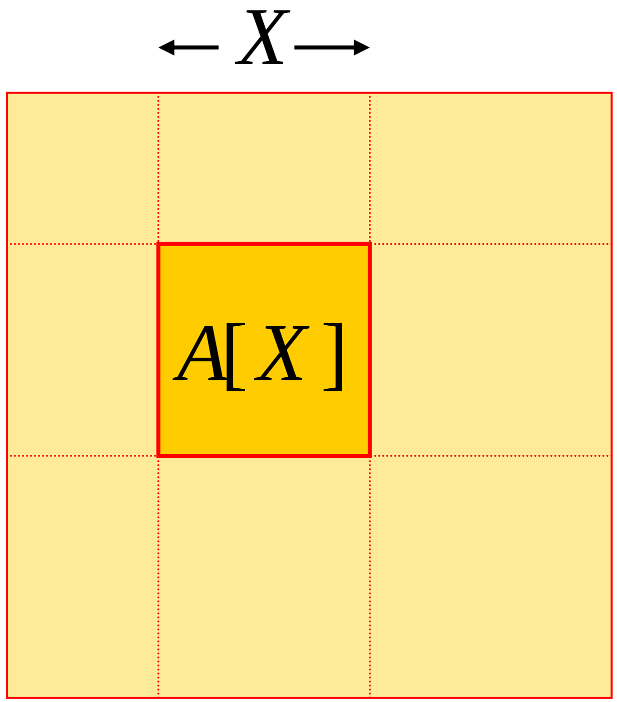
$h(X)$: Entropy of the Joint Distribution

$$h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y)$$

Conditional Mutual Information ≥ 0

Positive Definite Symmetric Matrices

$A =$



$$f(\phi) = 0$$

$$f(X) = \log \det A[X]$$

Ky Fan's Inequality

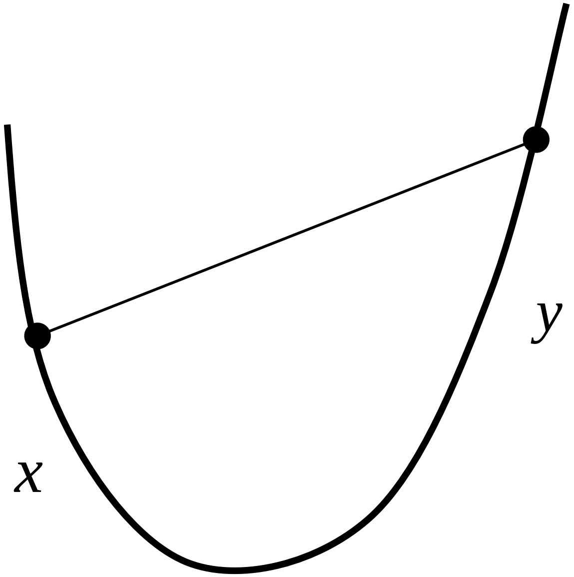
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

Extension of the Hadamard Inequality

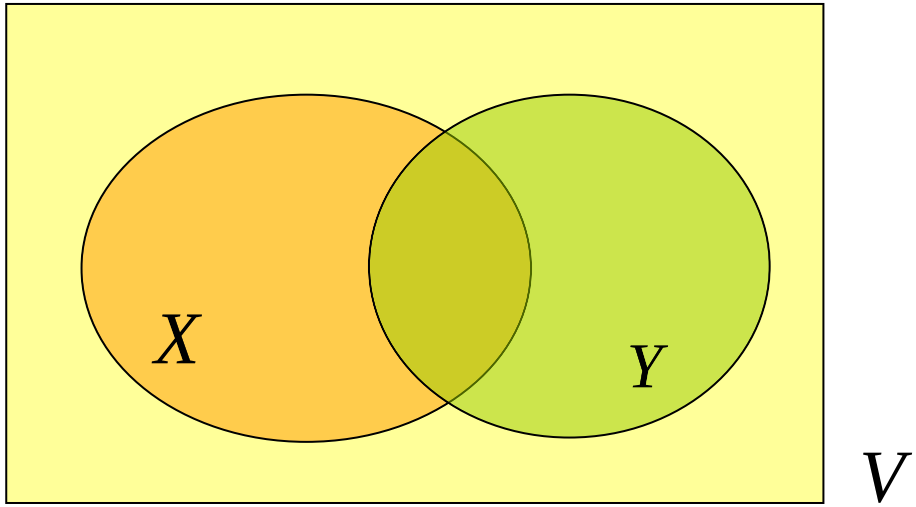
$$\det A \leq \prod_{i \in V} A_{ii}$$

Discrete Convexity

Convex Function

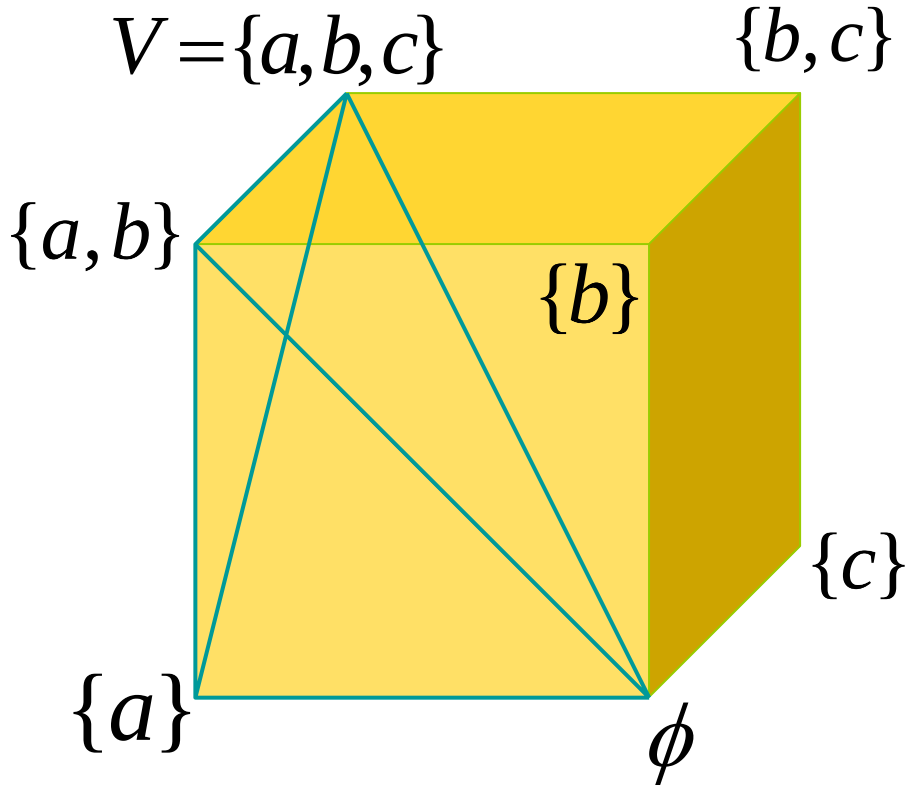


$$f(X)+f(Y)\geq f(X\cap Y)+f(X\cup Y)$$



Discrete Convexity

Lovász (1983)



$\hat{f}$: Linear Interpolation

$\hat{f}$: Convex

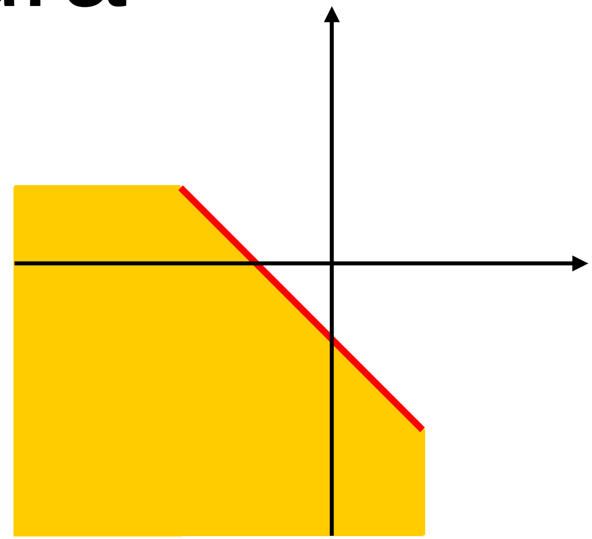
f : Submodular

Discrete Convex Analysis

Murota (2003)

Base Polyhedra

$$\mathbf{R}^V = \{x \mid V \to \mathbf{R}\}$$



$$x(Y) = \sum_{v \in Y} x(v)$$

$$f(\phi) = 0$$

Submodular Polyhedron

$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \leq f(Y)\}$$

Base Polyhedron

$$B(f) = \{x \mid x \in P(f), x(V) = f(V)\}$$

Greedy Algorithm

Edmonds (1970)
Shapley (1971)

$$\xleftarrow{\quad\quad} L(v) \xrightarrow{\quad\quad}$$

$$v_1 \quad v_2 \quad \cdots \quad \bigcirc \quad v \quad \bigcirc \quad \cdots \quad \bigcirc \quad v_n$$

$$y(v) = f(L(v)) - f(L(v) - \{v\}) \qquad (v \in V)$$

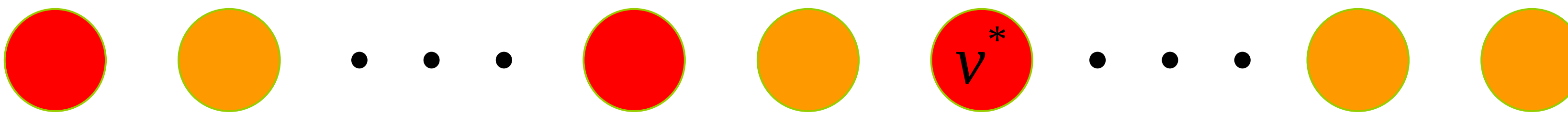
y : Extreme Base

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y(v_1) \\ y(v_2) \\ \vdots \\ y(v_n) \end{bmatrix} = \begin{bmatrix} f(L(v_1)) \\ f(L(v_2)) \\ \vdots \\ f(L(v_n)) \end{bmatrix}$$

Greedy Algorithm

$\boxed{y \in B(f)?}$ $\qquad y(X) \leq f(X), \quad \forall X \subseteq V$

Induction on $|X|$



$$y(X \setminus \{v^*\}) \leq f(X \setminus \{v^*\})$$

Submodularity

$$\begin{aligned} y(X) &= y(X \setminus \{v^*\}) + y(v^*) \\ &\leq f(X \setminus \{v^*\}) + f(L(v^*)) - f(L(v^*) \setminus \{v^*\}) \\ &\leq f(X) \end{aligned}$$

Linear Optimization

$$\boxed{\max\{\langle p, x\rangle \mid x \in B(f)\}?}$$

$$p \in \mathbf{R}_{+}^{V}$$

$$\max\{\langle p, x\rangle \mid x \in P(f)\}?$$

$$\langle p, x\rangle := \sum_{v\in V} p(v)x(v)$$

Greedy Algorithm with $p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n)$

$$y(v) = f(L(v)) - f(L(v) - \{v\}) \qquad (v \in V)$$

y : Optimal

$$\boxed{\hat{f}(p) = \max\{\langle p, x\rangle \mid x \in B(f)\}} \longrightarrow \hat{f} \text{ : Convex}$$

Linear Optimization

Dual LP

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{X \subseteq V} q_X f(X) & \\ \text{subject to} & \displaystyle\sum_{X \ni v} q_X = p(v), & \forall v \in V, \\ & q_X \geq 0, & \forall X \subseteq V. \end{array}$$

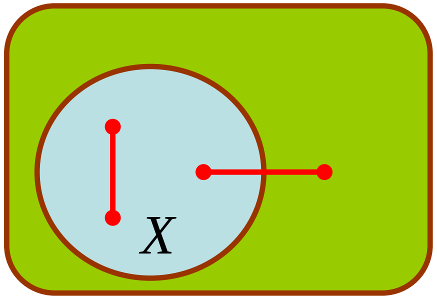
$$q_X := \begin{cases} p(v_j) - p(v_{j+1}) & (X = L(v_j)) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\sum_{X \subseteq V} q_X f(X) = \sum_{j=1}^{n} [p(v_j) - p(v_{j+1})] f(L(v_j)) = \hat{f}(p)$$

Graph Orientation

$G=(V,E)$: Graph $\qquad b:V\to \mathbf{Z}_+$

$e(X)$: Number of Edges Incident to X.



Hakimi [1965]

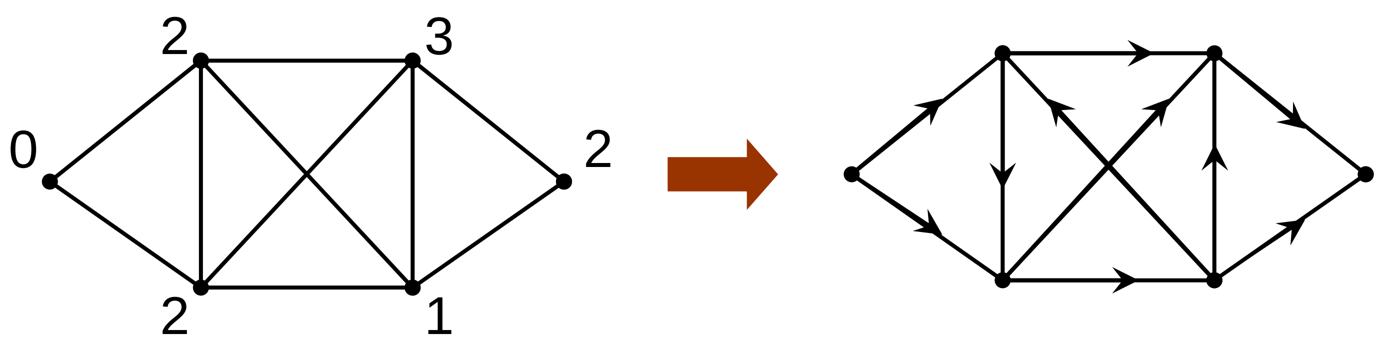
There exists an orientation $\vec{G}$ with in-deg$(v)=b(v)$ for every $v\in V$.

$\Longleftrightarrow$

$$b(X)\le e(X),\quad \forall X\subseteq V,$$
$$b(V)=e(V).$$

$b\in B(e)$

e: Submodular

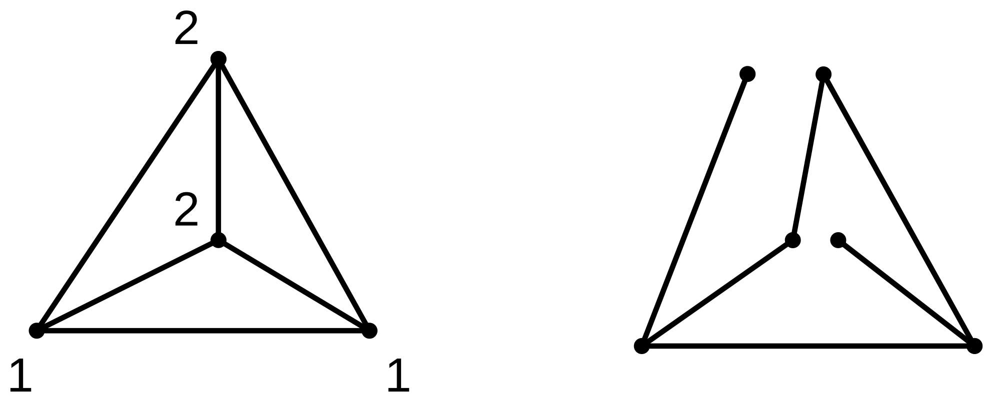


Connected Detachment

$G=(V,E)$: Connected Graph $\qquad b: V \to \mathbf{Z}_+$

Detachment

$$G=(V,E) \longrightarrow \hat{G}=(W,E):$$



Split each vertex $v \in V$ into $b(v)$ vertices. Each edge should be incident to some corresponding vertices.

Connected Detachment

Theorem (Nash-Williams [1985])

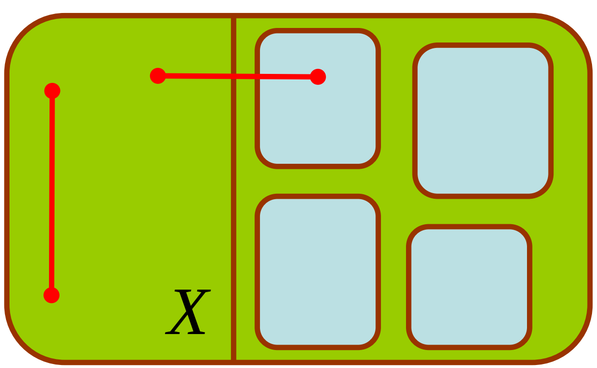
There exists a connected b-detachment of G.

$$\Longleftrightarrow \; b(X) \le e(X) - c(X) + 1, \quad \forall X \subseteq V.$$

$c(X)$: Number of Connected Components in $G \setminus X$.

$\Rightarrow$ Consider an b-detachment.

Shrink each connected component in $G \setminus X$.



Number of vertices: $b(X) + c(X)$.

Number of edges: $e(X)$.

If the resulting graph is connected,

$$b(X) + c(X) \le e(X) + 1.$$

Connected Detachment

$f(X) := e(X) - c(X) + 1$ Submodular

$f(V) = |E| + 1, \quad f(\phi) = 0.$

$\Leftarrow$ $\boxed{b \in P(f)}$ $\quad \exists h \in B(f), \ \ h \geq b.$

$s \in V$

$$y(v) := \begin{cases} h(v) & (v \neq s) \\ h(s) - 1 & (v = s) \end{cases}$$

$\boxed{y \in B(e)}$

$\exists$Orientation $\vec{G}$ with in-deg $(v) = y(v), \ \ \forall v \in V.$

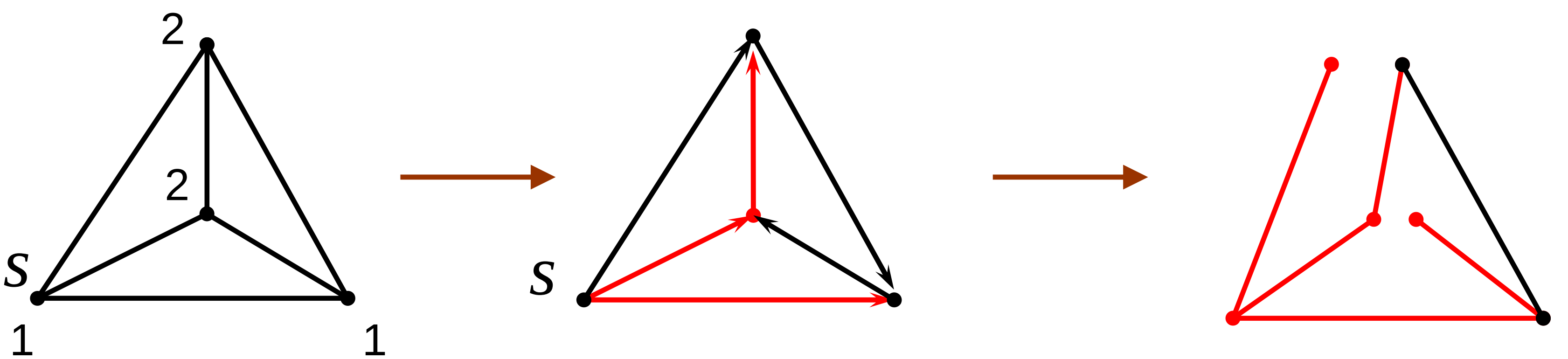
R: Set of vertices reachable from s.

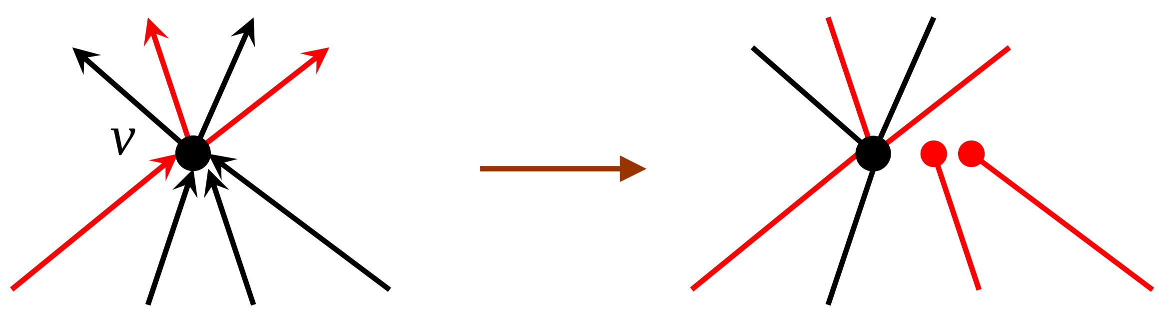
$$\underline{e(R)} = y(R) = h(R) - 1 \leq \underline{f(R) - 1}$$

$\boxed{R = V}$

Connected Detachment

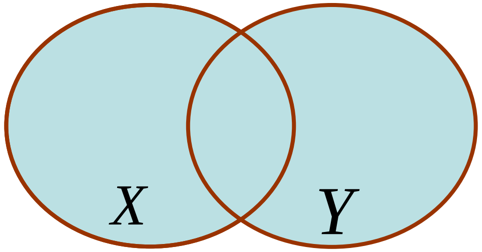
An orientation connected from a root s such that in-deg$(v) \geq b(v)$ for every $v \neq s$ and in-deg$(s) \geq b(s) - 1$.





$\geq r(v)$

Intersecting Submodular Functions



Intersecting:

$$X \cap Y \neq \phi, \quad X \setminus Y \neq \phi, \quad Y \setminus X \neq \phi.$$

$$f : 2^V \rightarrow \mathbf{R}$$

Intersecting Submodular:

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

$\forall X, Y \subseteq V$: Intersecting

Intersecting Submodular Functions

$f: 2^V \to \mathbf{R}$ Intersecting Submodular $\quad f(\phi) = 0$

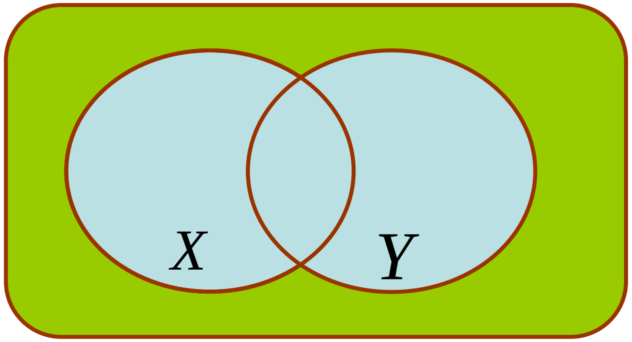
$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \le f(Y)\}$$

Theorem (Lovász [1977])

There exists a fully submodular function $\tilde{f}: 2^V \to \mathbf{R}$ such that $P(f) = P(\tilde{f})$.

$$\tilde{f}(X) = \min\{\sum_{i=1}^{k} f(X_i) \mid \{X_1, \ldots, X_k\} : \text{partition of } X\}$$

Crossing Submodular Functions



Crossing:

$$X \cap Y \neq \phi, \quad X \cup Y \neq V,$$

$$X \setminus Y \neq \phi, \quad Y \setminus X \neq \phi.$$

$$f : 2^V \to \mathbf{R}$$

Crossing Submodular:

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

$\forall X, Y \subseteq V$: Crossing

Crossing Submodular Functions

$f : 2^V \to \mathbf{R}$ Crossing Submodular $f(\phi) = 0$

$$B(f) = \{x \mid x \in \mathbf{R}^V, x(V) = f(V), \forall Y \subseteq V, x(Y) \le f(Y)\}$$

Theorem (Frank [1982], Fujishige [1984])

There exists a fully submodular function $\tilde{f} : 2^V \to \mathbf{R}$ such that $B(f) = B(\tilde{f})$, provided that $B(f)$ is nonempty.

Bi-truncation Algorithm Frank & Tardos [1988].

Graph Orientation

$G=(V,E)$: Graph $\quad\quad b:V\to\mathbf{Z}_+$

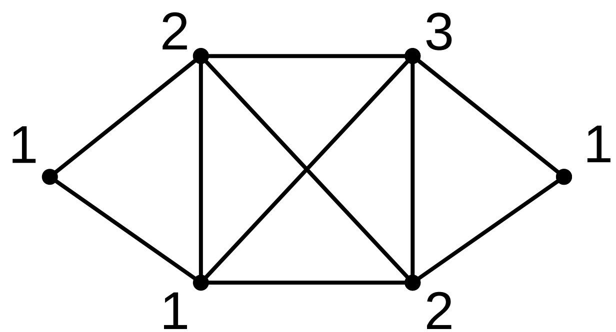
$b\in B(f)$

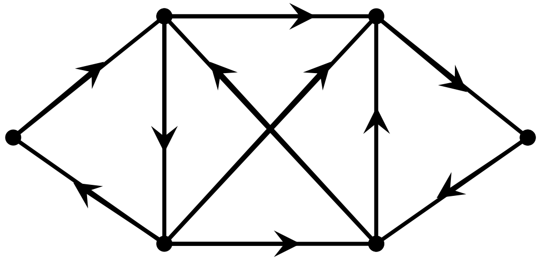
$e(X)$: Number of Edges Incident to X.

There exists an k-arc-connected orientation $\vec{G}$ with in-deg$(v)=b(v)$ for every $v\in V$.

$\Longleftrightarrow$

$$b(X)\le e(X)-k,\quad \forall X\subseteq V,$$
$$b(V)=e(V).$$





Graph Orientation

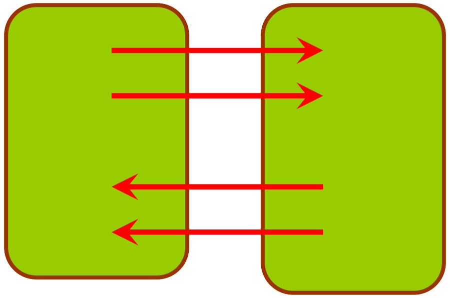
When is $B(f)$ nonempty?

Theorem (Nash-Williams [1969])

There exists an k-arc-connected orientation of G.

$\Updownarrow$

$G:\ 2k$-edge-connected



$$x(v) := d(v)/2 \quad (v \in V)$$

$$x \in B(f)$$

Minimax Acyclic Orientation

$G=(V,E)$: Graph

Find an acyclic orientation that minimizes the maximum in-degree