# Submodular Function Minimization

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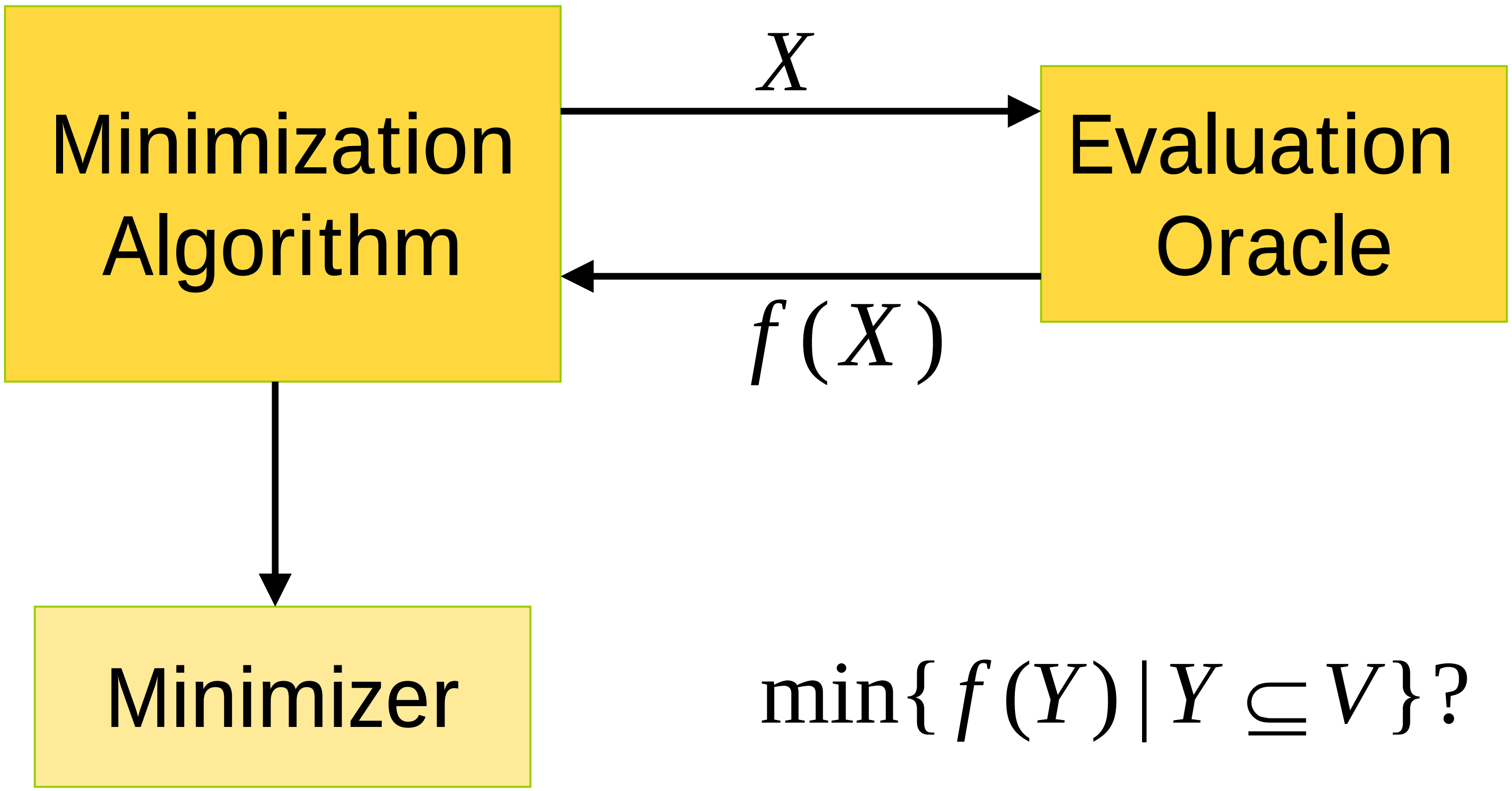
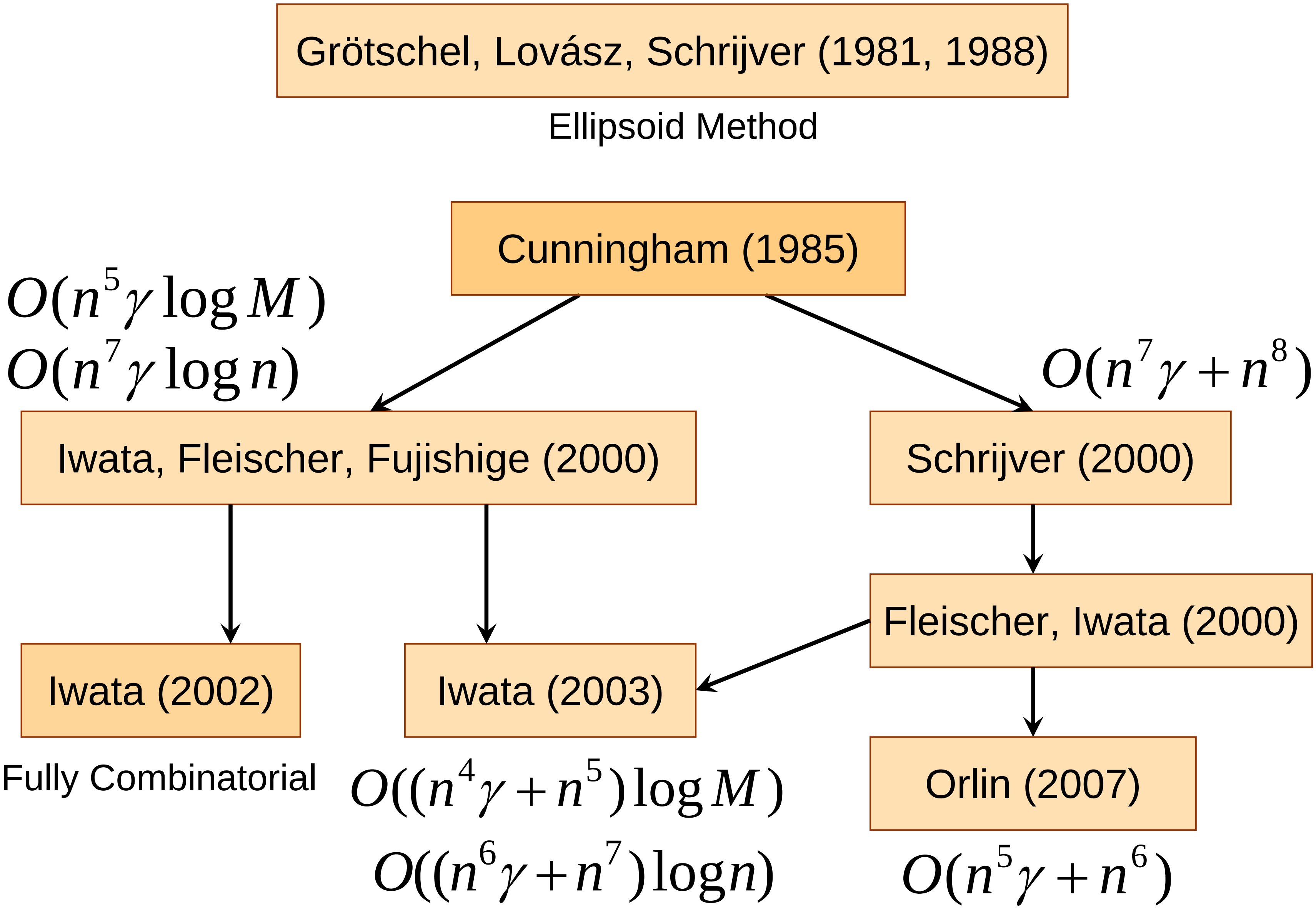
# Submodular Function Minimization

Assumption: $f(\phi) = 0$

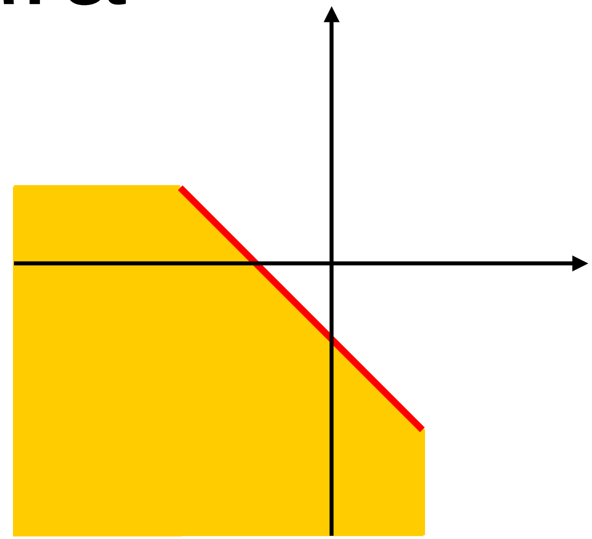



$$\min\{f(Y) \mid Y \subseteq V\}?$$

# Submodular Function Minimization



# Base Polyhedra



$$\mathbf{R}^V = \{x \mid V \to \mathbf{R}\}$$

$$x(Y) = \sum_{v \in Y} x(v)$$

$$f(\phi) = 0$$

Submodular Polyhedron

$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \leq f(Y)\}$$

Base Polyhedron

$$B(f) = \{x \mid x \in P(f), x(V) = f(V)\}$$

# Min-Max Theorem

**Theorem** Edmonds (1970)

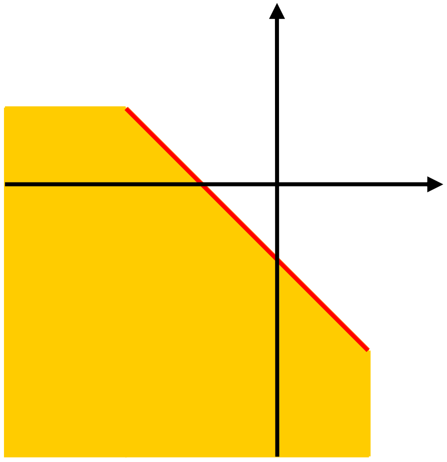
$$\min_{Y \subseteq V} f(Y) = \max\{z(V) \mid z \in P(f), z \leq 0\}$$

$$= \max\{x^{-}(V) \mid x \in B(f)\}$$

$$x^{-}(v) := \min\{0, x(v)\}$$

$$z(V) \leq z(Y) \leq f(Y)$$

$$x^{-}(V) \leq x(Y) \leq f(Y)$$



# Min-Max Theorem

$$\min_{Y \subseteq V} f(Y) = \max\{ z(V) \mid z \in P(f), z \leq 0 \}$$
$$= \max\{ x^-(V) \mid x \in B(f) \}$$

$$f^\circ(X) := \min\{ f(Y) \mid Y \subseteq X \}$$ Submodular

$$P(f^\circ) \subseteq P(f)$$

$$z \in B(f^\circ) \Rightarrow z(V) = f^\circ(V) = \min_{Y \subseteq V} f(Y)$$

# Combinatorial Approach

Extreme Base $y_i \in B(f)$

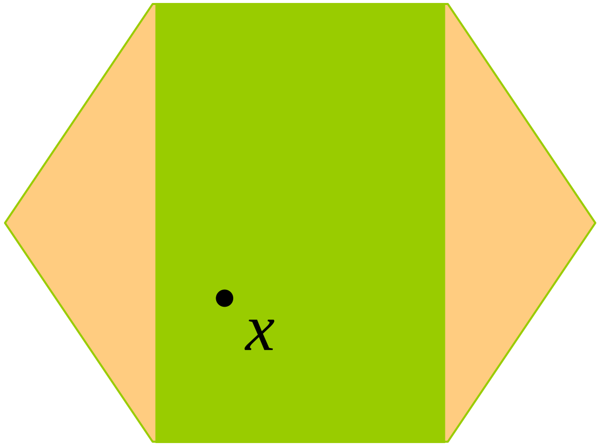
Convex Combination

$$x = \sum_{i \in I} \lambda_i y_i$$

Cunningham (1985)

$$O(n^6 M \gamma \log nM)$$

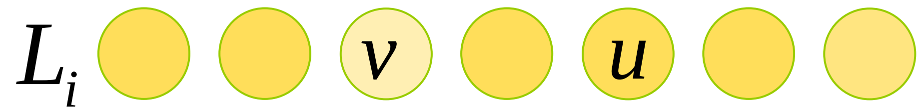
$$M = \max_{X \subseteq V} | f(X) |$$




Gaussian Elimination

# Schrijver's Algorithm

$$x = \sum_{i \in I} \lambda_i y_i$$



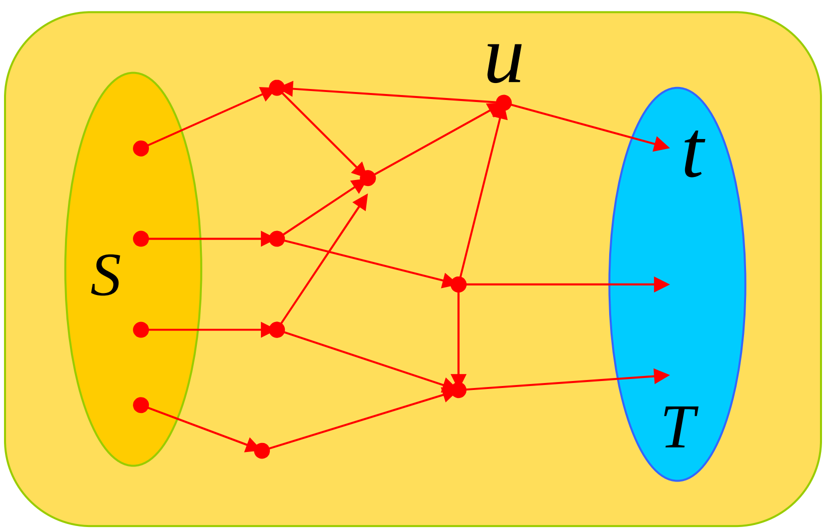

$$G_I = (V, A_I)$$

$$A_I = \{(u,v) \mid \exists i \in I, v \prec_i u\}$$

$d(v)$: distance from $S$

$d(t)$: maximun in $T$

$d(u) = d(t) - 1$

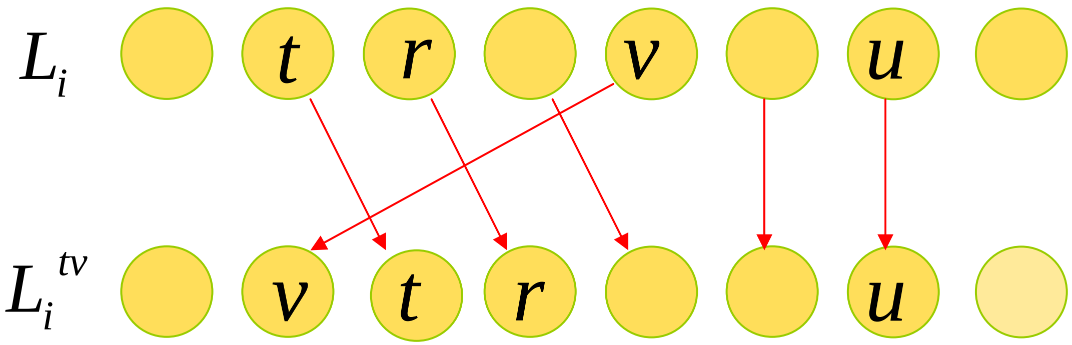


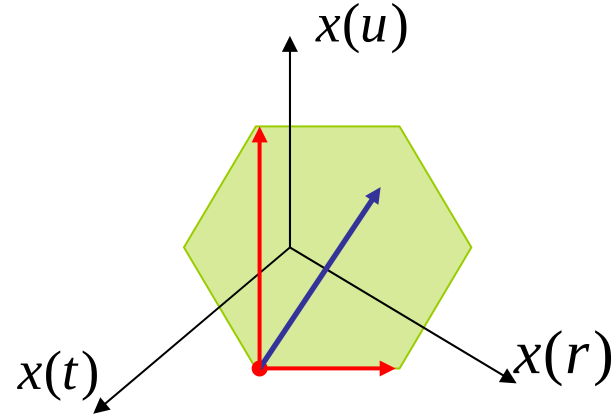

$$S = \{v \mid x(v) < 0\} \qquad T = \{v \mid x(v) > 0\}$$

$$x(u) := x(u) + \alpha$$

$$x(t) := x(t) - \alpha$$

# Schrijver's Algorithm




$$
\begin{array}{c|cccc|}
 & r & & & u \\
\hline
t & - & \boxed{-} & \cdots & - \\
r & + & - & \cdots & - \\
 & 0 & + & \ddots & \vdots \\
 & \vdots & \ddots & \ddots & - \\
u & 0 & \cdots & 0 & +
\end{array}
\begin{bmatrix} \xi_r \\ \vdots \\ \vdots \\ \xi_u \end{bmatrix}
=
\begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

$y_i^{tv} - y_i$

$$\alpha := \min\{x(u), \frac{\lambda_i}{\sum_v \xi_v}\}$$

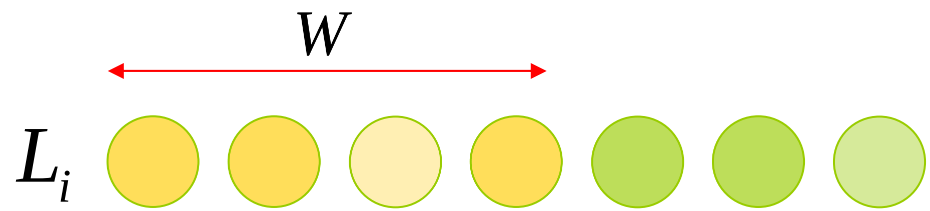
$$\lambda_{i_v} := \alpha \xi_v$$

$$\lambda_i := \lambda_i - \alpha \sum_v \xi_v$$

$$\xi \geq 0$$

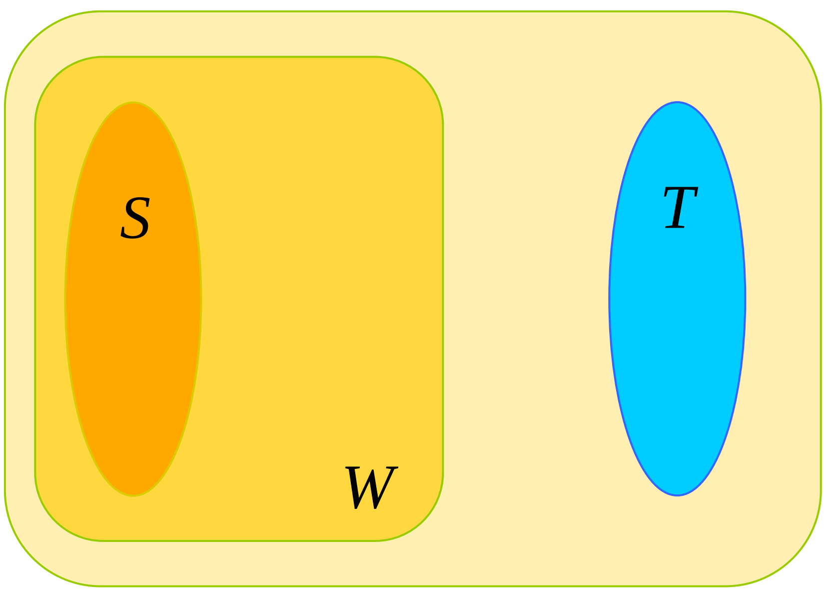
# Schrijver’s Algorithm

No Path from $S$ to $T$




$$W := \{v \mid v : \text{Reachable from } S\}$$

$$y_i(W) = f(W), \ \forall i \in I$$




$$\underline{x^-(V)} = x(W) = \sum_{i \in I} \lambda_i y_i(W) = \underline{f(W)}$$

$$S = \{v \mid x(v) < 0\} \quad T = \{v \mid x(v) > 0\}$$

$$O(n^7 \gamma + n^8)$$

# IFF Scaling Algorithm

$$\boxed{\delta \approx \frac{M}{n^2} \longrightarrow \delta < \frac{1}{n^2}}$$

$\left\lfloor \dfrac{f_\delta(X)}{\delta} \right\rfloor$ : Submodular

$$f_\delta(X) = f(X) + \underline{\delta\,|X| \cdot |V \setminus X|}$$

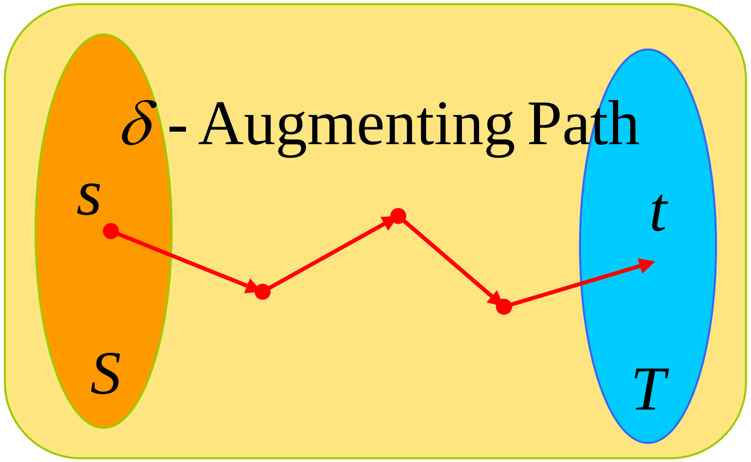
Cut Function

$\varphi$ : Flow in the Complete Digraph $\varphi(u,v) \le \delta$

$$x = \sum_{i \in I} \lambda_i y_i$$

$$\boxed{z = x + \partial\varphi}$$

Increase $z^-(V)$

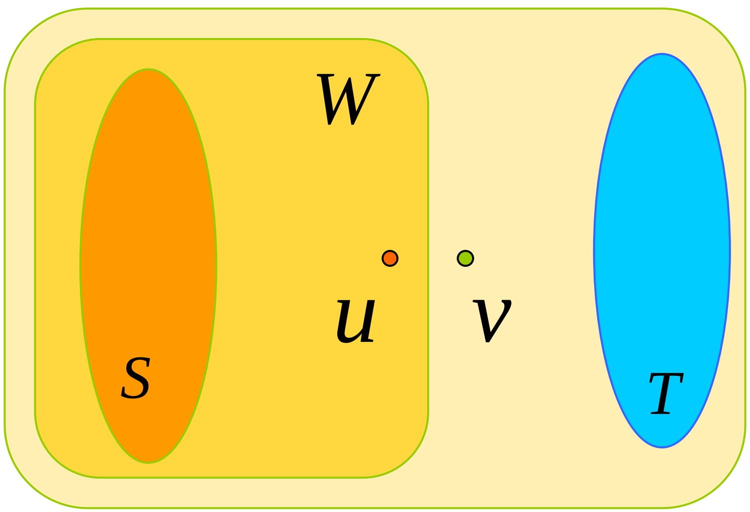


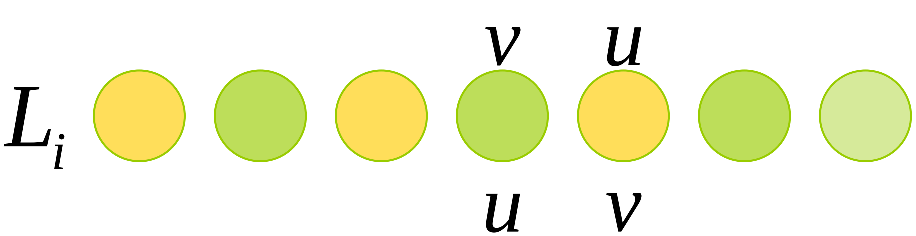

$S = \{v \mid z(v) < -\delta\}$ $\quad$ $T = \{v \mid z(v) > \delta\}$

# IFF Scaling Algorithm

No Path from $S$ to $T$

$$W := \{v \mid v : \text{Reachable from } S\}$$



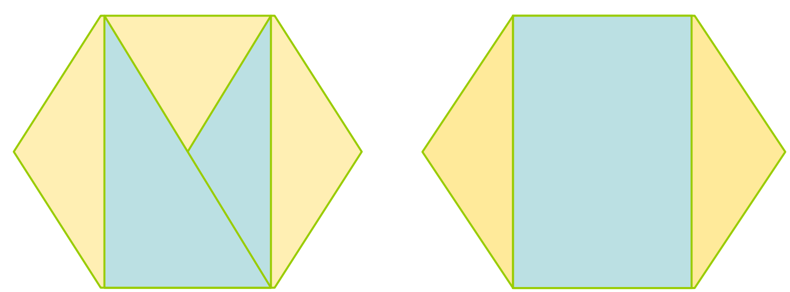

Double-Exchange

$$y_i := y_i + \beta(\chi_u - \chi_v)$$

$$x := x + \alpha(\chi_u - \chi_v)$$

$$\alpha := \min\{\delta, \lambda_i \beta\}$$

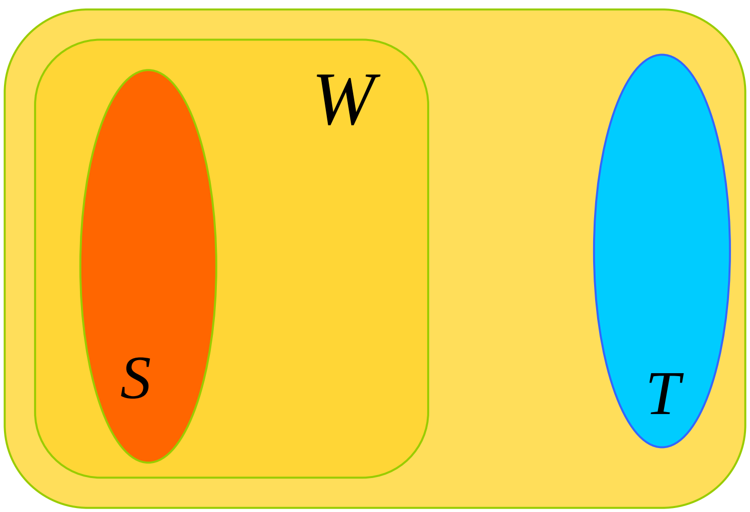
$\alpha = \lambda_i \beta$ $\alpha < \lambda_i \beta$

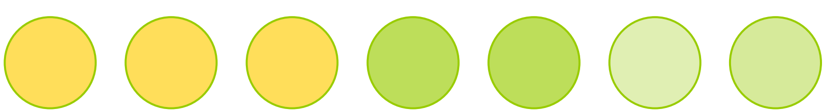


Saturating Nonsaturating

# IFF Scaling Algorithm

| |
|---|
| No Path from $S$ to $T$ |
| No Active Triple$(i,u,v)$ |




$L_i$ 

$$y_i(W) = f(W), \ \forall i \in I$$

$$x(W) = \sum_{i \in I} \lambda_i y_i(W) = f(W)$$

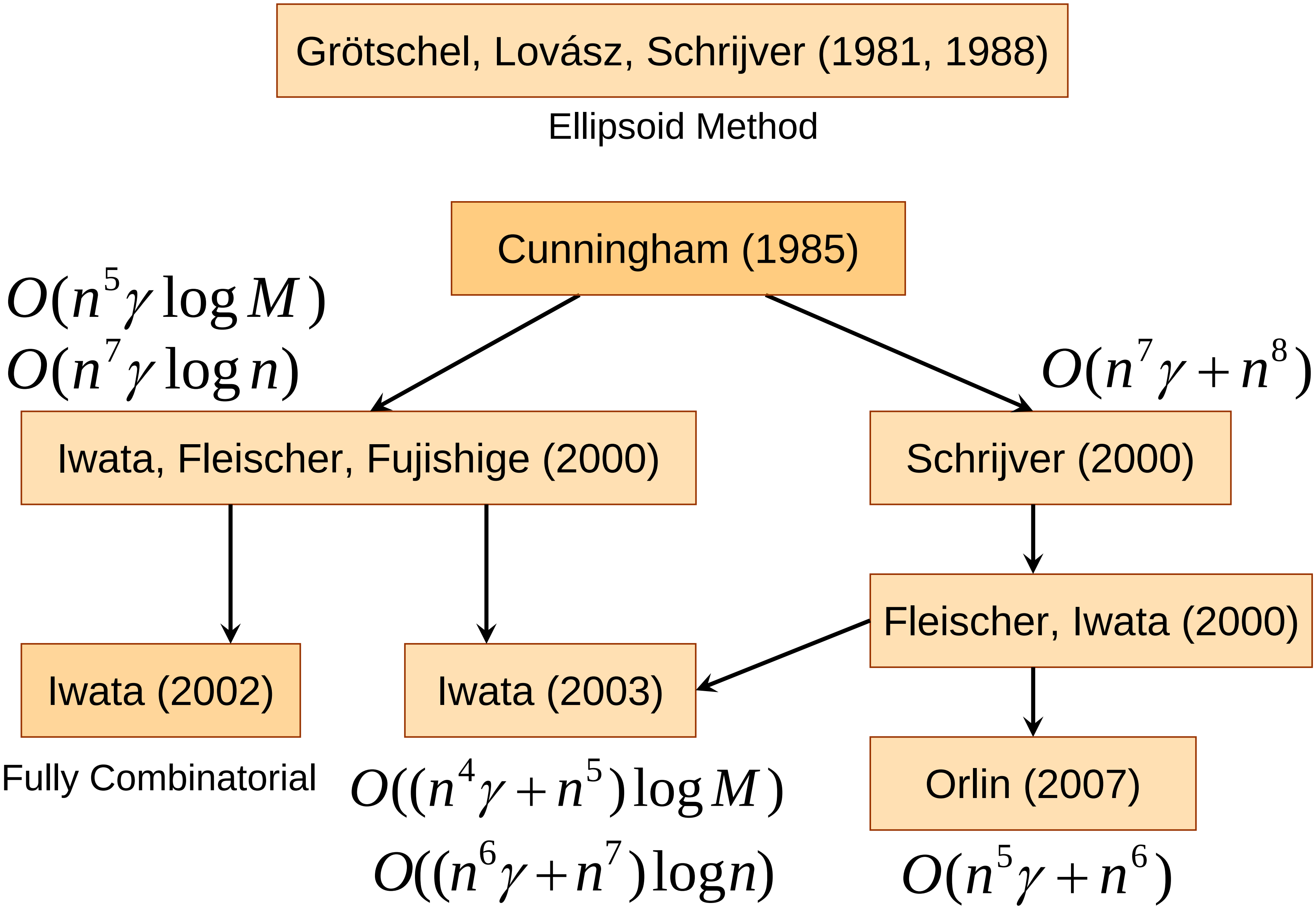
$$z^-(V) \geq f(W) - n\delta$$

$$x^-(V) \geq f(W) - n^2\delta$$

$$\delta < \frac{1}{n^2} \Rightarrow f(W) : \text{Min}$$

$$O(n^5 \gamma \log M)$$

# Submodular Function Minimization



# Open Problems

- A Fully Combinatorial SFM Algorithm without Scaling.

- A Lower Bound on the Number of Oracle Calls for General SFM.

# Fully Combinatorial Algorithm

Addition, Subtraction, Comparison

Oracle Call (Function Evaluation)

Multiplication $\alpha \in \mathbf{R}, \mu \in \mathbf{Z} \longrightarrow$ Compute $\mu\alpha$

Division $\alpha \in \mathbf{R}, \beta \in \mathbf{R} \longrightarrow$ Compute $q = \lceil \alpha / \beta \rceil$

- Neglect the Gaussian Elimination Step.
- Use Nonnegative Integer Combination Instead of Convex Combination.
- Choose a Step Length Appropriately.

# Pointers

- S. Fujishige: *Submodular Functions and Optimization*, North-Holland, 2005.

- S. T. McCormick: Submodular Function Minimization, *Discrete Optimization*, K. Aardal, G. Nemhauser, R. Weismantel, eds., Handbooks in Operations Research, Elsevier, 2005.

# Pointers

- S. Iwata: Submodular Function Minimization, *Mathematical Programming*, 112 (2008), pp. 45-64