

# Submodular Function Minimization

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# Submodular Functions and Polyhedra

$$f : 2^V \rightarrow \mathbf{R} \qquad f(\emptyset) = 0$$

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y), \quad \forall X, Y \subseteq V$$

Submodular Polyhedron

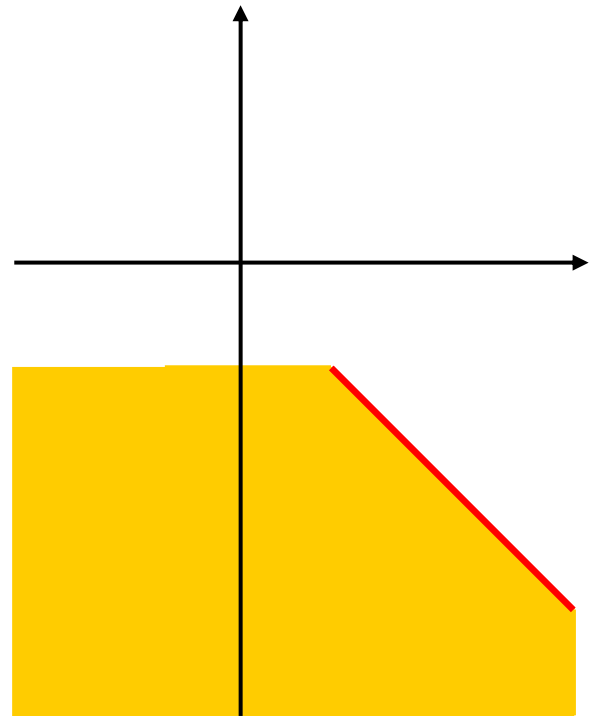
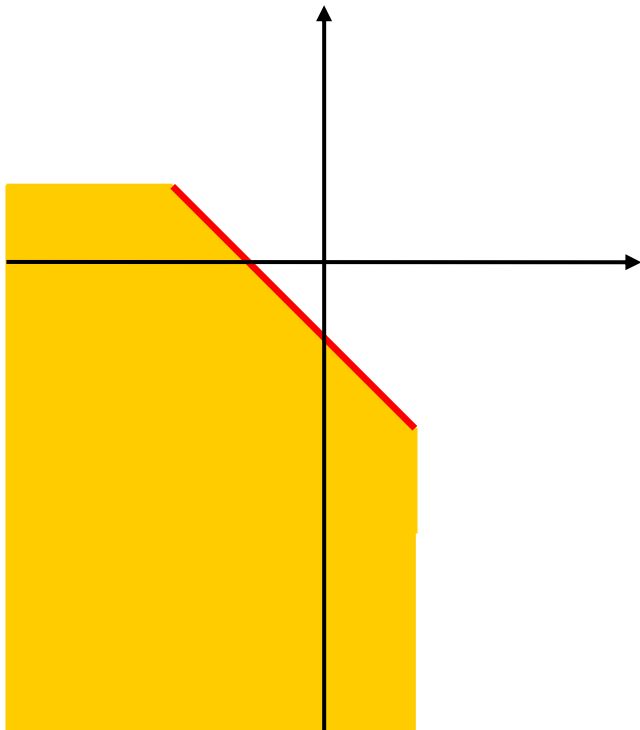
$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \leq f(Y)\}$$

Base Polyhedron

$$B(f) = \{x \mid x \in P(f), x(V) = f(V)\}$$

# Combinatorial Approach

$$\min_{Y \subseteq V} f(Y) = \max \{ x^-(V) \mid x \in B(f) \}$$



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## Convex Combination

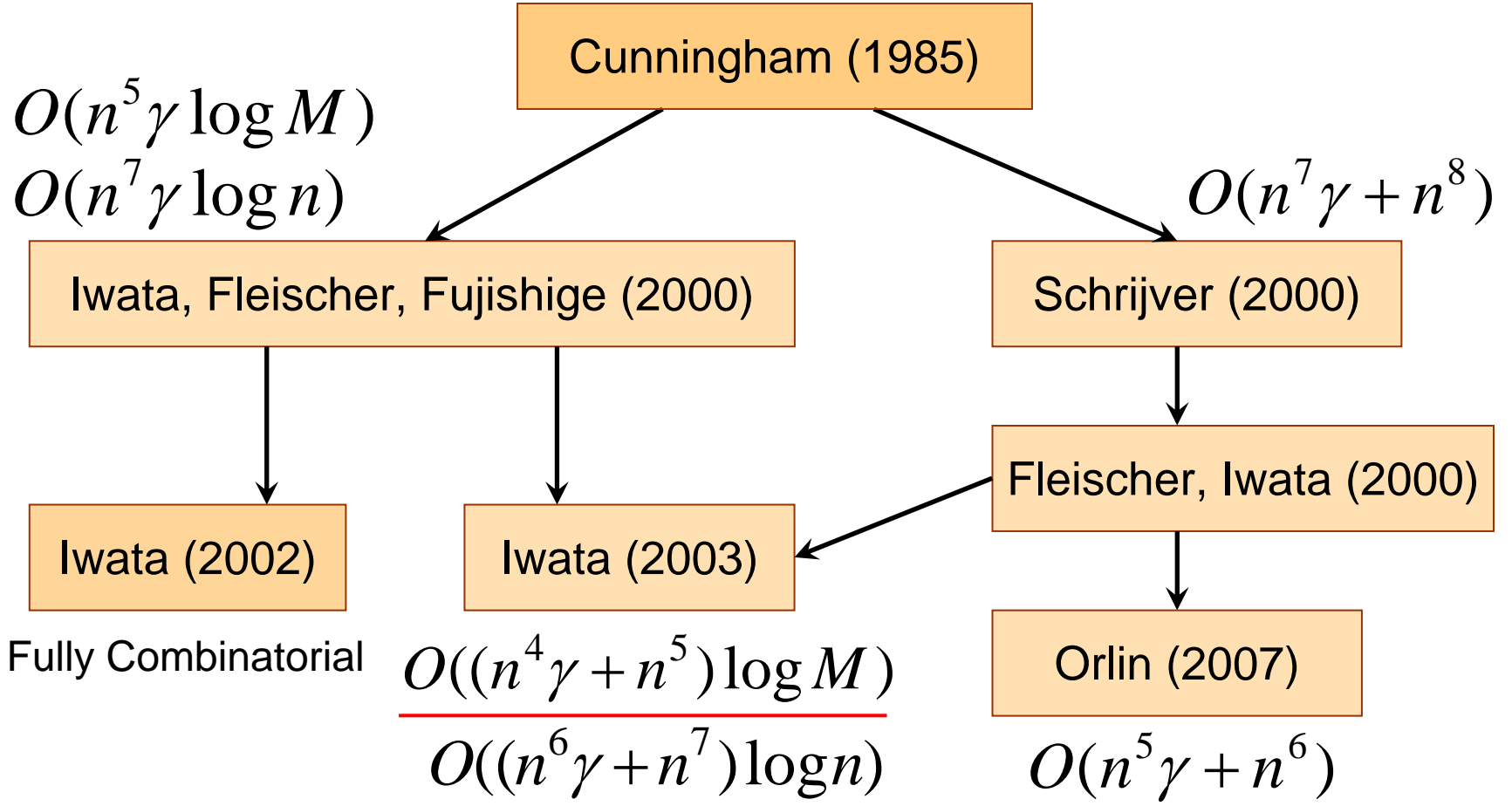
$$x = \sum_{i \in I} \lambda_i y_i$$

$y_i \in B(f)$ : Extreme Base Generated by the Greedy Algorithm with an Linear Ordering  $L_i$  in  $V$ .

# Submodular Function Minimization

Grötschel, Lovász, Schrijver (1981, 1988)

Ellipsoid Method





# Partial Order of an Extreme Base

$y \in B(f) : \text{Extreme}$

Bixby, Cunningham, Topkis (1985)

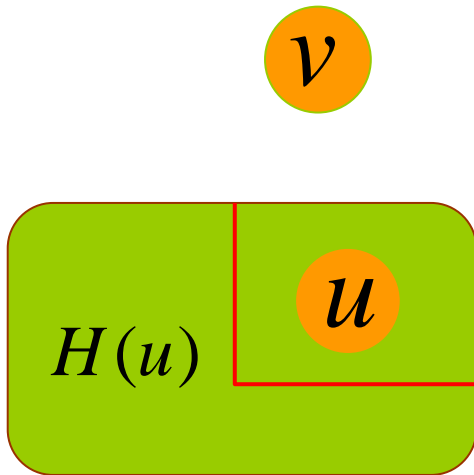
$v_1 \quad v_2 \quad \cdot \quad \cdot \quad \cdot \quad v_n$

$X : \text{Tight} \quad y(X) = f(X)$

$y(L(v)) = f(L(v)), \quad \forall v \in V$

Represent all tight sets

$\longrightarrow G(y)$



$H(u) : \text{Maximal Ideal Excluding } u.$

Test if  $H(u) \cup \{v\}$  is tight.

If not, then  $v \longrightarrow u$ .

# Finding All Minimizers

Extreme Base

$$y_i \in B(f)$$

Partial Order (DAG)

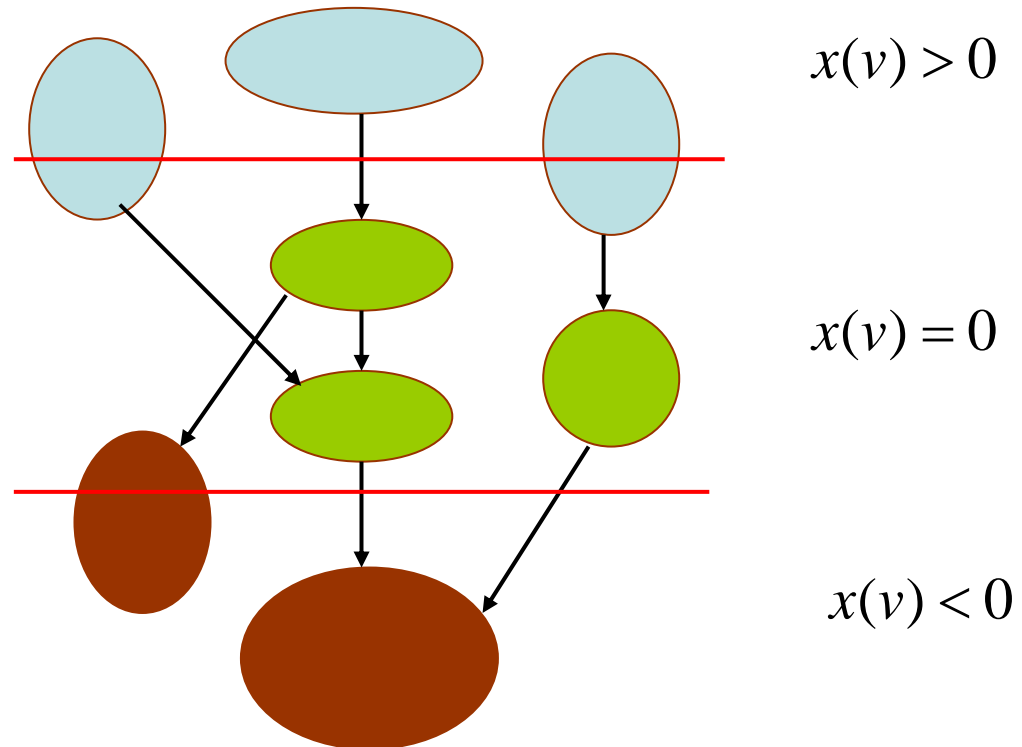
$$G(y_i)$$

Convex Combination

$$x = \sum_{i \in I} \lambda_i y_i$$

$$(\lambda_i > 0, \forall i \in I)$$

$G(x)$  : Superposition of  $G(y_i)$   
SCC Decomposition



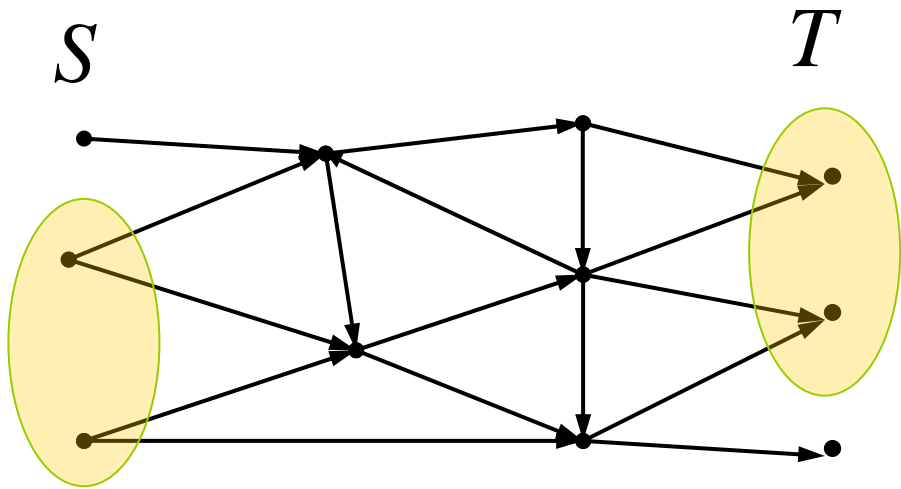


# Applications

- Evacuation Problem
- Multi-terminal Source Coding
- Multi-class Queueing System
- Constraint Satisfaction Problem
- Potts Model (Anglès d'Auriac, Iglói,  
Preissmann, Sebő [2002])
- Testing Branch-width  
(Oum & Seymour [2007])

# Evacuation Problem ( Dynamic Flow )

Hoppe, Tardos (2000)



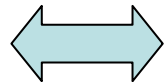
$c(a)$  : Capacity

$\tau(a)$  : Transit Time

$b(v)$  : Supply/Demand

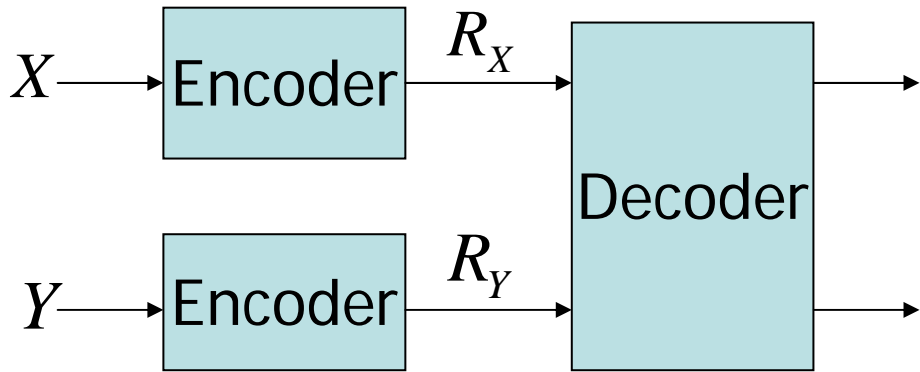
$o(X)$  : Maximum Amount of Flow from  $X \cap S$  to  $T \setminus X$ .

Feasible

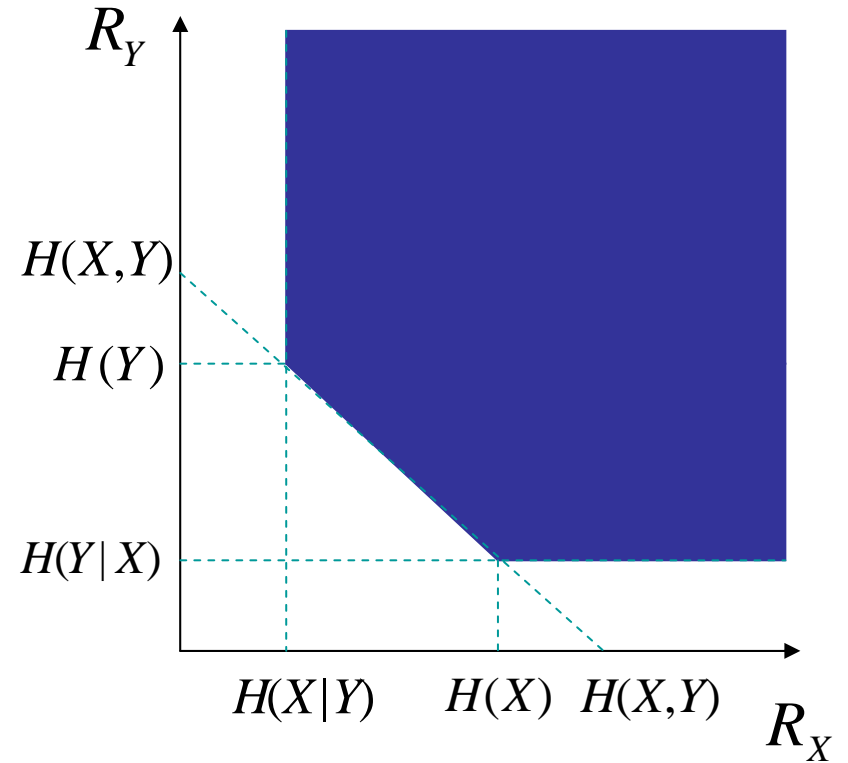


$$b(X) \leq o(X), \forall X \subseteq S \cup T$$

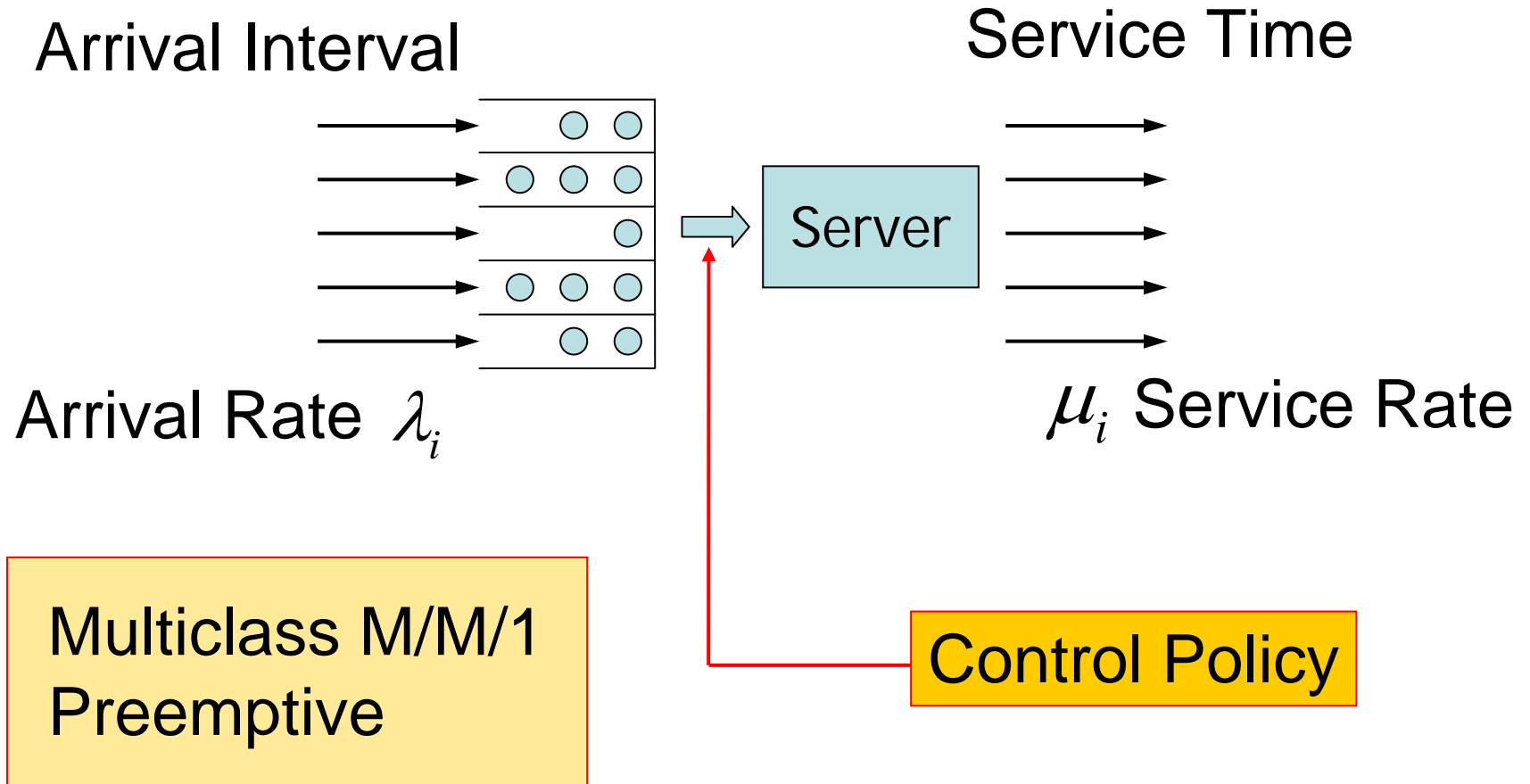
# Multi-terminal Source Coding



Slepian, Wolf (1973)



# Multi-class Queueing Systems



# Performance Region

$S_j$  : Expected Staying Time of a Job in  $j$

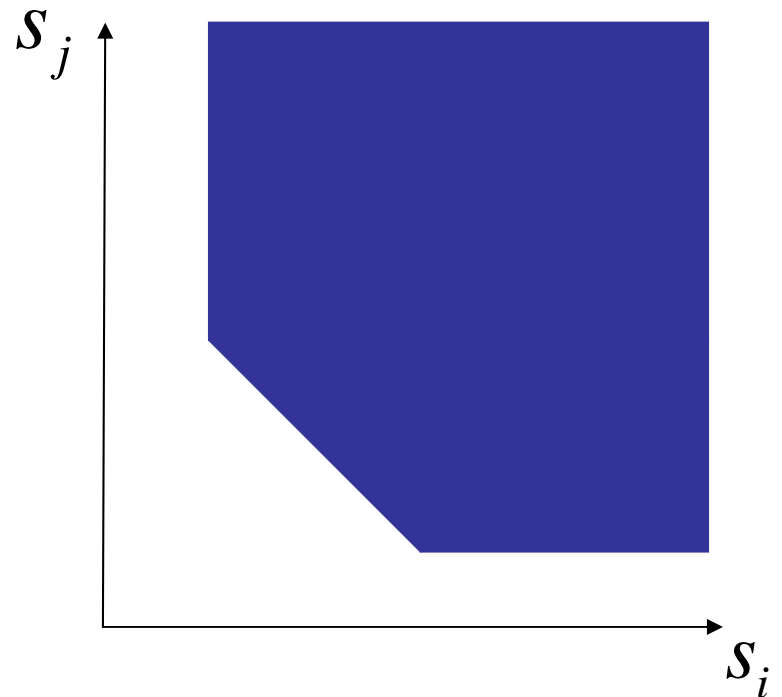
$S$  : Achievable



$$\sum_{i \in X} \rho_i S_i \geq \frac{\sum_{i \in X} \rho_i / \mu_i}{1 - \sum_{i \in X} \rho_i}, \forall X \subseteq V$$

Coffman, Mitrani (1980)

$$\rho_i := \lambda_i / \mu_i, \quad \sum_{i \in V} \rho_i < 1$$



# A Class of Submodular Functions

$$x, y, z \in \mathbb{R}_+^V$$

Itoko & Iwata (2005)

$h$ : Nonnegative, Nondecreasing, Convex

$$f(X) = z(X) - y(X)h(x(X)) \quad (X \subseteq V)$$

Submodular

$$\sum_{i \in X} \rho_i S_i \geq \frac{\sum_{i \in X} \rho_i / \mu_i}{1 - \sum_{i \in X} \rho_i}, \quad \forall X \subseteq V$$

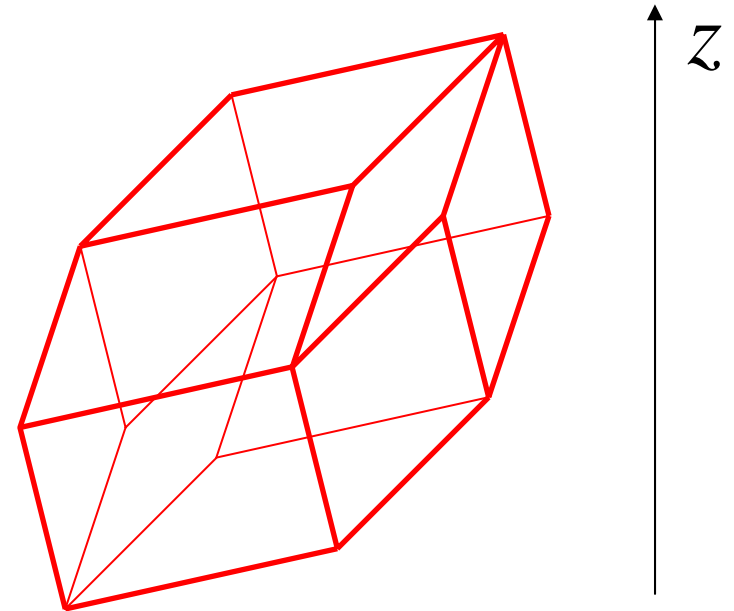
$$z_i := \rho_i S_i \quad y_i := \frac{\rho_i}{\mu_i}$$
$$x_i := \rho_i \quad h(x) := \frac{1}{1-x}$$

# Zonotope in 3D

$$w(X) = (x(X), y(X), z(X))$$

$$Z = \text{conv}\{w(X) \mid X \subseteq V\}$$

Zonotope



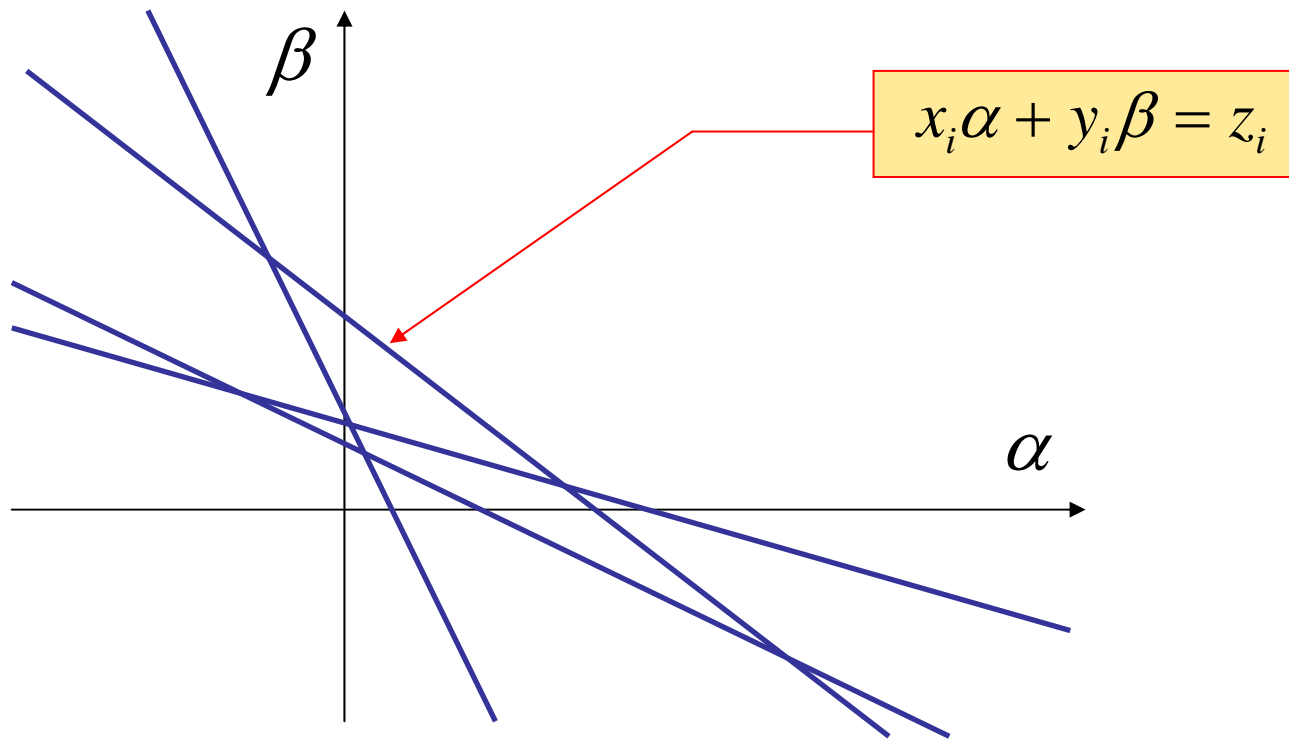
$$\tilde{f}(x, y, z) = z - yh(x)$$

$$\min\{f(X) \mid X \subseteq V\}$$

$$= \min\{\tilde{f}(x, y, z) \mid (x, y, z) : \text{Lower Extreme Point of } Z\}$$

Remark:  $\tilde{f}(x, y, z)$  is NOT concave!

# Line Arrangement



Enumerating All the Cells

Topological Sweeping Method  
Edelsbrunner, Guibas (1989)

→  $O(n^2)$