A SOLUTION OF THE PROBLEM OF VAN DEN ESSEN AND SHPILRAIN II

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Let k be a field and let $p \in k[x_1, ..., x_n]$ be a polynomial. We say that p is a coordinate polynomial, if it can be included in a generating set of cardinality n of the algebra $k[x_1, ..., x_n]$. Of course, if p is a coordinate polynomial, then it is irreducible and its zero-set is isomorphic to k^{n-1}.

Let $\phi : k[x_1, ..., x_n] \to k[x_1, ..., x_n]$ be a polynomial automorphism. It is easy to see that if f is a coordinate polynomial, then so is $\phi(f)$. Conversely, Arno van den Essen and Vladimir Shpilrain in the paper [2] have stated the following:

Problem 1. *Let k be a field of characteristic zero. Is it true that every endomorphism of $k[x_1, ..., x_n]$ taking any coordinate polynomial to a coordinate one is actually an automorphism?*

This is an interesting problem which has connection with the famous Jacobian Conjecture. In the paper [5] Problem 1 was solved in the affirmative for the complex field $\mathbb{C}$. Naturally, one can ask the following refined version of Problem 1:

Problem 2. *Let k be a field of characteristic zero. Is it true that every endomorphism of $k[x_1, ..., x_n]$ taking any linear polynomial to a coordinate one is actually an automorphism?*

For $k = \mathbb{C}$ this problem is still unsolved. When k is not algebraically closed, however, a counterexample to this problem was already constructed by Mikhalev-Yu-Zolotykh ([12]). This result suggests that the situation may be largely different depending on whether k is algebraically closed or not. From this point of view a full solution of Problem 1 seems to be interesting and important.

Here we modify our old approach and using a recent result of Li Y. and Yu J. T. we obtain a full solution of Problem 1:

Theorem. *Let k be a field of characteristic zero. Let ϕ be an endomorphism of $k[x_1, ..., x_n]$. Assume that ϕ takes any coordinate polynomial to a coordinate one. Then ϕ is an automorphism.*

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