Transitivity of automorphisms of Gizatullin surfaces

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The description of automorphism groups of affine algebraic surfaces poses an open problem. Danilov and Gizatullin studied in their pioneer works [DG1] and [DG2] normal affine surfaces, which can be completed by a zigzag, i. e. a chain of smooth rational curves. A classical result on Gizatullin surfaces states, that the automorphism group acts on the surface with a big orbit, i. e. this action admits an open orbit with a finite complement. Besides $\mathbb{C}^* \times \mathbb{C}^*$ these are the only surfaces with this property. Thus, the action of the automorphism group on other surfaces cannot be transitive.
There have been many approaches to study Gizatullin surfaces and their the automorphism group. Flenner, Kaliman and Zaidenberg showed that Gizatullin surfaces with a distinguished and rigid extended divisor have at most two conjugacy classes of $\mathbb{A}^1$-fibrations (cf. [FKZ1]) and also gave a full description of the conjugacy classes of $\mathbb{A}^1$-fibrations on special Gizatullin surfaces (cf. [FKZ2]). Blanc and Dubouloz associated to every Gizatullin surface V a (not necessary finite) graph $\mathcal{F}_V$, which reflects the structure of the automorphism group of V. They showed that, except of Gizatullin surfaces with boundary $[[0, -1, (-2)_n]]$, $\mathrm{Aut}(V)$ is generated by automorphisms of $\mathbb{A}^1$-fibrations if and only if $\mathcal{F}_V$ is a tree (cf. [BD]).
Clearly, in the singular case the action of the automorphism group cannot be transitive. Considering some smooth Gizatullin surfaces like the affine plane, the Danielewski surfaces $\{xy - P(z) = 0\} \subseteq \mathbb{A}^3$ or the special Gizatullin surfaces, one can show that this action is always transitive. The following question arises:

Question: Are there smooth Gizatullin surfaces with a non-transitive action of the automorphism group?

I discovered a class of smooth Gizatullin surfaces with the desired property. Examples of such surfaces yield a subclass of Gizatullin surfaces with a distinguished and rigid extended divisor (cf. theorem below). These surfaces have the pleasant property, that they admit at most two projective models and, if the boundary divisor $D^{\geq 2}$ or the configuration invariant is not symmetric, the automorphism group is generated by automorphisms of $\mathbb{A}^1$-fibrations.

Denoting by $G(A) = \{\alpha \in \mathbb{C}^* \mid \alpha \cdot A = A\}$ for a finite set $A \subseteq \mathbb{C}^*$, the following result can be shown:

Theorem 1. *Let V be a smooth Gizatullin surface with a distinguished and rigid extended divisor. Denoting by $(X, D = C_0 \triangleright \cdots \triangleright C_n)$ a standard completion of V and by $A_i = \{P_{i,1}, \cdots, P_{i,r_i}\} \subseteq C_i \backslash (C_{i-1} \cup C_{i+1}) \cong \mathbb{C}^*$, $3 \leq i \leq n-1$, the base point sets of the feathers $F_{i,j}$, $1 \leq j \leq r_i$, the following holds:*

(1) For $4 \leq i \leq n-2$ we let $B_{i,1}, \ldots, B_{i,k_i}$ be the orbits of the $G(A_i)$-action on A_i and

$$O_{i,j} := \bigcup_{1 \leq l \leq r_i; P_{i,l} \in B_{i,j}} F_{i,l} \cap F_{i,l}^{\vee} \quad , \quad 1 \leq j \leq k_i,$$

as well as

$$O_0 := V \setminus \left(\bigcup_{4 \le i \le n-2, 1 \le j \le r_i} F_{i,j} \cap F_{i,j}^{\vee} \right).$$

The set O_0 *is the big orbit of the natural action of* $\mathrm{Aut}(V)$ *on* V *and the sets* $O_{i,j}$ *are invariant under* $\mathrm{Aut}(V)$*.*

(2) *If* $F(V)$ *is the fix point set of the natural action of* $\mathrm{Aut}(V)$ *on* V*, then*

$$\bigcup_{4 \le i \le n-2, k_i = r_i, 1 \le j_i \le r_i} F_{i,j_i} \cap F_{i,j_i}^{\vee} \subseteq F(V).$$

(3) *If at most two of the* r_i *are non-zero, then the sets* O_0 *and the* $O_{i,j}$ *are the orbits of* $\mathrm{Aut}(V)$ *and in assertion (2) equality holds.*

It is a classical result that the automorphism groups of the affine plane, the Danielewski surfaces and toric surfaces can be written as an amalgamated product of two automorphism subgroups. For Gizatullin surfaces with a distinguished and rigid extended divisor there is also a similar presentation af the automorphism group.

References:

[DG1] V. I. Danilov, M. H. Gizatullin, *Automorphisms of affine surfaces I*, Izv. Akad. Nauk SSSR Ser. Mat. 39: 3 (1975), 148 - 166.

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[FKZ1] H. Flenner, S. Kaliman, M. Zaidenberg, *Uniqueness of* $\mathbb{C}^*$*- and* $\mathbb{C}_+$*-actions on Gizatullin surfaces*, arXiv: 0706.2261v1.

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