Automorphism Groups of Affine Varieties and Vector Fields

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"Groups of Automorphisms in Birational and Affine Geometry"

Levico Terme, October 29 - November 3, 2012

Notation

1. Our base field is $\mathbb{C}$, the field of complex numbers;
2. $\mathcal{O}(X)$ is the algebra of regular functions on the variety X;
3. The topology is always the ZARISKI-topology;

Ind-varieties and ind-groups

Definition ("Infinite dimensional variety", SHAFAREVICH 1966)

An *ind-variety* is a set $\mathfrak{V}$ together with a filtration $\mathfrak{V}_1 \subset \mathfrak{V}_2 \subset \cdots \subset \mathfrak{V}$ such that

1. $\mathfrak{V} = \bigcup_i \mathfrak{V}_i$,
2. Every $\mathfrak{V}_i$ is a variety, and $\mathfrak{V}_i \subset \mathfrak{V}_{i+1}$ is closed.

Definition (continued)

- Affine ind-varieties;
- Morphisms of ind-varieties, products, ...;
- Topology of an ind-variety:
  $A \subseteq \mathfrak{V}$ closed $:\Longleftrightarrow$ $A \cap \mathfrak{V}_k \subseteq \mathfrak{V}_k$ closed for all k;
- Ind-groups $\mathfrak{G}$.

Examples of ind-varieties and ind-groups

Example (Ind-varieties)

- Vector spaces of countable dimension, e.g. $\mathrm{Mor}(X, \mathbb{A}^n) = \mathcal{O}(X)^n$, X an affine variety;
- Locally closed subsets of ind-varieties, e.g. $\mathrm{Mor}(X, Y)$, X, Y affine varieties;
- Countable set: *discrete* ind-varieties.

Theorem

Let X be an affine variety. Then $\mathrm{Aut}(X)$ *has the structure of an affine ind-group, with the usual universal properties.*

Automorphism group of affine *n*-space

For $g \in \mathrm{Aut}(\mathbb{A}^n)$, $g = (g_1, \ldots, g_n)$, define $\deg g := \max_i\{\deg g_i\}$ and put

$$\mathrm{Aut}(\mathbb{A}^n)_k := \{g \in \mathfrak{G}(n) \mid \deg g \leq k\}.$$

Lemma

$\mathrm{Aut}(\mathbb{A}^n)_k \subset \mathrm{End}(\mathbb{A}^n)_k$ *is locally closed and affine.*

In general, we show that $\mathrm{Aut}(X) \subset \mathrm{End}(X) \times \mathrm{End}(X)$ is locally closed and affine.

We do not know if $\mathrm{Aut}(X) \subset \mathrm{End}(X)$ is locally closed!

Example (Ind-groups)

- $\mathrm{GL}(\mathbb{C}[x_1, \ldots, x_n])$;
- Closed subgroups of ind-groups, e.g. the standard constructions like center, centralizer, normalizer, ...
- Countable groups, e.g. $\mathbb{Z}^n$, or any character group $X(G)$;
- "Discrete" subgroups of $\mathfrak{G}$, e.g. the braid group B_3 appears as a discrete subgroup of $\mathrm{Aut}(\mathbb{A}^3)$.

Remarks and Questions

- For a "general" X the group $\mathrm{Aut}(X)$ is trivial.
- Every finite group appears as $\mathrm{Aut}(X)$ (cf. JELONEK, 1992).
- Is there an X with $\mathrm{Aut}(X)$ discrete, or $\simeq \mathbb{Z}$?

Locally finite subsets

Let V be any $\mathbb{C}$-vector space. Define

- $\mathrm{LEnd}(V)$, the algebra of linear endomorphisms of V;
- $\mathrm{GL}(V)$, the group of linear automorphisms.

Definition

$S \subset \mathrm{LEnd}(V)$ *locally finite* $:\Longleftrightarrow$ every $v \in V$ is contained in a finite dimensional S-stable subspace.

Remark

$S \subset \mathrm{GL}(V)$ locally finite $:\Longleftrightarrow$ $\langle S \rangle \subset \mathrm{GL}(V)$ locally finite.

Algebraic elements

From now on the varieties are affine, and $\mathfrak{G} = \bigcup_k \mathfrak{G}_k$ denotes an affine ind-group.

Definition

- $\varphi \in \mathrm{End}(X)$ *algebraic* $:\Longleftrightarrow$ $\varphi^* \in \mathrm{LEnd}\,\mathcal{O}(X)$ locally finite.
- $g \in \mathfrak{G}$ *algebraic* $:\Longleftrightarrow$ $\langle g \rangle$ bounded degree, i.e. $\langle g \rangle \subset \mathfrak{G}_k$ for some k $\Longleftrightarrow$ $\overline{\langle g \rangle}$ is an algebraic group.

- $\overline{\langle g \rangle} \simeq D$, $\mathbb{C}^+$ or $D \times \mathbb{C}^+$, where D is a *diagonalizable* group (i.e. closed subgroup of a torus) with D/D^0 cyclic;
- Can define *unipotent* and *semisimple* elements;
- Have a *Jordan decomposition* $g = g_s \cdot g_u$, well-behaved under homomorphisms.

Algebraically generated groups

Definition

A subgroup $G \subseteq \mathfrak{G}$ generated by a family $(G_i)_{i \in I}$ of connected algebraic subgroups $G_i \subset \mathfrak{G}$ is called *algebraically generated*.

Proposition (Arzhantsev-Flenner-Kaliman-et al)

Let $G = \langle G_i \mid i \in I \rangle \subseteq \mathrm{Aut}(X)$. Then every G-orbit is open in its closure, and there is a finite sequence $j_1, \ldots, j_m$ such that $Gx = G_{j_1} \cdots G_{j_m} x$ for all $x \in X$.

Corollary

G and its closure $\overline{G}$ have the same orbits on X.

The DE JONQUIÈRE subgroup

$$\mathfrak{J}(n) := \{g = (g_1, \dots, g_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid g_i = g_i(x_i, x_{i+1}, \dots, x_n)\}$$

It follows that $g_i = a_i x_i + h_i(x_{i+1}, \dots, x_n)$.

Proposition (FURTER)

The subgroup $\langle \mathfrak{J}(n)_k \rangle$ is bounded. In particular, $\mathfrak{J}(n) = \bigcup_k \overline{\langle \mathfrak{J}(n)_k \rangle}$ is a union of closed algebraic subgroups.

Question

Assume that every element of the ind-group $\mathfrak{G}$ is algebraic. Is then $\mathfrak{G}$ a union of closed algebraic subgroups?

We have some partial results (H.K. & IMMANUEL STAMPFLI).

The Lie algebra of an ind-group

For an ind-group $\mathfrak{G} = \bigcup \mathfrak{G}_k$ the space

$$\operatorname{Lie} \mathfrak{G} := \bigcup T_e \mathfrak{G}_k$$

has the structure of a Lie algebra with the usual properties.

If $\mathfrak{G} = \operatorname{Aut}(X)$, then there is a canonical homomorphism

$$\nu \colon \operatorname{Lie} \mathfrak{G} \to \operatorname{Vec}(X), \quad A \mapsto -\xi_A,$$

where $\operatorname{Vec}(X)$ the Lie algebra of algebraic vector fields on X. (As usual, we use the orbit map $\mu_x \colon \mathfrak{G} \to X$, $g \mapsto gx$, to define $\xi_A(x) := (d\mu_x)_e(A)$.)

Vector fields

Example

$$\mathrm{Vec}(\mathbb{A}^n) = \{\xi = \sum_i f_i \frac{\partial}{\partial x_i} \mid f_i \in \mathbb{C}[x_1, \ldots, x_n]\}$$

Proposition

The map $\nu\colon \mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Vec}(\mathbb{A}^n)$ *induces an isomorphism of* $\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$ *with*

$$\mathrm{Vec}^c(\mathbb{A}^n) := \{\xi = \sum_i f_i \frac{\partial}{\partial x_i} \mid \operatorname{div} \xi \in \mathbb{C}\} \subset \mathrm{Vec}(\mathbb{A}^n)$$

where $\operatorname{div} \xi := \sum_i \frac{\partial f_i}{\partial x_i}$.

What happens in general?

Theorem (K-Zaidenberg)

The map $\nu\colon \operatorname{Lie}\operatorname{Aut}(X) \to \operatorname{Vec}(X)$ *is injective.*

Question

What is the relation between closed ind-subgroups of $\operatorname{Aut}(X)$ and Lie subalgebras of $\operatorname{Vec}(X)$?

Theorem (Cohen-Draisma, 2003)

Let $L \subset \operatorname{Vec}(X)$ *be a finite dimensional Lie subalgebra. Then* $L \subset \operatorname{Lie} G$ *for an algebraic group* $G \subset \operatorname{Aut}(X)$ *if and only if* L *is locally finite, as a subset of* $\operatorname{LEnd}(\mathcal{O}(X))$.

Example

If $m \in \mathbb{C}[x_1, \ldots, x_n]$ is a monomial not containing x_i, then

$$u_{i,m} := (x_1, \ldots, x_i + m, \ldots, x_n) \in \mathrm{End}(\mathbb{A}^n)$$

is a unipotent automorphism of $\mathbb{A}^n$. It defines a subgroup $U_{i,m} := \overline{\langle u_{i,m} \rangle}$ isomorphic to $\mathbb{C}^+$, and

$$U_{i,m} \subset \mathrm{SAut}(\mathbb{A}^n) := \{g \in \mathrm{Aut}(\mathbb{A}^n) \mid \mathrm{Jac}\, g = 1\}.$$

Lemma

The vector fields $m\frac{\partial}{\partial x_i}$ associated to the subgroups $U_{i,m}$ generate $\mathrm{Vec}^0(\mathbb{A}^n)$. Hence $\mathrm{Lie}\, \overline{\langle U_{i,m} \rangle} = \mathrm{Lie}\, \mathrm{SAut}(\mathbb{A}^n)$.

Question

Is $\overline{\langle U_{i,m} \rangle} = \mathrm{SAut}(\mathbb{A}^n)$?

Put $U := \{(x + sy^2, y)\}$, $V := \{(x, y + sx^2)\}$, both $\simeq \mathbb{C}^+$. Then

- $G = \langle U, V \rangle = U * V \Rightarrow$ only algebraic subgroups are U, V.
- $L := \langle x^2 \frac{\partial}{\partial y}, y^2 \frac{\partial}{\partial x} \rangle$ contains many locally nilpotent VF.

Proposition (Andriy Regeta)

The following are equivalent:

(i) *All Lie subalgebras $L \subset \mathrm{Vec}^0(\mathbb{A}^2)$ isomorphic to $\mathfrak{sl}_2$ are conjugate under $\mathrm{Aut}(\mathbb{A}^2)$.*

(ii) *The Jacobian conjecture holds for $\mathbb{A}^2$.*

Actions of ind-groups

If the ind-group $\mathfrak{G}$ acts on the affine variety X, $\rho\colon \mathfrak{G} \to \operatorname{Aut}(X)$, we get a canonical homomorphism

$$d\rho\colon \operatorname{Lie}\mathfrak{G} \to \operatorname{Vec}(X).$$

Questions

- Is $\operatorname{Ker} d\rho = \operatorname{Lie}\operatorname{Ker}\rho$?
- Is the action ρ determined by $d\rho$ in case $\mathfrak{G}$ is connected?
- What can we say about the image of $d\rho$, e.g. is $\operatorname{Im} d\rho = \operatorname{Lie}\overline{\rho(\mathfrak{G})}$?

RAMANUJAM-connected subgroups

Definition (RAMANUJAM 1964)

$Y \subseteq \mathfrak{V}$ is *R-connected* $:\Longleftrightarrow$ for $y_1, y_2 \in Y$ there is an irreducible algebraic subvariety $C \subset \mathfrak{V}$ such that $y_1, y_2 \in C \subset Y$.

E.g. algebraically generated subgroups are R-connected.

Proposition (RAMANUJAM 1964)

Let $G \subseteq \mathrm{Aut}(X)$ be a R-connected subgroup. Then one of the following holds:

(i) *G is a closed algebraic subgroup;*
(ii) *G contains algebraic subvarieties of arbitrary large dimension.*

The Lie algebra of an algebraically generated group

Proposition (K-ZAIDENBERG)

Let $G = \langle G_i \mid i \in I \rangle \subseteq \operatorname{Aut}(X)$ *be algebraically generated, and let* $L \subseteq \operatorname{Vec}(X)$ *be the Lie algebra generated by the* $\operatorname{Lie} G_i$, $i \in I$. *Then*

1. L *is stable under* G *and* $\overline{G}$;
2. L *depends only on* G *and not on the generating subgroups* G_i;
3. $L \subseteq \operatorname{Lie} \overline{G}$, *and this is an ideal;*
4. *Every vector field in* L *is tangent to the* G*-orbits.*

Questions

$G = \langle G_i \mid i \in I \rangle \subseteq \overline{G} \subseteq \mathrm{Aut}(X)$, $L = \langle \mathrm{Lie}\, G_i \rangle \subseteq \mathrm{Vec}(X)$.

- Is $L = \mathrm{Lie}\, \overline{G}$?
- Does $\mathrm{Lie}\, H \subseteq L$ imply that $H \subseteq \overline{G}$?

Theorem (K-Zaidenberg)

Let $G = \langle G_i \mid i \in I \rangle \subseteq \mathrm{Aut}(X)$ *and* $L = \langle \mathrm{Lie}\, G_i \rangle \subseteq \mathrm{Vec}(X)$ *be as above. Then*

$$L \text{ is finite dimensional} \iff G \text{ is an algebraic group}$$

And then $L = \mathrm{Lie}\, G$.