Unirationality and existence of infinitely transitive models

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The subject described below deals both with birational geometry and with affine algebraic geometry. So first we recall some basic notions and facts from these domains, and then pass to our main question. The ground field is assumed to be algebraically closed of characteristic 0.

1. Unirationality and rational connectedness

Definition. A variety X of dimension n is said to be *rational* (resp. *unirational*) if there exists a birational (resp. dominant) morphism $\mathbb{P}^n \dashrightarrow X$.

Definition. A variety X is said to be *rationally connected* if for every two generic points in X there exists an irreducible rational curve in X joining these points.

If X is assumed to be smooth, this condition is equivalent to the fact that every two generic points can be joined by a chain of rational curves.

Clearly, every unirational variety is rationally connected. However nobody knows an example of a variety which is rationally connected but not unirational.

Recall that X is Fano if $-K_X$ is ample.

Theorem 1 ([4], [8], [11])**.** *Every Fano variety (over an algebraically closed field of characteristic zero) is rationally connected.*

By Kollar [7], the hypersurfaces of degree d in $\mathbb{P}^{n+1}$ for $\frac{2}{3}(n+3) < d \leqslant n$ are rationally connected but not rational. One may expect them to be such examples of rationally connected non-unirational varieties.

2. Infinite transitivity of affine varieties

In this section we let X be an affine algebraic variety. By $\mathrm{Aut}(X)$ we mean the group of algebraic automorphisms of X. We are interested in one-parameter unipotent subgroups H in $\mathrm{Aut}(X)$. Such a group is isomorphic (as an abstract group) to $(\Bbbk, +)$, and its action on X can be given via a locally nilpotent derivation on $\Bbbk[X]$ (see [5]). This makes their use extremely convenient.

Definition. X is said to be flexible if for every smooth point $x \in X_{\mathrm{reg}}$ the tangent space T_xX is generated by the tangent vectors to the orbits of $\{H \subset \mathrm{Aut}(X) \mid H \simeq (\Bbbk, +)\}$.

Definition. By $\mathrm{SAut}(X)$ we mean a subgroup in $\mathrm{Aut}(X)$ generated by all $\{H \subset \mathrm{Aut}(X) \mid H \simeq (\Bbbk, +)\}$.

Definition. An affine variety X is said to be *infinitely transitive* if for every $m \in \mathbb{N}$ and every two m-tuples of points $(P_1, \ldots, P_m)$ and $(Q_1, \ldots, Q_m)$ in X_{reg} there exists an automorphism $g \in \mathrm{SAut}(X)$ such that for all i we have $g(P_i) = Q_i$.

Theorem 2 ([1])**.** *For an affine variety X of dimension at least 2, the following conditions are equivalent:*

(i) X is flexible;

(ii) $\mathrm{SAut}(X)$ *acts on X transitively;*

(iii) $\mathrm{SAut}(X)$ *acts on X infinitely transitively.*

Let us give some examples of infinitely transitive varieties.

Example 1. Suspension over the affine space (see [6]). Let $f \in \Bbbk[x_1, \ldots, x_k]$ be a nontrivial polynomial, $k \geqslant 2$. Then the hypersurface $\{uv - f(x_1, \ldots, x_k) = 0\} \subset \mathbb{A}^{k+2}$ is infinitely transitive.

Example 2. Normal affine cones over flag varieties (see [2]). Consider a flag variety G/P, where G is a semisimple algebraic group, and P is a parabolic subgroup in G. Then every normal affine cone X over G/P is flexible and its special automorphism group $\mathrm{SAut}(X)$ acts infinitely transitively on the smooth locus X_{reg}.

Example 3. The analogous conclusion holds if X is any non-degenerate affine toric variety of dimension at least 2 (see [2]).

Example 4. Suspensions over infinite transitive varieties (see [2]). Suppose that an affine variety X is flexible and either $X = \mathbb{A}^1$, or $\dim X \geq 2$ and the special automorphism group $\mathrm{SAut}(X)$ acts infinitely transitively on the smooth locus X_{reg}. Then any suspension over X also has properties of flexibility and infinite transitivity of the special automorphism group. As above, we mean by suspension the subvariety $\overline{X} \subset X \times \mathbb{A}^2$ given by the equation $uv = f(\bar{x})$, $f \in \Bbbk[X]$ a non-constant function.

Example 5 ([9])**.** Affine cones over del Pezzo surfaces of degree 4 and 5 are infinitely transitive.

But not all infinitely-transitive varieties are rational:

Example 6. Homogeneous spaces (see [1]). There exists a finite subgroup F of $SL(4)$ such that the quotient space $SL(4)/F$ is not rational (and even not stably rational). Due to [1] and [10] such a variety is flexible, hence infinitely transitive.

Theorem 3 ([1])**.** *Every infinitely transitive variety is unirational.*

3. Stable infinite transitivity

Definition. An algebraic variety Y is said to be stably infinitely transitive if there exists $n \in \mathbb{N}$ and a birational model X of $Y \times \Bbbk^n$ which is affine and infinitely transitive.

Conjecture 4. *Any unirational variety X is stably birationally infinitely transitive.*

Every stably infinitely transitive variety is rationally connected. Our main motivation is to distinguish unirational varieties among rationally connected ones.

Theorem 5 ([3])**.** *Let Y be a smooth projective variety of dimension n. Assume that there exist n morphisms $\pi_i : Y \to Z_i$ (we name them cancellations) satisfying the following:*

1. *for each i, Z_i is a normal projective variety, the general fiber of π_i is isomorphic to $\mathbb{P}^1$, and such that π_i admits a section over an open subset in Z_i;*
2. *for the general point $y \in Y$, the tangent vectors to the fibers of π_i $T_{F_{1},y}, \ldots, T_{F_{n},y}$ span the tangent space $T_y(Y)$.*

Then X is stably birationally infinitely transitive.

The second condition is similar to the flexibility property.

4. Examples

Let us present some examples of stably birationally infinitely transitive varieties.

Example 7 (see [3])**.** Let us start with the projective space $\mathbb{P}^n$, $n \geq 2$, and a finite group $\Gamma \subset PGL_{n+1}(\Bbbk)$. Notice that the quotient $\mathbb{P}^n/\Gamma$ is stably birationally infinitely transitive. Indeed, let us replace Γ by its finite central extension $\tilde{\Gamma}$ acting linearly on $V := \Bbbk^{n+1}$, so that $V/\tilde{\Gamma} \approx \mathbb{P}^n/\Gamma \times \mathbb{P}^1$. Further, form the product $V \times V$ with the diagonal $\tilde{\Gamma}$-action, and take the quotient $V' := (V \times V)/\tilde{\Gamma}$. Then, projecting on the first factor we get $V' \approx V \times V/\tilde{\Gamma}$, and similarly for the second factor. This implies that V' admits $2n + 2$ cancellations (cf. Theorem 5). Hence V' is stably birationally infinitely transitive by Theorem 5.

Example 8 (see [3])**.** *Cubic hypersurfaces.* Let $X_3 \subset \mathbb{P}^{n+1}$, $n \geq 2$, be a smooth cubic. Then it is stably birationally infinitely transitive.

Example 9 (see [3])**.** *Quartic hypersurfaces.* Let $X_4 \subset \mathbb{P}^n$, $n \geq 4$, be a quartic hypersurface with a line $L \subset X_4$ of double singularities. Then X_4 is stably birationally infinitely transitive.

Example 10 (see [3])**.** *Complete intersections.* Let $X_{2\cdot 2\cdot 2} \subset \mathbb{P}^6$ be a smooth complete intersection of three quadrics. Then $X_{2\cdot 2\cdot 2}$ is stably birationally infinitely transitive.

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