Notes for the Trento School *Geometry of Special Varieties*
Trento, 10–13/9/2007

Francesco Russo
Departamento de Matemática
Universidade Federal de Pernambuco
Cidade Universitária
50670-901 Recife–PE (Brasil)
frusso@dmat.ufpe.br

Preface

The aim of these notes is to provide an introduction to some classical and recent results and techniques in projective algebraic geometry. We treat the geometrical properties of varieties embedded in projective space, their secant and tangent lines, the behaviour of tangent linear spaces, the algebro-geometric and topological obstructions to their embedding into smaller projective spaces, the classification in the extremal cases.

These are classical themes in algebraic geometry and the renewed interest at the beginning of the '80 of the last century came from some conjectures posed by Hartshorne, [**H2**], from an important connectedness theorem of Fulton and Hansen, [**FH**], and from its new and deep applications to the geometry of algebraic varieties, as shown by Fulton, Hansen, Deligne, Lazarsfeld and Zak, [**FH**], [**FL**], [**D2**], [**Z2**].

We shall try to illustrate these themes and results during the course and with more details through these notes, also pointing out simple proofs of some important theorems and some new results via the theory of deformations of rational curves on algebraic varieties (Mori's Theory) and via the theory of degenerations, see [**CMR**], [**CR**], [**Ru2**], [**IR1**], [**IR2**].

A standard reference on some topics treated here is [**Z2**], which influenced the presentation of some parts of the book, altough the proofs and the general philosophy of important classification results differ substantially from Zak's original ones.

Ringraziamenti

Innanzitutto sono molto riconoscente agli organizzatori della Scuola/Workshop, Gianfranco Casnati, Claudio Fontanari, Roberto Notari e Sonia Spreafico per avermi invitato ad offrire questo corso e al CIRM per finanziare questa iniziativa.

Il CNPq (Conselho Nacional de Desenvolmimento Cientifico e Tecnologico do Brasil) e il PRONEX-Algebra Comutativa e Geometria Algebrica hanno finanziato fin troppo generosamente negli ultimi otto anni vari miei progetti di ricerca su questi argomenti, sia come borsista, sia con fondi diretti e grant di vario tipo. Esprimo qui la mia gratitudine per la fiducia concessa, spero almeno parzialmente ricambiata.

Quanto scritto in queste note é frutto di quanto appreso da vari amici, colleghi e professori e, in molte occasioni, dai *classici*, antichi e moderni. Un grazie speciale a Ciro (Ciliberto) e Paltin (Ionescu) per l' interazione intensa in questi ultimi anni. Molti dei risultati originali presentati in queste note sono frutto del nostro lavoro o di nostre discussioni.

Recife, 7 settembre 2007 Francesco

Contents

Introduction

Let us quote excerpts from Hilbert presentation of projective geometry in [**HCV**]:

"*..... we shall learn about geometrical facts that can be formulated and proved without any measurement or comparison of distances or of angles. It might be imagined that no significant properties of a figure could be found if we do without measurement of distances and angles and that only vague statements could be made. And indeed research was confined to the metrical side of geometry for a long time, and questions of the kind we shall discuss in this chapter arose only later, when the phenomena underlying perspective painting were being studied scientifically. Thus, if a plane figure is projected from a point onto another plane, distances and angles are changed, and in addition, parallel lines may be changed into lines that are not parallel; but certain essential properties must nevertheless remain intact, since we could not otherwise recognize the projection as being a true picture of the original figure. In this way, the process of projecting led to a new theory, which was called projective geometry because of its origins. Since the 19th century, projective geometry has occupied a central position in geometric research. With the introduction of homogeneous coordinates, it became possible to reduce the theorems of projective geometry to algebraic equations in much the same way that Cartesian coordinates allow this to be done for the theorems of metric geometry. But projective analytic geometry is distinguished by the fact that it is far more symmetrical and general than metric analytic geometry, and when one wishes, conversely, to interpret higher algebraic relations geometrically, one often transforms the relations into homogeneous form and interprets the variables as homogeneous coordinates, because the metric interpretation in Cartesian coordinates would be too unwieldy.*"

Classical algebraic Geometers, antique and modern, taught and teach to us also to experiment the *live rapport* with *the objects one studies* and showed us the *concrete intuition*, described by Hilbert in his preface to the book *Geometry and the Imagination*, [**HCV**]:

"*In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations. As to geometry, in particular, the abstract tendency has here led to the magnificent systematic theories of Algebraic Geometry, of Riemannian Geometry, and of Topology; these theories make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is still as true today as it ever was that intuitive understanding plays a major role in geometry. And such concrete intuition is of great value not only far the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry. In this book, it is our purpose to give a presentation of geometry, as it stands today, in its visual, intuitive aspects. With the aid of visual imagination we can illuminate the manifold facts and problems of geometry, and beyond this, it is possible in many cases to depict the geometric outline of the methods of investigation and proof, without necessarily entering into the details connected with the strict definitions of concepts and with the actual calculations.*

In this manner, geometry being as many-faceted as it is and being related to the most diverse branches of mathematics, we may even obtain a summarizing survey of mathematics as a whole, and a valid idea of the variety of its problems and the wealth of ideas it contains."

I wish to end this introduction by also quoting the beginning of the book of Georg R. Kempf, [**Ke1**]:

"*Algebraic geometry is a mixture of the ideas of two Mediterrean cultures. It is the superposition of the Arab science of the lightning calculation of the solutions of equations over the Greek art of position and shape. This tapestry was originally woven by on European soil and is still being refined under the influence of international fashion. Algebraic geometry studies the delicate balance between the geometrically plausible and the algebraic possible. Whenever one side of this mathematical teeter–tooter outweighs the other, one immediately loses interest and runs off in search of a more exciting amusement*".

CHAPTER 1

Tangent cones, tangent spaces, tangent stars; secant, tangent and tangent star variety to an algebraic variety

1.1. Tangent cones to an algebraic variety and associated varieties

Let X be an algebraic variety, or more generally a scheme of finite type, over a fixed algebraically closed field K. Let $x \in X$ be a closed point. We briefly recall the definitions of *tangent cone to X at x* and of *tangent space to X at x*. For more details one can consult [**Mu**] or [**Sh**].

1.1.1. DEFINITION. (**Tangent cone at a point**). Let $U \subset X$ be an open affine neighbourhood of x, let $i : U \to \mathbb{A}^N$ be a closed immersion and let U be defined by the ideal $I \subset K[X_1, \ldots, X_N]$. There is no loss of generality in supposing $i(x) = (0, \ldots, 0) \in \mathbb{A}^N$. Given $f \in K[X_1, \ldots, X_N]$ with $f(0, \ldots, 0) = 0$, we can define the *leading form* of f, f^*, as the lower degree homogeneous polynomial in its expression as a sum of homogenous polynomials in the variables X_i's. Let

$$I^* = \{\text{the ideal generated by the "leading form"} f^*, \text{ for all } f \in I\}.$$

Then

$$\mathcal{C}_x X := \operatorname{Spec}(K[X_1, \ldots, X_N]/I^*),$$

is called *the affine tangent cone to X at x*.

It could seem that it depends on the choice of $U \subset X$ and on the choice of $i : U \to \mathbb{A}^N$. It is not the case because if $(\mathcal{O}_x, m_x)$ is the local ring of regular functions of X at x, then it is immediate to see that

$$(k[X_1, \ldots, X_N]/I^*) \simeq \operatorname{gr}(\mathcal{O}_x) := \bigoplus_{n \geq 0} \frac{m_x^n}{m_x^{n+1}}.$$

This fact simply says that we can calculate $\mathcal{C}_x X$ by choosing an arbitrary set of generators of I and moreover that the definition is *local*. It should be noticed that $\mathcal{C}_x X$ is a scheme, which can be neither irreducible nor reduced as the examples of plane cubic curves with a node and with a cusp show. We now get a geometrical interpretation of this cone and see some of its properties.

Since $\mathcal{C}_x X$ is *locally* defined by homogeneous forms, it can be naturally projectivized and thought as a subscheme of $\mathbb{P}^{N-1} = \mathbb{P}(\mathbb{A}^N)$. If we consider the blow-up of $x \in U \subset \mathbb{A}^N$, $\pi : \operatorname{Bl}_x U \to U$, then $\operatorname{Bl}_x U$ is naturally a subscheme of $U \times \mathbb{P}^{N-1} \subset \mathbb{A}^N \times \mathbb{P}^{N-1}$ and the exceptional divisor $E := \pi^{-1}(x)$ is naturally a subscheme of $x \times \mathbb{P}^{N-1}$. With these identifications one shows that $E \simeq \mathbb{P}(\mathcal{C}_x X) \subset \mathbb{P}^{N-1}$ as schemes, see [**Mu**, pg. 225]. In particular, if X is equidimensional at x, then $\mathcal{C}_x X$ is an equidimensional scheme of dimension $\dim(X)$. Moreover, we deduce the following geometrical definition:

$$\mathcal{C}_x X = \overline{\bigcup_{y \in U} \lim_{y \to x} < y, x >}.$$

The cone $\mathcal{C}_x X$ can also be described geometrically in this way, see [**Sh**].

If $X \subset \mathbb{P}^N$ is quasi-projective, we define *the projective tangent cone to X at x*, indicated by $C_x X$, as the closure of $\mathcal{C}_x X \subset \mathbb{A}^N$ in $\mathbb{P}^N$, where $x \in U = \mathbb{A}^N \cap X$ is a suitable chosen affine neighbourhood. The same geometrical definition holds, remembering of the scheme structure,

$$C_xX = \overline{\bigcup_{y\in U} \lim_{y\to x} < y, x >} \subset \mathbb{P}^N.$$

We now recall the definition of tangent space to X at $x \in X$.

1.1.2. DEFINITION. (**Tangent space at a point; Tangent variety to a variety**). Let notation be as in the previous definition. Given $f \in K[X_1, \ldots, X_N]$ with $f(0, \ldots, 0) = 0$, we can define the *linear term* of f, f^{lin}, as the degree one homogeneous polynomial in its expression as a sum of homogenous polynomials in the variables X_i's. In other words, $f^{\text{lin}} = \sum_{i=1}^N \frac{\partial f}{\partial X_i}(\mathbf{0})X_i$. Let

$$I^{\text{lin}} = \{\text{the ideal generated by the } \textit{linear terms} f^{\text{lin}}, \text{ for all } f \in I\}.$$

Then

$$\mathcal{T}_xX := \operatorname{Spec}(K[X_1, \ldots, X_N]/I^{\text{lin}}),$$

is called *the affine tangent space to X at x*.

Geometrically it is the locus of *tangent lines to X at x*, where a line through x is tangent to X at x if it is tangent to the hypersurfaces $V(f) = 0$, $f \in I$, i.e. if the multiplicity of intersection of the line with $V(f)$ at $(0, \ldots, 0)$ is greater than one. In particular this locus is a linear subspace of $\mathbb{A}^N$ being an intersection of linear subspaces.

Since $I^{\text{lin}} \subseteq I^*$, we deduce the inclusion of schemes

$$\mathcal{C}_xX \subseteq \mathcal{T}_xX;$$

and that $\mathcal{T}_xX$ is the smallest *linear* subscheme of $\mathbb{A}^N$ containing $\mathcal{C}_xX$ as a subscheme (and not only as a set!). In particular for every $x \in X$ $\dim(\mathcal{T}_xX) \geq \dim(X)$ holds.

We recall that a point $x \in X$ is non-singular if and only $\mathcal{C}_xX = \mathcal{T}_xX$. Since $\mathcal{T}_xX$ is reduced and irreducible and since $\mathcal{C}_xX$ is of dimension $\dim(X)$, we have that $x \in X$ is non-singular if and only if $\dim(\mathcal{T}_xX) = \dim(X)$.

Once again there is an intrinsic definition of $\mathcal{T}_xX$

$$(K[X_1, \ldots, X_N]/I^{\text{lin}}) \simeq S(m_x/m_x^2),$$

where $S(m_x/m_x^2)$ is the symmetric algebra of the K-vector space m_x/m_x^2.

If $X \subset \mathbb{P}^N$ is a quasi-projective variety, we define *the projective tangent space to X at x*, indicated by T_xX, as the closure of $\mathcal{T}_xX \subset \mathbb{A}^N$ in $\mathbb{P}^N$, where $x \in U = \mathbb{A}^N \cap X$ is a suitable chosen affine neighbourhood. Then T_xX is a linear projective space naturally attached to X and clearly $C_xX \subseteq T_xX$ as schemes. We also set, for a (quasi)-projective variety $X \subset \mathbb{P}^N$,

$$TX = \bigcup_{x\in X} T_xX,$$

the variety of tangents, or *the tangent variety of X*.

At a non-singular point $x \in X \subset \mathbb{P}^N$, the equality $C_xX = T_xX$ says that every tangent line to X at x is the *limit* of a secant line $< x, y >$ with $y \in X$ approaching x. For singular points this is not the case as one sees in the simplest examples of singular points of a hypersurface.

An interesting question is to investigate what are the limits of secant lines $< y_1, y_2 >$, $y_i \in X$, $y_1 \neq y_2$, when y_i, $i = 1, 2$, approaches a fixed $x \in X$. As we will immediately see for a non-singular point $x \in X$, every tangent line to X at x arises in this way but for singular points this is not the case. These limits generate a cone, the tangent star cone to X at x, which contains but does not usually coincide with C_xX (or $\mathcal{C}_xX$). From now on we restrict ourselves to the projective setting since we will not treat local questions related to tangent star cones but the situation can be "localized". Firstly we introduce the notion of secant variety to a variety $X \subset \mathbb{P}^N$.

1.1.3. DEFINITION. (**Secant varieties to a variety**). For simplicity let us suppose that $X \subset \mathbb{P}^N$ is a closed irreducible subvariety.

Let

$$S_X^0 := \{((x_1, x_2), z) \ : \ z \in < x_1, x_2 >\} \subset ((X \times X) \setminus \Delta_X) \times \mathbb{P}^N.$$

The set is locally closed so that taken with the reduced scheme structure it is a quasi-projective irreducible variety of dimension $\dim(S_X^0) = 2\dim(X)+1$. Recall that, by definition, it is a $\mathbb{P}^1$-bundle over $(X \times X) \setminus \Delta_X$, which is irreducible. Let S_X be its closure in $X \times X \times \mathbb{P}^N$. Then S_X is an irreducible projective variety of dimension $2\dim(X) + 1$, called *the abstract secant variety to* X. Let us consider the projections of S_X onto the factors $X \times X$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & S_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times X & & \mathbb{P}^N. \end{tikzcd}$$

The secant variety to X, SX, is the scheme-theoretic image of S_X in $\mathbb{P}^N$, i.e.

$$SX = p_2(S_X) = \overline{\bigcup_{x_1 \neq x_2 \, , \, x_i \in X} < x_1, x_2 >} \subseteq \mathbb{P}^N,$$

which is an irreducible algebraic variety of dimension $s(X) \leq \min\{2\dim(X) + 1, \ N\}$, the variety swept out by the secant lines to X.

Let now $k \geq 1$ be a fixed integer. We can generalize the construction to the case of $(k + 1)$-secant $\mathbb{P}^k$, i.e. to the variety swept out by the linear spaces generated by $k + 1$ independent points on X.

Define

$$(S_X^k)^0 \subset \underbrace{X \times \ldots \times X}_{k+1} \times \mathbb{P}^N$$

as the locally closed irreducible set

$$(S_X^k)^0 := \{((x_0, \ldots, x_k), z) \ : \ \dim(< x_0, \ldots, x_k >) = k \ , \ z \in < x_0, \ldots, x_k >\}.$$

Let S_X^k, *the abstract* k*-secant variety of* X, be

$$\overline{(S_X^k)^0} \subset \underbrace{X \times \ldots \times X}_{k+1} \times \mathbb{P}^N.$$

The closed set S_X^k is irreducible and of dimension $(k + 1)\dim(X) + k$. Consider the projections of S_X^k onto the factors $\underbrace{X \times \ldots \times X}_{k+1}$ and $\mathbb{P}^N$,

$$\begin{tikzcd} & S_X^k \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times \ldots \times X & & \mathbb{P}^N. \end{tikzcd}$$

The k*-secant variety to* X, S^kX, is the scheme-theoretic image of S_X^k in $\mathbb{P}^N$, i.e.

$$S^kX = p_2(S_X^k) = \overline{\bigcup_{x_i \in X \, , \, \dim(< x_0, \ldots, x_k >) = k} < x_0, \ldots, x_k >} \subseteq \mathbb{P}^N.$$

It is an irreducible algebraic variety of dimension $s_k(X) \leq \min\{N, \ (k + 1)\dim(X) + k\}$.

We are now in position to define the last cone attached to a point $x \in X$. This notion was introduced by Johnson in [**Jo**] and further studied extensively by Zak. Algebraic properties of tangent star cones and of the algebras related to them are investigated in [**SUV**].

1.1.4. DEFINITION. (**Tangent star at a point; Variety of tangent stars**, [**Jo**]). Let $X \subset \mathbb{P}^N$ be an irreducible projective variety.

The abstract variety of tangent stars to X, T_X^*, is defined by the following cartesian diagram

$$\begin{tikzcd} T_X^* \arrow[r, hook] \arrow[d, "p"'] & S_X \arrow[d, "p_2"] \\ \Delta_X \arrow[r, hook] & X \times X \end{tikzcd}.$$

The tangent star to X *at* x, T_x^*X, is defined by

$$T_x^*X := p_2(p^{-1}((x,x))) \subseteq \mathbb{P}^N.$$

It is a scheme which can be described geometrically as follows:

$$T_x^*X = \overline{\bigcup_{(x_1,x_2)\in X\times X\setminus\Delta_X} \lim_{x_i\to x} < x_1, x_2 >} \subset \mathbb{P}^N.$$

The variety of tangent stars to X is by definition

$$T^*X = p_2(T_X^*) \subseteq \mathbb{P}^N,$$

so that by construction

$$T^*X \subseteq SX.$$

Moreover letting only one point varying we deduce

$$C_xX \subseteq T_x^*X.$$

It is also clear that the limit of a secant line is a tangent line, i.e. that

$$T_x^*X \subseteq T_xX.$$

By what we have defined and studied we deduce that for a point $x \in X \subset \mathbb{P}^N$, there is the following relation between the cones we attached to X at x:

$$C_xX \subseteq T_x^*X \subseteq T_xX.$$

Moreover a point $x \in X$ is non-singular if and only if $C_xX = T_x^*X = T_xX$. We immediately show in the following class of examples that at singular points strict inequalities can hold, i.e. at singular points there could exist tangent lines which are not limit of secant lines.

1.1.5. EXAMPLE. (**Singular points for which** $C_xX \subsetneq T_x^*X \subsetneq T_xX$). Let $Y \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$ be an irreducible, non-degenerate variety in $\mathbb{P}^N$. Consider a point $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ and let $X := S(p, Y)$ be *the cone over* Y *of vertex* p, i.e.

$$S(p,Y) = \bigcup_{y\in Y} < p, y > .$$

Then X is an irreducible, non-degenerate variety in $\mathbb{P}^{N+1}$. In fact, modulo a projective transformation, the variety X is defined by the same equations of Y, now thought as homogeneous polynomials with one variable more; in particular $\dim(X) = \dim(Y) + 1$.

The line $< p, y >$ is contained in X for every $y \in Y$, so that $X \subset T_pX$ and therefore $\mathbb{P}^N =< Y >\subset T_pX$. Since $p \in T_pX$, we get

$$T_pX = \mathbb{P}^{N+1}. \tag{1.1.1}$$

It follows from the definition of tangent cone to a variety that

$$C_pS(p,Y) = S(p,Y).$$

We also have that

$$S(p,SY) = SX. \tag{1.1.2}$$

Indeed, by projecting from p onto $\mathbb{P}^N$, it is clear that a general secant line to X projects onto a secant line to Y, proving $SX \subseteq S(p,SY)$. On the contrary if we get a general point $q \in S(p,SY)$, by definition it projects onto a general point $q' \in SY$, which belongs to a secant line $< p_1', p_2' >$, $p_i' \in Y$. The plane $< p, p_1', p_2' >$ contains the point q, while the lines $< p, p_i' >$, $i = 1,2$, are contained in X by definition of cone; hence through q there pass infinitely many secant lines to X, yielding $S(p,SY) \subseteq SX$. The claim is proved.

The above argument proves the following general fact:

$$T_p^*S(p,Y) = S(p,SY).$$

Indeed by definition $T_p^*X \subseteq SX = S(p,SY)$ as schemes. On the other hand, by fixing two general points $p_1, p_2 \in X$, $p_1 \neq p_2$, $p_i \neq p$, the plane $< p, p_1, p_2 >$ is contained in T_p^*X as it is easily seen by varying the velocity of approaching p of two points $q_i \in < p, p_i >$. By the generality of the points p_i we get the inclusion $SX \subseteq T_p^*X$ as schemes and the proof of the claim.

As an immediate application one constructs example of irreducible singular varieties X with a point $p \in \mathrm{Sing}(X)$ for which

$$C_pX \subsetneq T_p^*X \subsetneq T_pX.$$

One can take as $Y \subset \mathbb{P}^4 \subset \mathbb{P}^5$ an irreducible, smooth, non-degenerate curve in $\mathbb{P}^4$ and consider the cone X over Y of vertex $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$. Then $C_pX = S(p,Y) = X$, $T_p^*X = S(p,SY) = SX$ is an hypersurface in $\mathbb{P}^5$, because SY is an hypersurface in $\mathbb{P}^4$, while $T_pX = \mathbb{P}^5$. Every variety Y such that $SY \subsetneq \mathbb{P}^N$ (see the exercises at the end of the chapter or take $N > 2\dim(Y) + 1$) will produce analogous examples.

1.2. Join of varieties

We generalize to arbitrary irreducible varieties $X, Y \subset \mathbb{P}^N$ the notion of "cone" or of "join" of linear spaces.

Let us remember that if $L_i \simeq \mathbb{P}^{N_i} \subseteq \mathbb{P}^N$, $i = 1,2$, is a linear subspace, then

$$< L_1, L_2 > := \bigcup_{x_i \in L_i\ ,\ x_1 \neq x_2} < x_1, x_2 >,$$

is a linear space called *the join of* L_1 *and* L_2. It is the smallest linear subspace of $\mathbb{P}^N$ containing L_1 and L_2. By Grassmann formula we have

$$\dim(< L_1, L_2 >) = \dim(L_1) + \dim(L_2) - \dim(L_1 \cap L_2), \tag{1.2.1}$$

where as always $\dim(\emptyset) = -1$. This shows that the dimension of the join depends on the intersection of the two linear spaces.

On the other hand, if $X \subset \mathbb{P}^N \subset \mathbb{P}^{N+1}$ is an irreducible subvariety and if $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ is an arbitrary point, if we define as before

$$S(p,X) = \bigcup_{x \in X} < p, x >,$$

the cone of vertex p *over* X, then for every $z \in < p, x >$, $z \neq p$, we have by construction

$$T_zS(p,X) = < p, T_xX > = < T_pp, T_xX >, \tag{1.2.2}$$

i.e. the well known fact that the tangent space is constant along the ruling of a cone.

As we shall see in the next section, once we have defined the join of two varieties as the union of lines *joining* points of them, then we can *linearize* the problem looking at the tangent spaces and calculate the dimension of the *join* by looking at the affine cones over the varieties, exactly as in the proof of the formula (1.2.1). The dimension of the join of two varieties will depend on the intersection of a general tangent space of the first one with a general tangent space of the other one, a result known as Terracini Lemma, [**T1**]. Moreover a kind of property similar to the second tautological inequality in (1.2.2) will hold generically, at least in characteristic zero, see Theorem 1.3.1.

1.2.1. DEFINITION. (**Join of varieties; relative secant, tangent star and tangent varieties**). Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties.

Let

$$S^0_{X,Y} := \{((x,y,z)\,, x \neq y :\ z \in < x, y >\} \subset X \times Y \times \mathbb{P}^N.$$

The set is locally closed so that taken with the reduced scheme structure it is a quasi-projective irreducible variety of dimension $\dim(S^0_{X,Y}) = \dim(X) + \dim(Y) + 1$. Let $S_{X,Y}$ be its closure in $X \times Y \times \mathbb{P}^N$. Then $S_{X,Y}$ is an irreducible projective variety of dimension $\dim(X) + \dim(Y) + 1$, called *the abstract join of X and Y*. Let us consider the projections of $S_{X,Y}$ onto the factors $X \times Y$ and $\mathbb{P}^N$,

(1.2.3)
$$\begin{tikzcd} & S_{X,Y} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times Y & & \mathbb{P}^N. \end{tikzcd}$$

The join of X and Y, $S(X,Y)$, is the scheme-theoretic image of $S_{X,Y}$ in $\mathbb{P}^N$, i.e.

$$S(X,Y) = p_2(S_{X,Y}) = \overline{\bigcup_{x \neq y,\, x \in X,\, y \in Y} < x, y >} \subseteq \mathbb{P}^N;$$

it is an irreducible algebraic variety of dimension $s(X,Y) \leq \dim(X) + \dim(Y) + 1$, swept out by lines joining points of X with points of Y.

With this notation $S(X,X) = SX$ and $S(X, S^{k-1}X) = S^kX = S(S^lX, S^hX)$, if $h \geq 0$, $l \geq 0$, $h + l = k - 1$. Moreover, for arbitrary irreducible varieties X, Y and Z, we have $S(X, S(Y,Z)) = S(S(X,Y), Z)$.

When $Y \subseteq X \subset \mathbb{P}^N$ is an irreducible closed subvariety, the variety $S(Y,X)$ is usually *the relative secant variety of X with respect to Y*. Analogously, $T(Y,X) = \bigcup_{y \in Y} T_yX$. In this case by taking $\Delta_Y \subset Y \times X$ and by looking at (1.2.3), we can define $T^*_{Y,X} := p_1^{-1}(\Delta_Y) \subseteq S_{Y,X}$ to be *the abstract relative tangent star variety* and finally

(1.2.4)
$$T^*(Y,X) := p_2(T^*_{Y,X}) \subseteq S(X,Y)$$

to be *the relative tangent star variety*. If

$$T^*_y(Y,X) = p_2(p_1^{-1}(y \times y)) = \overline{\bigcup_{(y_1,x_1) \in Y \times X \setminus \Delta_Y} \lim_{\substack{y_1 \to y \\ x_1 \to y}} < y_1, x_1 >} \subset \mathbb{P}^N,$$

then $T^*(Y,X) = \bigcup_{y \in Y} T^*_y(Y,X)$. With this terminology, $T^*_y(y,X) = C_yX$ and $T^*_y(X,X) = T^*_yX$ for every $y \in X$. In particular $C_yX = T^*_y(y,X) \subseteq T^*_y(X,X) = T^*_yX$.

We furnish some immediate applications of the definition of join to properties of S^kX and to characterizations of linear spaces. For a variety $X \subseteq \mathbb{P}^N$, the linear space $< X > \subseteq \mathbb{P}^N$ is the *linear span of X in $\mathbb{P}^N$*, i. e. the smallest linear subspace of $\mathbb{P}^N$ containing X. The variety $X \subset \mathbb{P}^N$ is said to be *non-degenerated* if $< X > = \mathbb{P}^N$.

1.2.2. PROPOSITION. ([**P2**]) *Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties. The following holds:*

(1) *for every* $x \in X$,
$$Y \subseteq S(x,Y) \subseteq T_xS(x,Y) \subseteq T_xS(X,Y)$$
and in partiucular $< x, < Y >> \subseteq T_xS(x,Y)$.
(2) *if* $S^kX = S^{k+1}X$ *for some* $k \geq 0$*, then* $S^kX = \mathbb{P}^{s_k(X)} \subseteq \mathbb{P}^N$*;*
(3) *if* $\dim(S^{k+1}X) = \dim(S^kX) + 1$ *for some* $k \geq 0$*, then* $S^{k+1}X = \mathbb{P}^{s_{k+1}(X)}$ *so that* S^kX *is a hypersurface in* $\mathbb{P}^{s_{k+1}(X)}$*;*
(4) *if for some* $k \geq 0$ $S^{k+1}X \subset \mathbb{P}^N$ *is not a linear space, then* $S^kX \subseteq \mathrm{Sing}(S^{k+1}X)$.

PROOF. Exercise 1.6.2. □

To a non-degenerate irreducible closed subvariety $X \subset \mathbb{P}^N$ we can associate an ascending filtration of irreducible projective varieties, whose inclusion are strict by Proposition 1.2.2, and an integer $k_0 = k_0(X) \geq 1$:

$$X = S^0X \subsetneq SX \subsetneq S^2X \subsetneq \ldots \subsetneq S^{k_0}X = \mathbb{P}^N, \tag{1.2.5}$$

where k_0 is the least integer such that $S^kX = \mathbb{P}^N$.

The above immediate consequences of the definitions give also the following result, which was classically very well known, see for example [**P1**, footnote pg. 635], but considered as an open problem by Atiyah in [**At**, pg. 424]. The next result, via an argument of Atiyah, yields a proof of C. Segre's and Nagata's Theorem about the minimal section of a geometrically ruled surface, see [**Ln**].

1.2.3. COROLLARY. ([**P1**]) *Let* $C \subset \mathbb{P}^N$ *be an irreducible non-degenerate projective curve. Then* $s_k(C) = \min\{2k+1, N\}$.

Let $X \subset \mathbb{P}^N$ *be an irreducible non-degenerate projective variety of dimension* $n \geq 1$*. Let* $k < k_0$*. Then* $s_k(X) \geq n + 2k$ *for every* $k < k_0$*. If* $s_j(X) = n + 2j$ *for some* $j \geq 1$*,* $j < k_0$*, then* $s_k(X) = n + 2k$ *for every* $k \leq j$*. In particular if* $s_k(X) = n + 2k$ *for some* $k \geq 1$*,* $k < k_0$*, then* $s(X) = n + 2$ *and* $SX \subsetneq \mathbb{P}^N$.

PROOF. Exercise 1.6.2. □

We define and study linear projections with the terminology just introduced and generalize the dimension formula (1.2.1) to the case of arbitrary cones, at least in characteristic zero. In the next section we shall deal with the general case of join of two arbitrary varieties.

1.2.4. DEFINITION. (**Linear projections and** *linear* **cones**) Let $L = \mathbb{P}^l \subset \mathbb{P}^N$ be a fixed linear space, $l \geq 0$, and let $M = \mathbb{P}^{N-l-1}$ be a linear space skew to L, i.e. $L \cap M = \emptyset$ and $< L, M >= \mathbb{P}^N$. Let $X \subseteq \mathbb{P}^N$ be a closed irreducible variety not contained in L and let

$$\pi_L : X \dashrightarrow \mathbb{P}^{N-l-1} = M,$$

be the rational map defined on $X \setminus (L \cap X)$ by

$$\pi_L(x) =< L, x > \cap M.$$

The map is well defined by Grassmann formula, see (1.2.1). Let $X' = \overline{\pi_L(X)} \subset \mathbb{P}^{N-l-1}$ be the closure of the image of X by π_L. The whole process can be described with the terminology of joins. Indeed we have

$$X' = S(L, X) \cap M,$$

i.e. X' is the intersection of M with the *cone over* X *of vertex* L and moreover $S(L, X) = S(L, X')$. The projective differential of π_L is the projection of the tangent spaces from L, i.e. if $x \in X \setminus (L \cap X)$, then

$$d_{\pi_L}(T_xX) =< L, T_xX > \cap M \subseteq T_{\pi_L(x)}X',$$

as it is easily seen eventually passing to (local) coordinates. Clearly $S(L, T_xX) =< L, T_xX >$.

Suppose $L \cap X = \emptyset$, then we claim that $\pi_L : X \to X'$ is a finite morphism, which implies $\dim(X) = \dim(X')$. Being a morphism between projective varieties, it is sufficient to show that it has finite fibers by Stein Factorization. By definition for $x' \in X'$,

$$\pi_L^{-1}(x') = < L, x' > \cap X \subset < L, x' > = \mathbb{P}^{l+1}.$$

If there exists an irreducible curve $C \subset < L, x' > \cap X \subset < L, x' >$, then $\emptyset \neq L \cap C \subseteq L \cap X$, contrary to our assumption.

In particular for an arbitrary L, the dimension of X' does not depend on the choice of the position of M, except for the requirement $L \cap M = \emptyset$.

The relation $S(L, X) = S(L, X')$ allows us to calculate the dimension of the irreducible variety $S(L, X)$ for an arbitrary L. Exactly as in (1.2.2) for $z \in S(L, X) \setminus L$,

$$z \in < L, x > = < L, \pi_L(z) > = < L, x' >,$$

with $x \in X$ and $\pi_L(z) = \pi_L(x) = x' \in X'$. Since $S(L, X')$ is, modulo a projective transformation, the variety defined by the same homogeneous polynomials of X' now thought as polynomials in $N + 1$ variables, we have

$$T_z S(L, X) = < L, T_{\pi_L(z)} X' > \supseteq < L, T_x X > . \tag{1.2.6}$$

Taking $z \in S(L, X)$ general and recalling that $L \cap M = \emptyset$ we deduce:

$$\dim(S(L, X)) = \dim(< L, T_{\pi_L(z)} X' >) = \dim(X') + l + 1. \tag{1.2.7}$$

Suppose till the end of the subsection char$(K) = 0$. By generic smoothness, the differential map is surjective so that $T_{\pi_L(x)} X' = \pi_L(T_x X)$ for $x \in X$ general. In this case $\pi_L(x) = x' \in X'$ will be general on X' and finally

$$\dim(X') = \dim(T_{x'} X') = \dim(\pi_L(T_x X)) = \dim(X) - \dim(L \cap T_x X) - 1,$$

which combined with (1.2.7) gives the following generalization of (1.2.1):

$$\dim(S(L, X)) = \dim(L) + \dim(X) - \dim(L \cap T_x X), \tag{1.2.8}$$

$$x \in X \text{ general point.}$$

Moreover, we get the following refinement of (1.2.6)

$$T_z S(L, X) = < L, T_x X >, \tag{1.2.9}$$

$$x \in X \ , \ z \in < L, x > \ \text{general points.}$$

We have generalized the notion of cone over a variety lying in a skew space with respect to the vertex by taking $S(L, X)$ and shown that by projecting the variety X from the vertex L, we can find the description of it as an "usual" cone, $S(L, X')$.

Now we investigate under which condition a variety is a "cone", i.e. there exists a "vertex" $L \simeq \mathbb{P}^l \subseteq X$ such that $X = S(L, X) = S(L, X')$, if X' is the section with a general $\mathbb{P}^{N-l-1}$ skew with the "vertex" L. Clearly the "vertex" is not uniquely determined if we do not require some maximality condition. Let us begin with the definitions.

1.2.5. DEFINITION. (**Cone**; **vertex of a variety**) Let $X \subset \mathbb{P}^N$ be a closed (irreducible) subvariety. The variety is a *cone* if there exists $x \in X$ such that $S(x, X) = X$. Geometrically this means that given $y \in X$, $y \neq x$, the line $< x, y >$ is contained in X. In particular $x \in \bigcap_{y \in X} T_y X$.

This motivates the definition of vertex of a variety. Given $X \subset \mathbb{P}^N$ an irreducible closed subvariety, *the vertex of* X, $\mathrm{Vert}(X)$, is the set

$$\mathrm{Vert}(X) = \{\, x \in X \ : \ S(x, X) = X \,\}.$$

In particular a variety X is a cone if and only if $\mathrm{Vert}(X) \neq \emptyset$; by definition $S(X, Y) = X$ if and only if $Y \subseteq \mathrm{Vert}(X)$.

We list some obvious consequences and leave to the interested reader the pleasure of showing that the hypothesis on the characteristic of the base field are necessary.

1.2.6. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a closed irreducible variety of dimension $\dim(X) = n$. The following holds:*

(1)
$$\mathrm{Vert}(X) = \mathbb{P}^l \subseteq \bigcap_{x \in X} T_x X,$$
$l \geq -1$*;*

(2) *if* $\mathrm{codim}(\mathrm{Vert}(X), X) \leq 1$*, then* $\mathrm{Vert}(X) = X = \mathbb{P}^n \subset \mathbb{P}^N$*;*

(3) *if* $\dim(S(X, Y)) = \dim(X) + 1$*, then* $Y \subseteq \mathrm{Vert}(S(X, Y))$*;*

(4) *if* $\mathrm{char}(K) = 0$*,*
$$\mathrm{Vert}(X) = \bigcap_{x \in X} T_x X = \mathbb{P}^l,$$
$l \geq -1$. *In particular $\bigcap_{x \in X} T_x X \subseteq X$ (a non-obvious fact, which is false in positive characteristic!).*

(5) *suppose* $\mathrm{char}(K) = 0$ *and* $\emptyset \neq \mathrm{Vert}(X) \subsetneq X$*, then $X = S(\mathrm{Vert}(X), X')$ is a cone, where X' is the projection of X from* $\mathrm{Vert}(X)$ *onto a $\mathbb{P}^{N-l-1}$ skew to* $\mathrm{Vert}(X)$ $(\dim(X') = n - l - 1)$.

PROOF. Exercise 1.6.3. □

We end this section by putting in relation the projections of a variety and the dimension of its secant or tangent varieties.

If $L = \mathbb{P}^l \subset \mathbb{P}^N$ is a linear space and if $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^{N-l-1}$ is the projection onto a skew complementary linear space, then π_L restricts to a finite morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$, as soon as $L \cap X = \emptyset$. Assuming in principle that studying varieties whose codimension is small with respect to the dimension is easier (this is true from some points of view but not from others!), we can ask when this finite morphism is one-to-one, or a closed embedding. Let us examine these conditions.

1.2.7. PROPOSITION. *Let notation be as above. Then:*

(1) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is one-to-one if and only if $L \cap SX = \emptyset$;*
(2) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is unramified if and only if $L \cap TX = \emptyset$;*
(3) *the morphism $\pi_L : X \to \mathbb{P}^{N-l-1}$ is a closed embedding if and only if $L \cap SX = L \cap TX = \emptyset$.*

PROOF. The morphism $\pi_L : X \to X' \subseteq \mathbb{P}^{N-l-1}$ is one-to-one if and only there exists no secant line to X cutting the center of projection: $< L, x > = < L, y >$ if and only if $< x, y > \cap L \neq \emptyset$. It is ramified at a point $x \in X$ if and only if $T_x X \cap L = \emptyset$ by looking at the projective differential of π_L. A morphism is a closed embedding if and only if it is one-to-one and unramified. □

We must state the following well known result, which only takes into account that for smooth varieties the equality $TX = T^* X$ yields $TX \subseteq SX$.

1.2.8. COROLLARY. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible closed subvariety. If $N > \dim(SX)$, then X can be isomorphically projected into $\mathbb{P}^{N-1}$. In particular if $N > 2\dim(X) + 1$, then X can isomorphically projected into $\mathbb{P}^{N-1}$.*

One could ask what is the meaning of $L \cap T^*X = \emptyset$. This means that π_L (or $d(\pi_L)$) restricted to $T^*_x X$ is finite for every $x \in X$. This is the notion of J-unramified morphism, where J stands for Johnson [**Jo**], and it can be expressed in terms of affine tangent stars, see [**Z2**]. We take the above condition as the definition of J-unramified projection. In particular, if $L \cap SX = \emptyset$, then π_L is one-to-one and J-unramified and it is said to be *a J-embedding*. If the projection $\pi_L : X \to X' \subset \mathbb{P}^{N-l-1}$ is a J-embedding, then $\mathrm{Sing}(\pi_L(X)) = \pi_L(\mathrm{Sing}(X))$ so that X' does not acquire singularities from the projection.

It is clearly weaker than the usual notion of embedding and it is well behaved to study the projections of singular varieties. For example take $C \subset \mathbb{P}^4 \subset \mathbb{P}^5$ a smooth non-degenerate curve in $\mathbb{P}^4$ and let $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$. If $X = S(p, C)$ is the cone over C, then $T_pX = \mathbb{P}^5$, see (1.1.1), and X cannot be projected isomorphically in $\mathbb{P}^4$. Since $SX = S(p, SC)$, see (1.1.2), is an hypersurface in $\mathbb{P}^5$, there exists a point $q \in \mathbb{P}^5 \setminus X$ such that $\pi_q : X \to X'$ is a J-embedding and $X' = S(\pi_q(p), C)$ is a cone over C of vertex $\pi_q(p) = p'$. In this example the morphism π_q is one-to-one and unramified outside the vertex of the cones and maps the tangent star at p, $T^*_p X = S(p, SC)$, m-to-one onto $\mathbb{P}^4$, where $m = \deg(S(p, SC)) = \deg(SC) = \binom{d-1}{2} - g$, $d = \deg(C)$, g the genus of C.

The conditions $L \cap S(Y, X) = \emptyset$, respectively $L \cap T^*(Y, X) = \emptyset$ or $L \cap T(Y, X) = \emptyset$, with $Y \subseteq X$, mean that π_L is one-to-one in a neighbourhood of Y, respectively is J-unramified in a neighbourhood of Y or unramified in a neighbourhood of Y.

1.3. Terracini Lemma and its first applications

As we have seen the definition of secant variety is the "join" of X with itself and it is not clear how to calculate the dimension of SX, see exercise 1.6.1, or more generally the dimension of $S(X, Y)$. In fact, the circle of ideas, which allowed Alessandro Terracini to solve the problem of calculating the dimension of SX, or more generally of S^kX, originated exactly from the study of examples like the ones considered in exercise 1.6.1 and from the pioneering work of Gaetano Scorza, [**S1**] and [**S4**]. Let Terracini explain us this process, by quoting the beginning of [**T1**]:

É noto, [**dP**]*, che la sola V_2, non cono, di S_r, i cui S_2 tangenti si incontrano a due a due, é, se $r \geq 5$, la superficie di VERONESE; e che questa superficie,* [**Sev**]*, é pure caratterizzata dall' essere, in un tale S_r, la sola, non cono, le cui corde riempono una V_4. Recentemente lo SCORZA,* [**S3**, pg. 265]*, disse di aver ragione di credere, sebbene non gli fosse venuto fatto di darne una dimostrazione, che le V_3 di S_7, o di uno spazio piú ampio, le cui corde non riempiono una V_7 << rientrino >> tra le V_3 a spazi tangenti mutuamente secantisi. Ora si puó dimostrare, piu' precisamente, che queste categorie di V_3 coincidono, anzi, piu' in generale, che: Se una V_k di S_r ($r > 2k$) gode di una delle due proprietá, che le corde riempiano una varietá di dimensione $2k - i$ ($i \geq 0$), o che due qualsiansi S_k tangenti si seghino in uno S_i, gode pure dell' altra. Questo teorema, a sua volta, non é se non un caso particolare di un teorema piú generale che ora dimostreremo, teorema che pone in relazione l' eventuale abbassamento di dimensione della varietá degli S_h $(h + 1)$-seganti di una V_k immersa in uno spazio di dimensione $r \geq (h + 1)k + h$, coll' esistenza di $h + 1$ qualsiansi suoi S_k tangenti in uno spazio minore dell' ordinario.*

To calculate the dimension of $S(X, Y)$ in a simple way and to determinate the relation between $T_zS(X, Y)$, T_xX and T_yY, where $z \in< x, y >$, $z \neq x$, $z \neq y$, $x \neq y$, we recall the definition of affine cone over a projective variety $X \subset \mathbb{P}^N$.

Let $\pi : \mathbb{A}^{N+1} \setminus \mathbf{0} \to \mathbb{P}^N$ be the canonical projection. If $X \subset \mathbb{P}^N$ is a closed subvariety, we indicate by $C_{\mathbf{0}}(X)$ *the affine cone over* X, i.e. $C_{\mathbf{0}}(X) = \pi^{-1}(X) \cup \mathbf{0}$ is the affine variety cut out by the homogeneous polynomials in $N + 1$ variables defining X in $\mathbb{P}^N$. If $\mathbf{x} \neq \mathbf{0}$ is a point such that $\pi(\mathbf{x}) = x \in X$, then

$$\pi(T_{\mathbf{x}} C_{\mathbf{0}}(X)) = T_x X.$$

Moreover, if $L_i = \pi(U_i)$, $i = 1, 2$, U_i vector subspace of $\mathbb{A}^{N+1}$, then by definition $< L_1, L_2 >= \pi(U_1 + U_2)$, where $+ : \mathbb{A}^{N+1} \times \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$ is the vector space operation. Therefore, thought as a morphism of algebraic varieties, the differential of the sum coincides with the operation, i.e.

$$d_{(\mathbf{x},\mathbf{y})} : \mathcal{T}_{(\mathbf{x},\mathbf{y})}(\mathbb{A}^{N+1} \times \mathbb{A}^{N+1}) = \mathcal{T}_{\mathbf{x}}\mathbb{A}^{N+1} \times \mathcal{T}_{\mathbf{y}}\mathbb{A}^{N+1} \to \mathcal{T}_{\mathbf{x}+\mathbf{y}}\mathbb{A}^{N+1}$$

is the sum of the corresponding vectors.

With the above notation we have

$$\overline{C_{\mathbf{0}}(X) + C_{\mathbf{0}}(Y)} = C_{\mathbf{0}}(S(X,Y)). \tag{1.3.1}$$

We are now in position to prove the so called Terracini Lemma. The original proof of Terracini relies on the study of the differential of the second projection morphism $p_2 : S_{X,Y} \to S(X,Y)$. Here we follow Ådlandsvik, [**Åd**], to avoid the "difficulty", if any, of writing the tangent space at a point $(x, y, z) \in S^0_{X,Y}$. When writing $z \in < x, y >$, we always suppose $x \neq y$.

1.3.1. THEOREM. (**Terracini Lemma**) *Let $X, Y \subset \mathbb{P}^N$ be irreducible subvarieties. Then:*

(1) *for every $x \in X$, for every $y \in Y$, $x \neq y$, and for every $z \in < x, y >$,*

$$< T_xX, T_yY > \subseteq T_zS(X,Y);$$

(2) *if* char$(K) = 0$, *there exists an open subset U of $S(X,Y)$ such that*

$$< T_xX, T_yY >= T_zS(X,Y)$$

for every $z \in U$, $x \in X$, $y \in Y$, $z \in < x, y >$. In particular

$$\dim(S(X,Y)) = \dim(X) + \dim(Y) - \dim(T_xX \cap T_yY)$$

for $x \in X$ and $y \in Y$ general points.

PROOF. The first part follows from equation (1.3.1) and from the interpretation of the differential of the affine sum. The second part from generic smoothness applied to the affine cones over X, Y and $S(X,Y)$. □

Since we have quoted the original form given by Terracini, let us state it as an obvious corollary.

1.3.2. COROLLARY. ([**T1**]) *Let $X \subset \mathbb{P}^N$ be an irreducible subvariety of $\mathbb{P}^N$. Then:*

(1) *for every $x_0, \ldots, x_k \in X$ and for every $z \in < x_0, \ldots, x_k >$,*

$$< T_{x_0}X, \ldots, T_{x_k}X > \subseteq T_zS^kX;$$

(2) *if* char$(K) = 0$, *there exists an open subset U of S^kX such that*

$$< T_{x_0}X, \ldots, T_{x_k}X >= T_zS^kX$$

for every $z \in U$, $x_i \in X$, $i = 0, \ldots, k$, $z \in < x_0, \ldots, x_k >$. In particular

$$\dim(SX) = 2\dim(X) - \dim(T_xX \cap T_yX)$$

for $x, y \in X$ general points.

The first application we give is the so called *Trisecant Lemma*. Let us recall that a line $l \subset \mathbb{P}^N$ is said to be a *trisecant line to $X \subset \mathbb{P}^N$* if length$(l \cap X) \geq 3$.

1.3.3. PROPOSITION. (**Trisecant Lemma**) *Let $X \subset \mathbb{P}^N$ be a non-degenerate, irreducible closed subvariety. Suppose* char(K)=*0 and* codim$(X) > k$. *Then a general $(k+1)$-secant $\mathbb{P}^k$, $< x_0, \ldots, x_k >= L = \mathbb{P}^k$, is not $(k+2)$-secant, i.e. $L \cap X = \{x_0, \ldots, x_k\}$ as schemes. In particular, if* codim$(X) > 1$, *the projection of X from a general point on it, $\pi_x : X \dashrightarrow X' \subset \mathbb{P}^{N-1}$, is a birational map.*

PROOF. We claim that it is sufficient to prove the result for $k = 1$. Indeed X is not linear so that by taking a general $x \in X$ and projecting X from this point we get a non-degenerate, irreducible subvariety $X' = \pi_x(X) \subset \mathbb{P}^{N-1}$ with $\operatorname{codim}(X') = \operatorname{codim}(X) - 1 > k - 1$. If the general $L =< x_0, \dots, x_k >$ as above were $k+2$-secant, by taking $x = x_k$, the linear space $< \pi_x(x_0), \dots, \pi_x(x_{k-1}) >= \mathbb{P}^{k-1} = L'$ would be a general k-secant $\mathbb{P}^{k-1}$, which results to be $(k+1) = ((k-1)+2)$-secant. So we can assume $k = 1$ and we set $n = \dim(X)$.

Take $x \in X \setminus \operatorname{Vert}(X)$. Then a general secant line through x, $l =< x, y >$, is not tangent to X neither at x nor at y. If l is a trisecant line then necessarily it exists $u \in (l \cap X) \setminus \{x, y\}$. Consider the projection of X from x. Since $x \notin \operatorname{Vert}(X)$, if $X' = \pi_x(X) \subset \mathbb{P}^{N-1}$, then $\dim(X') = \dim(X)$ and $\pi_x(y) = \pi_x(u) = x'$ is a general smooth point of X'. By generic smoothness

$$< x, T_{x'}X' >=< x, T_yX >=< x, T_uX >$$

so that T_yX and T_uX are hyperplanes in $< x, T_{x'}X' >= \mathbb{P}^{n+1}$, yielding

$$\dim(T_yX \cap T_uX) = n - 1.$$

Taking $z \in< x, y >=< y, u >$ general, we have a point in the set U specified in Corollary 1.3.2 so that $\dim(SX) = \dim(T_zSX) = \dim(< T_yX, T_uX >) = n+1$. This implies $\operatorname{codim}(X) = 1$ by Proposition 1.2.2 part 3). The last part follows from the fact that a generically one-to one morphism is birational if char(K)=0, being generically étale. □

As a second application we reinterpret Terracini Lemma as tangency of tangent space to higher secant varieties at a general point along the locus described on X by the secant spaces passing through the point. To this aim we first define tangency along a subvariety and then the entry loci described above, studying their dimension.

1.3.4. DEFINITION. (**Tangencies along a subvariety**) Let $Y \subset X$ be a closed (irreducible) subvariety of X and let $L = \mathbb{P}^l \subset \mathbb{P}^N$, $l \geq \dim(X)$, be a linear subspace.

The linear space L is said to be *tangent to X along Y* if for every $y \in Y$

$$T_yX \subseteq L,$$

i.e. if and only if $T(Y, X) \subseteq L$.

The linear space L is said to be *J-tangent to X along Y* if for every $y \in Y$

$$T_y^*X \subseteq L.$$

The linear space L is said to be *J-tangent to X with respect to Y* if for every $y \in Y$

$$T_y^*(Y, X) \subseteq L,$$

i.e. if and only if $T^*(Y, X) \subseteq L$.

Clearly if L is tangent to X along Y, it is also J-tangent to X along Y and if L is J-tangent to X along Y it is also J-tangent to X with respect to Y.

In the case $L = \mathbb{P}^{N-1}$, the scheme-theoretic intersection $L \cap X = D$ is a divisor, i.e. a subscheme of pure dimension $\dim(X) - 1$. By definition, for every $y \in D$, we have $T_yD = T_yX \cap L$ so that, if X is a smooth variety, $L = \mathbb{P}^{N-1}$ is tangent to X exactly along $\operatorname{Sing}(D) = \{y \in D : \dim(T_yD) > \dim(D)\}$.

We define the important notions of entry loci and k-secant defect and we study their first properties.

1.3.5. DEFINITION. (**Entry loci and k-secant defect δ_k**) Let $X \subset \mathbb{P}^N$ be a closed irreducible non-degenerate subvariety. Let us recall the diagram defining the higher secant varieties S^kX as join of X with

$S^{k-1}X$:

$$\begin{tikzcd} & S^k_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times S^{k-1}X & & \mathbb{P}^N. \end{tikzcd}$$

Let us define $\phi : X \times S^{k-1}X \to X$ to be the projection onto the first factor of this product.

Then, for $z \in S^kX$, *the k-entry locus of X with respect to z* is the scheme theoretic image

$$\Sigma^k_z = \Sigma^k_z(X) := \phi(p_1(p_2^{-1}(z))). \tag{1.3.2}$$

Geometrically, the support of Σ^k_z is the locus described on X by the $(k+1)$-secant $\mathbb{P}^k$ of X passing through $z \in S^kX$. If $z \in S^kX$ is general, then through z there passes an ordinary $(k+1)$-secant $\mathbb{P}^k$, i.e. given by $k+1$ distinct points on X and we can describe the support of Σ^k_z in this way

$$(\Sigma^k_z)_{\text{red}} = \overline{\{x \in X\ :\ \exists x_1, \dots, x_k \text{ distinct and } z \in < x, x_1, \dots, x_k >\}}.$$

Moreover, by the theorem of the dimension of the fibers for general $z \in S^kX$, the support of Σ^k_z is equidimensional and every irreducible component contains ordinary $\mathbb{P}^k$'s since necessarily $\operatorname{codim}(X) > k$, see Proposition 1.3.3. If char(K)=0, then for general $z \in S^kX$ the scheme $p_1^{-1}(z)$ is smooth so that Σ^k_z is reduced.

To recover the scheme structure of Σ^k_z geometrically, one could define Π_z as the locus of $(k+1)$-secant $\mathbb{P}^k$'s through z and define $\Sigma^k_z = \Pi_z \cap X$ as schemes. For example if through $z \in SX$ there passes a unique tangent line l to X, then in this way we get $\Pi_z = l$ and $\Sigma_z = l \cap X$ the point of tangency with the double structure.

Let us study the dimension of Σ^k_z for $z \in S^kX$ general. Before let us remark that if $x \in \Sigma^k_z$ is a general point in an irreducible component, $z \in S^kX$ general, then, as sets,

$$\phi^{-1}(x) = \{y \in S^{k-1}X\ :\ z \in < x, y >\} = < z, x > \cap S^{k-1}X \neq \emptyset$$

and $\dim(\phi^{-1}(x)) = 0$ because $z \in S^kX \setminus S^{k-1}X$ by the generality of z.

Then we define *the k-secant defect of* X, $1 \leq k \leq k_0(X)$, $\delta_k(X)$, as the integer

$$\delta_k(X) = \dim(\Sigma^k_z) = \dim(p_1(p_2^{-1}(z))) = s_{k-1}(X) + \dim(X) + 1 - s_k(X), \tag{1.3.3}$$

where $z \in S^kX$ is a general point.

For $k = 1$, we usually put $\Sigma_z = \Sigma^1_z$, $z \in SX$, and $\delta(X) = \delta_1(X) = 2\dim(X) + 1 - \dim(SX)$; for $k = 0$, $\delta_0(X) = 0$.

Let us reinterpret Terracini Lemma with these new definitions.

1.3.6. COROLLARY. (**Tangency along the entry loci**) *Let X be an irreducible non-degenerate closed subvariety. Let $k < k_0(X)$, i.e. $S^kX \subsetneq \mathbb{P}^N$, and let $z \in S^kX$ be a general point. Then:*

(1) *the linear space T_zS^kX is tangent to X along $(\Sigma^k_z)_{\text{red}} \setminus \operatorname{Sing}(X)$;*
(2) $\delta_k(X) < \dim(X)$*;*
(3) $\delta_{k_0}(X) = \dim(X)$ *if and only if* $s_{k_0-1}(X) = N - 1$*, i. e. if and only if* $S^{k_0-1}X$ *is a hypersurface;*
(4)
$$s_k(X) = (k+1)(n+1) - 1 - \sum_{i=1}^{k} \delta_i(X) = \sum_{i=0}^{k} (\dim(X) - \delta_i(X) + 1);$$
(5) (cfr. 1.2.3) *if X is a curve,* $s_k(X) = 2k + 1$.

PROOF. Part 1) is clearly a restatement of part 1) of Corollary 1.3.2 when we take into account the geometrical properties of Σ^k_z, $z \in S^kX$ general, described in the definition of entry loci and the fact that the locus of tangency of a linear space is closed in $X \setminus \operatorname{Sing}(X)$, see also Definition 1.4.7. Recall that if char(K)=0, the scheme Σ^k_z is reduced.

If $\dim(\Sigma_z^k) = \delta_k(X) = \dim(X)$, then a general tangent space to S^kX would be tangent along X and X would be degenerated.

With regard to 3), we remark that $\delta_{k_0}(X) = s_{k_0-1}(X) + \dim(X) + 1 - N$ so that $\dim(X) - \delta_{k_0}(X) = N - 1 - s_{k_0-1}(X)$.

Part 4) is an easy computation by induction, while part 5) follows from part 4) since for a curve $\delta_k(X) < \dim(X)$ yields $\delta_k(X) = 0$. $\square$

1.3.7. REMARK. The statement of part 1) cannot be improved. Take for example a cone $X \subset \mathbb{P}^5$ of vertex a point $p \in \mathbb{P}^5 \setminus \mathbb{P}^4$ over a smooth non-degenerate projective curve $C \subset \mathbb{P}^4$. If $z \in S(p, SC) = SX$ is general and if $z \in < x, y >$, $x, y \in X$, it is not difficult to see that $\Sigma_z(X) =< p, x > \cup < p, y >$. The hyperplane T_zSX is tangent to X at x and y by Terracini Lemma, so that it is tangent to X along the rulings $< p, x >$ and $< p, y >$ minus the point p. Since $T_pX = \mathbb{P}^5$, the hyperplane T_zSX is not tangent to X at p (neither J-tangent to X at p).

A phenomenon studied classically firstly by Scorza, [**S1**], [**S2**], [**S4**], and then by Terracini, [**T3**] is the case in which imposing tangency of a hyperplane at $k+1$ general points, $k \geq 0$, of a variety $X \subset \mathbb{P}^N$ forces tangency along a positive dimensional variety, even if $\delta_k(X) = 0$. Indeed, Terracini Lemma says that if $\delta_k(X) > 0$, $k < k_0(X)$, than a hyperplane tangent at $k+1$ points, becomes tangent along the corresponding entry locus. The interesting and exceptional behaviour occurs for varieties with $\delta_k(X) = 0$. The first examples are the tangent developable to a non-degenerate curve or cones of arbitrary dimension. Indeed they have $\delta_0 = 0$ as every variety but by imposing tangency at a general point, we get tangency along the ruling passing through the point.

Varieties for which a hyperplane tangent at $k+1$, $k \geq 0$, general points is tangent along a positive dimensional subvariety are called *k-weakly defective varieties*, according to Chiantini and Ciliberto, [**CC1**]. In [**CC1**] many interesting properties of these varieties are investigated and a refined Terracini Lemma is proved, also putting in modern terms the classification of k-weakly defective irreducible surfaces obtained classically by Scorza, [**S2**], and Terracini, [**T3**]. Let us remark that, as shown in [**CC1**], for every $k \geq 1$ there exist smooth varieties of dimension greater than one which are k-weakly defective but have $\delta_k(X) = 0$ or $s_k(X) = \min\{N, (k+1)n + k\}$. We shall come back to these definitions and phenomena in subsection 1.5.2 and in section 2.3.

As another application, we study the dimension of the projection of a variety from linear subspaces generated by general tangent spaces. Terracini Lemma says that we are projecting from a general tangent space to the related higher secant variety. As we have seen when the center of projection L cuts the variety it is difficult to control the dimension of the image of X under projection because we do not know a priori how a general tangent space intersects L. In the case of $L = T_zS^{k-1}X$ this information is encoded in the dimension of S^kX and of the defect $\delta_k(X)$ as we immediately see. In section 4.1 of chapter 4 we shall see how the degree of the projections from T_zS^kX is related to the number of $(k+2)$-secant $\mathbb{P}^{k+1}$ passing through a general point of $S^{k+1}X$, a problem dubbed as *Bronowski's conjecture*, see [**B1**] and *loc. cit.*, and partially solved in [**CMR**] for $k = 1$ and in [**CR**] for arbitrary $k \geq 1$. Projections from tangent spaces, or more generally from T_zS^kX, were a classical tool of investigation, [**Ca**], [**En**], [**S1**], [**S4**], [**B1**], [**B2**], and were recently used to study classical and modern problems, see [**CC1**], [**CMR**], [**CR**], [**IR1**] and the presentation in §4.2 below.

1.3.8. PROPOSITION. (**Projections from tangent spaces**) Let $X \subset \mathbb{P}^N$ be an irreducible, non-degenerate closed subvariety. Let $n = \dim(X)$ and suppose char(K)=0 and $N \geq s_k$, $k \geq 1$, where $s_k = s_k(X)$. Set $\delta_k = \delta_k(X)$.

Let $x_1, \ldots, x_k \in X$ be k general points, let $L =< T_{x_1}, \ldots, T_{x_k} >$ and let $\pi_k = \pi_L : X \to X' \subset \mathbb{P}^{N-s_{k-1}(X)-1}$. Then $\dim(L) = s_{k-1}(X) = s_{k-1}$ and, if $X'_k = \pi_k(X) \subset \mathbb{P}^{N-s_{k-1}-1}$, then

(1) $\dim(X'_k) = s_k - s_{k-1} - 1 = n - \delta_k$;

(2) suppose $N \geq (k+1)n+k$ and and $s_{k-1} = kn+k-1$, i.e. that $\delta_{k-1} = 0$. Then $s_k = (k+1)n+k$ (or equivalently $\delta_k = 0$) if and only if $\dim(X'_k) = n$. In particular if $N = (k+1)n+k$ and if $s_{k-1} = kn+k-1$, then $S^kX = \mathbb{P}^{(k+1)n+k}$ if and only if $\pi_k : X \dashrightarrow \mathbb{P}^n$ is dominant.

PROOF. If $z \in < x_1, \dots, x_k >$ is a general point, then z is a general point of $S^{k-1}X$ and by Terracini lemma $s_{k-1} = \dim(T_xS^{k-1}X) = \dim(< T_{x_1}, \dots, T_{x_k} >)$. By (1.2.7) we get

$$\dim(X'_k) = \dim(S(T_zS^{k-1}X, X)) - s_{k-1} - 1 = s_k - s_{k-1} - 1 = n - \delta_k.$$

The other claims are only reformulations of part 1). □

1.4. Dual varieties and contact loci of general tangent linear spaces

Let $X \subset \mathbb{P}^N$ be a projective, irreducible non-degenerate variety of dimension n; let $\mathrm{Sm}(X) := X \setminus \mathrm{Sing}(X)$ be the locus of non-singular points of X. By definition $\mathrm{Sm}(X) = \{x \in X \,:\, \dim(T_xX) = n\}$.

If we take an hyperplane section of X, $Y = X \cap H$, where $H = \mathbb{P}^{N-1}$ is an arbitrary hyperplane, then for every $y \in Y$ we get

$$T_yY = T_yX \cap H. \tag{1.4.1}$$

Since Y is a pure dimensional scheme of dimension $n-1$, we see that $\mathrm{Sing}(Y) \setminus (\mathrm{Sing}(X) \cap H) = \{y \in Y \setminus \mathrm{Sing}(X) \cap Y \,:\, T_yX \subseteq H\}$, which is an open subset in the locus of points of X at which H is tangent. In particular to show that an hyperplane section has non-singular points, we have to exhibit an hyperplane H which is not tangent at all the points in which it intersects X. It naturally arises the need of patching together all the "bad" hyperplanes and eventually show that there always exists an hyperplane section of X, non-singular at least outside $\mathrm{Sing}(X)$. Since hyperplane can be naturally thought as points in the dual projective space $\mathbb{P}^{N^*}$, we can define a subvariety of $\mathbb{P}^{N^*}$ parametrizing hyperplane sections which are singular also outside $\mathrm{Sing}(X)$. This locus is the so-called *dual variety*.

1.4.1. DEFINITION. (**Dual variety**) Let $X \subset \mathbb{P}^N$ be as above and let

$$\mathcal{P}_X := \overline{\{(x, H) \,:\, x \in \mathrm{Sm}(X),\ T_xX \subseteq H\}} \subset X \times \mathbb{P}^{N^*},$$

the so called *conormal variety of* X.

Let us consider the projections of $\mathcal{P}_X$ onto the factors X and $\mathbb{P}^{N^*}$,

$$\begin{tikzcd} & \mathcal{P}_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \mathbb{P}^{N^*}. \end{tikzcd}$$

The dual variety to X, X^*, is the scheme-theoretic image of $\mathcal{P}_X$ in $\mathbb{P}^{N^*}$, i.e. the algebraic variety

$$X^* := p_2(\mathcal{P}_X) \subseteq \mathbb{P}^{N^*}.$$

The set $\mathcal{P}_X$ is easily seen to be a closed subset. For $x \in \mathrm{Sm}(X)$, we have $p_1^{-1}(x) \simeq (T_xX)^* = \mathbb{P}^{N-n-1} \subset \mathbb{P}^{N*}$. Then the set $\mathcal{P}_X$ is irreducible since

$$p_1^{-1}(\mathrm{Sm}(X)) \to \mathrm{Sm}(X)$$

is a $\mathbb{P}^{N-n-1}$-bundle and clearly $\dim(\mathcal{P}_X) = N-1$. Then $\dim(X^*) \leq N-1$ and the *dual defect of* X, $\mathrm{def}(X)$, is defined as

$$\mathrm{def}(X) = N - 1 - \dim(X^*) \geq 0.$$

A variety is said to be *reflexive* if the natural isomorphism between $\mathbb{P}^N$ and $(\mathbb{P}^{N^*})^*$ induces an isomorphism between $\mathcal{P}_X$ and $\mathcal{P}_{X^*}$. This clearly implies that the natural identification between $\mathbb{P}^N$ and $\mathbb{P}^{N**}$ induces an isomorphism $X \simeq X^{**} = (X^*)^*$.

Let us take $H \in X^*$. By definition

$$C_H := C_H(X) = p_2^{-1}(H) = \overline{\{x \in \mathrm{Sm}(X) \, : \, T_xX \subseteq H\}}$$

is exactly the closure of non-singular points of X where H is tangent to X, it is not empty so that $H \cap X$ is singular outside $\mathrm{Sing}(X)$. On the contrary if $H \notin X^*$, the hyperplane section $H \cap X$ can be singular only along $\mathrm{Sing}(X)$. This is the classical *Bertini Theorem.*

In particular we proved the following result.

1.4.2. THEOREM. *Let $X \subset \mathbb{P}^N$ be a projective, irreducible non-degenerate variety of dimension $n = \dim(X)$. Then for every $H \in (\mathbb{P}^N)^* \setminus X^*$ the divisor $H \cap X$ is non-singular outside* $\mathrm{Sing}(X)$.

In particular if X has at most a finite number of singular points $p_1, \ldots, p_m$, then for every $H \notin X^* \cup (p_1)^* \cup \ldots \cup (p_m)^*$, the hyperplane section $H \cap X$ is a non-singular subscheme of pure codimension 1.

Later we shall see that if $n \geq 2$, then every hyperplane section is connected. For non-singular varieties with $\dim(X) \geq 2$, the hyperplane sections with hyperplanes $H \notin X^*$, being connected and non-singular are also irreducible so that are irreducible non-singular algebraic varieties.

To justify the name of conormal variety for $\mathcal{P}_X$ and to get some practice with the definitions, we refer to Exercise 1.6.6.

As we have seen the dual varieties encode informations about the tangency of hyperplanes. Terracini Lemma says that linear spaces containing tangent spaces to higher secant varieties are tangent along $(\Sigma_z^k)_{\mathrm{red}} \setminus \mathrm{Sing}(X)$, see Corollary 1.3.6. Thus the maximal dimension of the fibers of $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N^*}$ is an upper bound for $\delta_k(X)$ as soon as $S^kX \subsetneq \mathbb{P}^N$, as we shall immediately see. More refined versions with the higher Gauss maps γ_m, see below, can be formulated but in those cases the condition expressed by the numbers $\varepsilon_m(X)$, which can be defined as below, is harder to control.

1.4.3. THEOREM. (**Dual variety and higher secant varieties**) *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Let $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N^*}$ be as above and let $\varepsilon(X) = \max\{\dim(p_2^{-1}(H)) \, , \, H \in X^*\}$. If $S^kX \subsetneq \mathbb{P}^N$, then $\delta_k(X) \leq \varepsilon(X)$. In particular if $p_2 : \mathcal{P}_X \to X^*$ is a finite morphism, then* $\dim(S^kX) = \min\{(k+1)n + k, N\}$.

PROOF. Let $z \in S^kX$ be a general point. There exists $x \in \Sigma_z^k(X) \cap \mathrm{Sm}(X)$ and moreover T_zS^kX is contained in a hyperplane H. Then

$$p_1(p_2^{-1}(H)) \supseteq \mathrm{Sing}(X \cap H) \setminus (\mathrm{Sing}(X) \cap H)$$

(and more precisely $\mathrm{Sing}(X \cap H) \setminus (\mathrm{Sing}(X) \cap H))$ contains the irreducible component of $\Sigma_z^k(X) \setminus (\mathrm{Sing}(X) \cap \Sigma_z^k(X))$ passing through x by Corollary 1.3.6. Then $p_1(p_2^{-1}(H))$ has dimension at least $\delta_k(X) = \dim(\Sigma_z^k(X))$ and the conclusion follows. □

1.4.4. COROLLARY. (cfr. Corollaries 1.2.3 and 1.3.6).

Let $X \subset \mathbb{P}^N$ be either an irreducible non-degenerate curve or a smooth non-degenerate complete intersection. Then

$$\dim(S^kX) = \min\{(k+1)n + k, N\}.$$

PROOF. By Exercise 1.6.6, we know that in both cases $p_2 : \mathcal{P}_X \to X^*$ is a finite morphism. □

More generally one would study the locus of points at which a general hyperplane is tangent, the so called *contact locus*. For reflexive varieties it is a linear space of dimension $\mathrm{def}(X)$. This is an interpretation of the isomorphism $X \simeq (X^*)^*$. One should be careful in the interpretation of the result: it does not mean that the hyperplane remains tangent along the whole "contact locus", see remark 1.3.7 and adapt it to the more general situation of a ruling of a cone. This is true only for non-singular varieties. In particulat reflexive varieties of positive dual defect contain positive dimensional families of linear spaces.

1.4.5. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a reflexive variety. Then for $H \in \mathrm{Sm}(X^*)$,*

$$p_2^{-1}(H) = \overline{\{x \in \mathrm{Sm}(X) \,:\, T_x X \subset H\}} = (T_H X^*)^* = \mathbb{P}^{\mathrm{def}(X)}.$$

The following result will not be proved here but the reader can consult [**Ha**, pg. 208] for an elementary and direct proof. It is considered a classical theorem, know at least to C. Segre.

1.4.6. THEOREM. (**Reflexivity Theorem**) *Let $X \subset \mathbb{P}^N$ be an irreducible variety. Suppose* char(K)=0*. Then X is reflexive.*

Another natural and similar problem is to know if a general tangent space to a variety X is tangent at more than one point. During the discussion we will always suppose char(K)=0 to avoid artificial problems, since the natural ones are enough interesting.

We have seen in exercise 1.6.6 that for irreducible curves a general tangent space is tangent only at one point. On the other hand if X is a cone over a curve, we know that a a general tangent space is tangent exactly along the ruling passing through the point. The unique common feature of irreducible algebraic varieties from this point of view seems to be the linearity of the locus of points at which a general linear space is tangent.

1.4.7. DEFINITION. (**Gauss maps**) Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension $n = \dim(X) \geq 1$, let $m \geq n$ and let $\mathbb{G}(m, N)$ be the Grassmanian parametrizing linear subspaces of dimension m in $\mathbb{P}^N$. Let

$$\mathcal{P}_X^m := \overline{\{((x, [L]) \;:\; x \in \mathrm{Sm}(X),\ T_x X \subseteq L\}} \subset X \times \mathbb{G}(m, N).$$

Let us consider the projections of $\mathcal{P}_X^m$ onto the factors X and $\mathbb{G}(m, N)$,

$$\begin{tikzcd} & \mathcal{P}_X^m \arrow[dl, "p_1"'] \arrow[dr, "\gamma_m"] & \\ X & & \mathbb{G}(m, N). \end{tikzcd}$$

The variety of m-dimensional tangent subspaces to X, X_m^*, is the scheme-theoretic image of $\mathcal{P}_X^m$ in $\mathbb{G}(m, N)$, i.e. the algebraic variety

$$X_m^* := \gamma_m(\mathcal{P}_X^m) \subset \mathbb{G}(m, N).$$

For $m = N - 1$, we recover the dual variety and its definition, while for $m = n$, we get the usual Gauss map $\mathcal{G}_X : X \dashrightarrow \mathbb{G}(n, N)$ which associates to a point $x \in \mathrm{Sm}(X)$ its tangent space $T_x X$. For such $x \in \mathrm{Sm}(X)$ $\mathcal{G}_X(x) := \gamma_n(x) = [T_x X]$.

If $X \subset \mathbb{P}^N$ is an hypersurface, then $n = N - 1$ and clearly the Gauss map $\mathcal{G}_X : X \dashrightarrow \mathbb{P}^{N^*} = \mathbb{G}(N-1, N)$ associates to a smooth point p of X its tangent hyperplane. Thus in coordinates the Gauss map is given by the formula

$$\mathcal{G}_X(p) = \left(\frac{\partial f}{\partial X_0}(p) : \ldots : \frac{\partial f}{\partial X_N}(p)\right).$$

The following result is once again a consequence of reflexivity and it is a generalization of Proposition 1.4.5 and of the properties of cones. One can consult [**Z2**, pg. 21] for a proof.

1.4.8. THEOREM. (**Linearity of general contact loci**) *Let $X \subset \mathbb{P}^N$ be an irreducible projective non-degenerate variety. Assume* char(K)=0*. The general fiber of the morphism $\gamma_m : \mathcal{P}_X^m \to X_m^*$ is a linear space of dimension* $\dim(\mathcal{P}_X^m) - \dim(X_m^*)$*. In particular the closure of a general fiber of $\mathcal{G}_X : X \dashrightarrow X_n^* \subset \mathbb{G}(n, N)$ is a linear space of dimension $n - \dim(\mathcal{G}_X(X))$ so that a general linear tangent space is tangent along an open subset of a linear space of dimension $n - \dim(\mathcal{G}_X(X))$.*

We now prove via Terracini Lemma a relation between X^* and $(S^kX)^*$ for every $k < k_0(X)$, assuming as always in this kind of problems that the condition char(K)=0 holds.

1.4.9. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Assume* char(K)=0 *and $SX \subsetneq \mathbb{P}^N$. Then $(SX)^* \subseteq \operatorname{Sing}(X^*) \subsetneq X^*$, i.e. a general bitangent hyperplane represents a singular point of X^*. More generally for a given $k \geq 2$ such that $k < k_0(X)$, we have $(S^kX)^* \subseteq \operatorname{Sing}((S^{k-1}X)^*) \subsetneq (S^{k-1}X)^*$, i.e. a general $(k+1)$-tangent hyperplane represents a singular point of $(S^{k-1}X)^*$.*

PROOF. Exercise 1.6.7. □

Recall that to a non-degenerate irreducible closed subvariety $X \subset \mathbb{P}^N$ we associated an ascending filtration of irreducible projective varieties, see (1.2.5),

$$X = S^0X \subsetneq SX \subsetneq S^2X \subsetneq \ldots \subsetneq S^{k_0}X = \mathbb{P}^N.$$

The above proposition says that at least over a filed of characteristic zero, there exists also a strictly descending dual filtration:

$$X^* \supsetneq \operatorname{Sing}(X^*) \supseteq (SX)^* \supsetneq \ldots \supseteq (S^{k_0-2}X)^* \supsetneq \operatorname{Sing}((S^{k_0-2}X)^*) \supseteq (S^{k_0-1}X)^*.$$

1.5. Tangent cones to k-secant varieties, degree of k-secant varieties and varieties of minimal k-secant degree

1.5.1. Notation and definitions. Let k be a non–negative integer and let S^kX be the *k–secant variety* of $X \subset \mathbb{P}^N$ defined above.

Let $\operatorname{Sym}^h(X)$ be the h–th symmetric product of X. One can consider the *abstract k–th secant variety* S^k_X of X, i.e. $S^k_X \subseteq \operatorname{Sym}^k(X) \times \mathbb{P}^N$ is the Zariski closure of the set of all pairs $([p_0, ..., p_k], x)$ such that $p_0, ..., p_k \in X$ are linearly independent points and $x \in < p_0, ..., p_k >$. One has the surjective map

$$p^k_X : S^k_X \to S^kX \subseteq \mathbb{P}^N$$

i.e. the projection to the second factor. Recall that:

$$s_k(X) := \dim(S^kX) \leq \min\{N, \dim(S^k_X)\} = \min\{N, n(k+1)+k\}. \tag{1.5.1}$$

We will denote by $h^{(k)}(X)$ the codimension of S^kX in $\mathbb{P}^N$, i.e. $h^{(k)}(X) := N - s^{(k)}(X)$.

The right hand side of (1.5.1) is called the *expected dimension* of $S^k(X)$ and will be denoted by $\sigma^{(k)}(X)$.

Notice that the general fibre of p^k_X is pure of dimension $f_k(X) = (k+1)n + k - s^{(k)}(X) \geq \delta_k(X)$ and moreover $f_k(X) = \delta_k(X)$ if and only if $\delta_j(X) = 0$ for every $j < k$, see Corollary 1.3.6. We will denote by $\mu_k(X)$ the number of irreducible components of the general fibre of p^k_X. In particular, if $s^{(k)}(X) = (k+1)n+k$, then p^k_X is generically finite and $\mu_k(X)$ is the degree of p^k_X, i.e. it is the number of $(k+1)$–secant $\mathbb{P}^k$'s to X passing through the general point of $S^k(X)$.

If $s_k(X) = (k+1)n + k$, we will denote by $\nu_k(X)$ the number of $(k+1)$–secant $\mathbb{P}^k$'s to X meeting the general $\mathbb{P}^{h^{(k)}(X)}$ in $\mathbb{P}^N$. Of course one has:

$$\nu_k(X) = \mu_k(X) \cdot \deg(S^kX) \tag{1.5.2}$$

and therefore:

$$\nu_k(X) = \mu_k(X) \quad \text{if} \quad N = s_k(X) = (k+1)n + k. \tag{1.5.3}$$

Let $X \subset \mathbb{P}^N$ be an irreducible, projective variety. Let k be a positive integer and let $p_1, ..., p_k$ be general points of X. We denote by $T_{X,p_1,...,p_k}$ the span of $T_{p_i}X, i = 1, ..., k$.

Thus with this notation, we can reinterpret Terracini Lemma in the following way: if $p_0, ..., p_k \in X$ are general points and $z \in < p_0, ..., p_k >$ is a general point, then:

$$T_z S^k X = T_{X, p_0, \ldots, p_k}.$$

Consider the projection of X with centre $T_{X,p_1,\ldots,p_k}$. We call this a *general k–tangential projection* of X, and we will denote it by $\pi_{X,p_1,\ldots,p_k}$ or simply by π_k. We will denote by X_k its image. By Terracini's lemma, the map π_k is generically finite to its image if and only if $s_k(X) = (k+1)n + k$. In this case we will denote by $d_{X,k}$ its degree.

In the same situation, the projection of X with centre the space $< p_1, ..., p_k >$ is called a *general k–internal projection* of X, and we will denote it by $t_{X,p_1,\ldots,p_k}$ or simply by t_k. We denote by X^k its image. We set $X_0 = X^0 = X$. Notice that the maps t_k are birational to their images as soon as $k < N - n = \mathrm{codim}(X)$.

Sometimes we will use the symbols X_k [resp. X^k] for k–tangential projections [resp. k–internal projections] relative to *specific*, rather than *general*, points. In this case we will explicitly specify this, thus we hope no confusion will arise for this reason.

We record the following:

1.5.1. LEMMA. *Let $X, Y \subset \mathbb{P}^N$ be closed, irreducible, subvarieties and let L be a linear subspace of dimension n which does not contain either X or Y. Let*

$$\pi = \pi_L : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-n-1}$$

be the projection from L and let X', Y' be the images of X, Y via π. Then:

$$\pi(S(X, Y)) = S(X', Y').$$

In particular, if L does not contain X, then for any non–negative integer k one has:

$$\pi(S^k X) = S^k X'.$$

PROOF. Exercise 1.6.8. □

The following lemma is an application of Terracini's lemma:

1.5.2. LEMMA. *Let $X \subset \mathbb{P}^N$ be an irreducible, projective variety. For all $i = 1, ..., k$ one has:*

$$h^{(k-i)}(X_i) = h^{(k)}(X)$$

whereas for all $i \geq 1$ one has:

$$h^{(k)}(X^i) = \max\{0, h^{(k)}(X) - i\}.$$

PROOF. Exercise 1.6.9. □

As an application of tangential projections, one could prove the foloowing result. See Exercise 1.5.3 for the notation on rational normal scrolls.

1.5.3. PROPOSITION. *Let $X = S(a_1, \ldots, a_n) \subset \mathbb{P}^N$ be a rational normal scroll of dimension n. Then:*

$$\dim(S^k X) = \min\{N, N + k + 1 - \sum_{1 \leq j \leq n;\ k \leq a_j} (a_j - k)\}.$$

In particular, if $N \geq (k+1)n + k$, then $s_k(X) = (k+1)n + k$ if and only if $a_1 \geq k$.

PROOF. It follows by induction using (1.6.3) and Terracini's lemma. We leave the details to the reader, see Exercise 1.6.10. □

For analogous results on the dimension of secant varieties to the Segre embedding of two projective spaces see Exercise 1.6.11.

Given positive integers n, d, we will denote by $V_{n,d}$ the image of $\mathbb{P}^n$ under the *d–Veronese embedding* of $\mathbb{P}^n$ in $\mathbb{P}^{\binom{n+d}{d}-1}$.

If X is a variety of dimension n and Y a subvariety of X, we will denote by $\mathrm{Bl}_Y(X)$ the blow–up of X along Y. If Y is a finite set $\{x_1, ..., , x_n\}$ we denote the blow–up by $\mathrm{Bl}_{x_1,...,x_n}(X)$.

1.5.2. Further remarks on weakly-defective varieties. The two next results are consequences of [**CC1**, Theorem 1.4] that we partially recall here.

1.5.4. THEOREM. ([**CC1**, Theorem 1.4]) *Let $X \subset \mathbb{P}^N$ be an irreducible, projective, non–degenerate variety of dimension n. Assume X is not k–weakly defective for a given k such that $N \geq (n+1)(k+1)$. Then, given $p_0, \dots, p_k$ general points on X, the general hyperplane H containing $T_{X,p_0,\dots,p_k}$ is tangent to X only at $p_0, \dots, p_k$. Moreover such a hyperplane H cuts on X a divisor with ordinary double points at $p_0, \dots, p_k$.*

The first consequence we are interested in is the following:

1.5.5. LEMMA. *Let $X \subset \mathbb{P}^N$ be an irreducible, projective, non–degenerate variety of dimension n, which is not k–weakly defective for a fixed $k \geq 1$ such that $N \geq (k+1)(n+1)$. Then a general k–tangential projection of X is birational to its image, i.e. $d_{X,k} = 1$. In particular, if $N \geq 2n+2$ and $X \subset \mathbb{P}^N$ is not 1-weakly defective general tangential projection of X is birational to its image.*

PROOF. Since X is not k-weakly defective, it is not l–defective for all $l \leq k$. Thus we have $s^{(l)}(X) = (l+1)n + l$ for all $l \leq k$, so that by Terracini's lemma $\tau_{X,p_1,\dots,p_l}$ is generically finite onto X_l for every $l \leq k$ and $p_1, ..., p_l$ general points on X. In particular this is true for $l = k$.

Suppose now that $d_{X,k} > 1$. Then, given a general point $p_0 \in X$ there is a point $q \in X \setminus (T_{X,p_1,\dots,p_k} \cap X)$, $q \neq p_0$, such that $\tau_{X,p_1,\dots,p_k}(p_0) = \tau_{X,p_1,\dots,p_k}(q) := x \in X_k$. This would imply that $T_{X,p_0,p_1,\dots,p_k}$ and $T_{X,q,p_1,\dots,p_k}$ coincide, since both these spaces project via $\tau_{X,p_1,\dots,p_k}$ onto $T_{X_k,x}$. In particular, the general hyperplane tangent to X at $p_0, p_1, \dots, p_k$ is also tangent at q. This contradicts Theorem 1.5.4. $\square$

We also note that Terracini's lemma and Theorem 1.5.4 imply that:

1.5.6. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be an irreducible, projective variety which is not k–weakly defective. If $N \geq (n+1)(k+1)$, then $\mu_k(X) = 1$. Equivalently, if $N \geq (k+1)(n+1)$ and if $\mu_k(X) > 1$, then $X \subset \mathbb{P}^N$ is k-weakly defective.*

1.5.3. Tangent cones to higher secant varieties. In this section we describe the tangent cone to the variety S^kX, at a general point of S^lX, where $0 \leq l < k$, and $X \subset \mathbb{P}^N$ is an irreducible, projective variety of dimension n. Our result can be seen as a generalization of Terracini's Lemma:

1.5.7. THEOREM. ([**CR**, Theorem 3.1]) *Let $X \subset \mathbb{P}^N$ be an irreducible, non–degenerate, projective variety and let $l, m \in \mathbb{N}$ be such that $l + m = k - 1$. If $z \in S^lX$ is a general point, then the cone $S(T_zS^lX, S^mX)$ is an irreducible component of $(C_zS^kX)_{\mathrm{red}}$. Furthermore one has:*

$$\mathrm{mult}_z(S^kX) \geq \deg(S(T_zS^lX, S^mX)) \geq \deg(S^mX_{l+1}).$$

PROOF. We assume that $S^lX \neq \mathbb{P}^r$, otherwise the assertion is trivially true.

The scheme C_zS^kX is of pure dimension $s_k(X)$. Let now $w \in S^mX$ be a general point. By Terracini's lemma and by the generality of $z \in S^lX$, we get:

$$\begin{aligned}\dim(S(T_zS^lX, S^mX)) &= \dim(S(T_zS^lX, T_wS^mX)) = \\ &= \dim(S(S^lX, S^mX)) = \dim(S^kX) = s_k(X)\end{aligned}$$

Thus, since $S(T_zS^lX, S^mX)$ is irreducible and reduced, it suffices to prove the inclusion $S(T_zS^lX, S^mX) \subseteq (C_zS^kX)_{\mathrm{red}}$.

Let again $w \in S^mX$ be a general point. We claim that $w \notin T_zS^lX$. Indeed $S^lX \neq \mathbb{P}^r$ and by (1.2.2):

$$\mathrm{Vert}(S^lX) := \bigcap_{y \in S^lX} T_yS^lX$$

is a proper linear subspace of $\mathbb{P}^r$. If the general point of S^mX were contained in $\mathrm{Vert}(S^lX)$, then $X \subseteq S^mX \subseteq \mathrm{Vert}(S^lX)$ and $X \subset \mathbb{P}^N$ would be degenerate, contrary to our assumption.

Since $w \notin T_zS^lX$, then z is a smooth point of the cone $S(w, S^lX)$. We deduce that:

$$< w, T_zS^lX >= T_zS(w, S^lX) = C_zS(w, S^lX \subseteq C_zS(S^mX, S^lX) = C_zS^kX.$$

By the generality of $w \in S^mX$ we finally have $S(T_zS^lX, S^mX) \subseteq C_zS^kX$. This proves the first part of the theorem.

To prove the second part, we remark that:

$$\mathrm{mult}_z(S^kX) = \deg(C_zS^kX) \geq \deg(S(T_zS^lX, S^mX)).$$

Now, if $p_0, ..., p_l \in X$ are general points, then $S(T_zS^lX, S^mX)$ is the cone with vertex T_zS^lX over $\tau_{X,p_0,\ldots,p_l}(S^mX)$, and, by Lemma 1.5.1 we have that $\tau_{X,p_0,\ldots,p_l}(S^mX) = S^mX_{l+1}$. Thus

$$\deg(S(T_zS^lX, S^mX)) \geq \deg(S^mX_{l+1}),$$

proving the assertion. □

1.5.4. A lower bound on the degree of secant varieties. The degree d of an irreducible non–degenerate variety $X \subset \mathbb{P}^r$ verifies the lower bound

$$d \geq \mathrm{codim}(X) + 1. \tag{1.5.4}$$

Varieties whose degree is equal to this lower bound are called varieties of *minimal degree*. As well known, they have nice geometric properties, e.g. they are rational (see [**EH**]). In the present section we will prove a lower bound on the degree of the k–secant variety to a variety X. This bound generalizes (1.5.4) and we will see that varieties X attaining it have interesting features which resemble the properties of minimal degree varieties.

Before proving the main result of this section, we need a useful lemma. For an irreducible variety $Z \subseteq \mathbb{P}^N$ we defined $t_{Z,p}$ as the projection from the the general point $p \in Z$ restricted to Z, i.e. $t_{Z,p} : Z \dashrightarrow t_{Z,p}(Z) = Z^1$. In this section we shall sometimes abuse notation by considering an arbitrary $p \in Z$ and also in this case we shall indicate by Z^1 the projection from p.

1.5.8. LEMMA. *Let $X \subset \mathbb{P}^N$ be an irreducible, non–degenerate, projective variety, let $k \geq 0$ be an integer such that $S^kX \neq \mathbb{P}^N$ and let $p \in X$ be an arbitrary point. Then one has:*

(i) $t_{S^kX,p}(S^kX) = S^kX^1$*;*
(ii) *the general point in X does not belong to* $\mathrm{Vert}(S^kX)$*;*
(iii) *if $p \in X \setminus (X \cap \mathrm{Vert}(S^kX)$, in particular if $p \in X$ is a general point, then $t_{S^k(X),p}$ is generically finite to its image S^kX^1 and $s_k(X) = s_k(X^1)$;*
(iv) *if $\delta_k(X) = 0$ and $p \in X \setminus (X \cap \mathrm{Vert}(S^kX)$, then $\delta(X^1) = 0$;*
(v) *if $p \in X \setminus (X \cap \mathrm{Vert}(S^kX)$ and if $\theta_k(X)$ denotes the degree of $t_{S^k(X),p}$, then:*

$$\deg(S^kX) = \theta_k(X) \cdot \deg(S^kX^1) + \mathrm{mult}_p(S^kX) \geq \deg(S^kX^1) + \mathrm{mult}_p(S^kX)$$

and

$$\mu_k(X^1) = \theta_k(X) \cdot \mu_k(X).$$

In particular:

(vi) *if* $p \in X \setminus (X \cap \mathrm{Vert}(S^kX)$ *and if*

$$\deg(S^kX) = \deg(S^kX^1) + \mathrm{mult}_p(S^kX)$$

then $\theta_k(X) = 1$, *i.e.* $t_{S^k(X),p} : S^kX \dashrightarrow S^kX^1$ *is birational and then* $\mu_k(X^1) = \mu_k(X)$*;*

(vii) *if, in addition,* $\mu_k(X^1) = 1$ *then also* $\mu_k(X) = 1$ *and* $\theta_k(X) = 1$.

PROOF. Part (i) follows by Lemma 1.5.1.

Since S^kX is a proper subvariety in $\mathbb{P}^N$, then $\mathrm{Vert}(S^kX)$ is a proper linear subspace of $\mathbb{P}^N$. This implies part (ii). Part (iii) is immediate.

Since $S^kX \neq \mathbb{P}^N$, if X is not k–defective, we have $s_k(X) = (k+1)n + k < r$. By part (iii) we have also $s_(X^1) = (k+1)n + k \leq r - 1$, i.e. X^1 is also not k–defective. This proves (iv).

The first assertion of part (v) is immediate. Furthermore we have a commutative diagram of rational maps:

$$\begin{array}{ccc} S^k_X & \overset{t}{\dashrightarrow} & S^k_{X^1} \\ p^k_X \downarrow & & \downarrow p^k_{X^1} \\ S^k(X) & \overset{t_{S^k(X),p}}{\dashrightarrow} & S^k(X^1) \end{array}$$

where t is determined, in an obvious way, by $t_{S^k(X),p}$. By the hypothesis, $t_{S^k(X)}$ has degree $\theta_k(X)$, whereas t is easily seen to be birational. Hence the conclusion follows. Parts (vi) and (vii) are now obvious. ☐

Now we come to the main result of this section:

1.5.9. THEOREM. ([**CR**, Theorem 4.2]) *Let* $X \subset \mathbb{P}^N$ *be an irreducible, non–degenerate, projective variety and let* $h := \mathrm{codim}(S^kX) > 0$. *Then:*

$$\deg(S^kX) \geq \binom{h+k+1}{k+1} \tag{1.5.5}$$

and, if $l = 0, ..., k$ *and* $x \in S^lX$ *is any point, then:*

$$\mathrm{mult}_x(S^kX) \geq \binom{h+k-l}{k-l}. \tag{1.5.6}$$

Suppose equality holds in (1.5.5) and $h \geq 1$. *Then:*

(i) *if* $x \in X$ *is a general point, one has:*

$$C_xS^kX = S(T_xX, S^{k-1}X), \quad \mathrm{mult}_x(S^k(X)) = \binom{k+h}{k};$$

(ii) *for every* m *such that* $1 \leq m \leq h$, *one has:*

$$\deg(S^kX^m) = \binom{h-m+k+1}{k+1};$$

(iii) *for every* m *such that* $1 \leq m \leq h$, *the projection from a general point* $x \in X^{m-1}$*:*

$$t_{S^kX^{m-1},x} : S^kX^{m-1} \dashrightarrow S^kX^m$$

is birational;

(iv) *for every m such that $1 \le m \le k$ one has:*

$$\deg(S^{k-m}X_m) = \binom{h+k-m+1}{k-m+1};$$

in particular X_k is a variety of minimal degree;

(v) *if X is not k–defective, then, for every m such that $1 \le m \le h$, also X^m is not k–defective and $\mu_k(X) = \mu_k(X^m)$;*

(vi) *if X is not k–defective then:*

$$d_{X,k} \le \mu_k(X).$$

PROOF. We make induction on both k and h. For $k = 0$ we have the bound 1.5.4 for the minimal degree of an algebraic variety, while for $h = 0$ the assertion is obvious for every k. Let us project X and S^kX from a general point $x \in X$. By Lemma 1.5.8, Theorem 1.5.7, Lemma 1.5.2 and by induction we get:

$$\deg(S^kX) \ge \deg(S^kX^1) + \mathrm{mult}_x(S^kX) \ge$$
$$\ge \deg(S^kX^1) + \deg(S^{k-1}X_1) \ge \binom{k+h}{k+1} + \binom{k+h}{k} = \binom{k+h+1}{k+1}$$

whence (1.5.5) follows. Let now $x \in S^l(X)$ be a general point, then by Theorem 1.5.7, Lemma 1.5.2 and by (1.5.5) one has:

$$\mathrm{mult}_x(S^kX) \ge \deg(S^{k-l-1}X_{l+1}) \ge \binom{k+h-l}{k-l}$$

proving (1.5.6) in this case. Of course (1.5.6) also holds if $x \in S^l(X)$ is any point.

If equality holds in (1.5.5), one immediately obtains assertions (i)–(iv) for $m = 1$. By an easy induction one sees that (i)–(iv) hold in general.

Assertion (v) follows by Lemma 1.5.8. As for (vi), consider the following commutative diagram:

$$\begin{array}{ccc} X & \overset{\tau_{X,k}}{\dashrightarrow} & X_k \\ t_{X,h} \downarrow & & \downarrow t_{X_k,h} \\ X^h & \overset{\tau_{X^h,k}}{\dashrightarrow} & \mathbb{P}^n. \end{array}$$

Notice that the vertical maps $t_{X,h}, t_{X_k,h}$ are birational being projections from h general points on a variety of codimension bigger than h. Thus one has:

$$d_{X,k} = d_{X^h,k}.$$

On the other hand, by Theorem 4.1.6 and Lemma 1.5.8 one has:

$$d_{X^h,k} \le \mu_k(X^h) = \mu_k(X)$$

which proves the assertion. ☐

1.5.10. REMARK. It is possible to improve the previous result. For example, using Lemma 1.5.8, one sees that (i) holds not only if $x \in X$ is general, but also if x is any smooth point of X not lying on $\mathrm{Vert}(S^k(X))$. Similar improvements can be found for (ii)–(v). We leave this to the reader, since we are not going to use it later.

1.5.11. DEFINITION. Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective variety of dimension n. Let k be a positive integer.

Let $k \geq 2$ be an integer. One says that X is *k–regular* if it is smooth and if there is no subspace $\Pi \subset \mathbb{P}^r$ of dimension $k-1$ such that the scheme cut out by Π on X contains a finite subscheme of length $\ell \geq k+1$. By definition 1–*regularity* coincides with smoothness.

We say that X *has minimal k–secant degree*, briefly X is an $\mathcal{M}^k$–variety, if $r = s^{(k)}(X) + h$, $h := \operatorname{codim}(S^k(X)) > 0$, and $\deg(S^k(X)) = \binom{h+k+1}{k+1}$ (compare with Theorem 1.5.9).

We say that X is a variety *with the minimal number of apparent $(k+1)$–secant $\mathbb{P}^{k-1}$'s*, briefly X is an $\mathcal{MA}^{k+1}_{k-1}$–variety, if $s^{(k)}(X) = (k+1)n + k$, $r = s^{(k)}(X) + h$, $h := \operatorname{codim}(S^k(X)) > 0$, and if $\nu_k(X) = \binom{h+k+1}{k+1}$ (compare with Theorem 1.5.9 and 1.5.2). In other words X is an $\mathcal{MA}^{k+1}_{k-1}$–variety if and only if it is not k–defective, is an $\mathcal{M}^k$–variety and $\mu_k(X) = 1$. For example, an $\mathcal{M}^k$–variety which is not k–weakly defective is an $\mathcal{MA}^{k+1}_{k-1}$–variety.

We say that X is a variety *with one apparent $(k+1)$–secant* $\mathbb{P}^{k-1}$, briefly X is an $\mathcal{OA}^{k+1}_{k-1}$–variety, if $r = s^{(k)}(X) = (k+1)n + k$ and $\mu_k(X) = 1$.

The terminology introduced in the previous definition is motivated by the fact that, for example, $\mathcal{OA}^{k+1}_{k-1}$–varieties are an extension of *varieties with one apparent double point* or OADP–varieties, classically studied by Severi [**Sev**] (for a modern reference see [**CMR**]).

1.6. Exercises

1.6.1. EXERCISE. Let K be a(n algebraically closed) field. Recall that the linear combination of two (symmetric) matrixes of rank 1 has rank at most 2 and that every (symmetric) matrix of rank 2 can be written as the linear combination of two (symmetric) matrixes of rank 1.

Deduce the following geometrical consequences for the secant varieties of the varieties described below.

(1) Let $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ be the 2-Veronese surface in $\mathbb{P}^5$. Identify $\mathbb{P}^5$ with
$$\mathbb{P}(\{A \in M(3;K) \ : \ A = A^t\}),$$
and show that $X = \{[A] \ : \ \operatorname{rk}(A) = 1\}$. Prove that $SX = TX = V(\det(A)) \subset \mathbb{P}^5$ is the cubic hypersurface given by the cubic polynomial $\det(A)$. Show that if $x_1, x_2 \in X$, then $T_{x_1}X \cap T_{x_2}X \neq \emptyset$ (Try to prove that if the points are general, then the intersection consists of a point). Prove that $\operatorname{Sing}(SX) = X$.

(2) Let $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ be the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$. Identify $\mathbb{P}^8$ with
$$\mathbb{P}(\{A \in M(3;K)\}),$$
and show that $X = \{[A] \ : \ \operatorname{rk}(A) = 1\}$. Prove that $SX = TX = V(\det(A)) \subset \mathbb{P}^8$ is the cubic hypersurface given by the cubic polynomial $\det(A)$. Show that if $x_1, x_2 \in X$, then $T_{x_1}X$ and $T_{x_2}X$ intersect at least along a line (prove that if the points are general, then the intersection consists of a line). Take H be a general hyperplane in $\mathbb{P}^8$ and let $Y := X \cap H$. Then Y is a smooth, irreducible, non-degenerate 3-fold $Y \subset \mathbb{P}^7$ such that $SY \subseteq SX \cap H$ so that $\dim(SY) \leq 6$ (in fact one can prove that $SY = SX \cap H$ and hence that $\dim(SY) = 6$). Prove that given $y_1, y_2 \in Y$, then $T_{y_1}Y \cap T_{y_2}Y \neq \emptyset$ (consists of a point if the points are general). Prove that $\operatorname{Sing}(SX) = X$.

Let $p \in \mathbb{P}^9 \backslash \mathbb{P}^8$, let $Z = S(p, X) \subset \mathbb{P}^9$ and let $X' = X \cap W$, with $W \subset \mathbb{P}^9$ a general hypersurface, not an hyperplane, not passing through p. Then X' is a smooth, irreducible, non-degenerate 4-fold such that $SX' = SZ = S(p, SX)$. Conclude that $\dim(SX') = 8$ and use the fact that Z is a cone over X to deduce that two general tangent spaces to X' intersect.

(3) Generalize the previous exercise and find the relation between $SX \subset \mathbb{P}^N$ and $SX' \subset \mathbb{P}^{N+1}$ for $X' \subset \mathbb{P}^{N+1}$ a general intersection of $Z = S(p, X) \subset \mathbb{P}^{N+1}$ with a general hypersurface $W \subset \mathbb{P}^{N+1}$, not passing through $p \in \mathbb{P}^{N+1} \backslash \mathbb{P}^N$.

1.6.2. EXERCISE. Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties. The following holds:

(1) for every $x \in X$,
$$Y \subseteq S(x,Y) \subseteq T_xS(x,Y) \subseteq T_xS(X,Y)$$
and in partiucular $< x, < Y >> \subseteq T_xS(x,Y)$.

(Hint: By definition of join we get the inclusion $S(x,Y) \subseteq S(X,Y)$ and hence $T_xS(x,Y) \subseteq T_xS(X,Y)$. Moreover for every $y \in Y$, $y \neq x$, the line $< x, y >$ is contained in $S(x,Y)$ and passes through x so that it is contained in $T_xS(x,Y)$ and part 1) easily follows.)

(2) if $S^kX = S^{k+1}X$ for some $k \geq 0$, then $S^kX = \mathbb{P}^{s_k(X)} \subseteq \mathbb{P}^N$; (Hint: Let $z \in S^kX$ be a smooth point of S^kX. From part 1) we get
$$X \subseteq T_zS(S^kX, X) = T_zS^{k+1}X = T_zS^kX = \mathbb{P}^{s_k(X)}.$$
Thus $S^kX \subseteq < X > \subseteq T_zS^kX = \mathbb{P}^{s_k(X)}$ so that $S^kX = < X > = \mathbb{P}^{s_k(X)}$ since S^kX and T_zS^kX are both irreducible varieties of dimension $s_k(X)$.)

(3) if $\dim(S^{k+1}X) = \dim(S^kX) + 1$ for some $k \geq 0$, then $S^{k+1}X = \mathbb{P}^{s_{k+1}(X)}$ so that S^kX is a hypersurface in $\mathbb{P}^{s_{k+1}(X)}$; (Hint: To prove part 3), take a general point $z \in S^{k+1}X \setminus S^kX$. For general $x \in X$ we get $S^kX \subsetneq S(x, S^kX) \subseteq S(X, S^kX) = S^{k+1}X$. Thus for general $x \in X$ we get $S(x, S^kX) = S^{k+1}X$ since $s_{k+1}(X) = s_k(X) + 1$. In particular $z \in S(x, S^kX)$ for $x \in X$ general, i.e. there exists $y \in S^kX$ such that $z \in < x, y > \subset S^{k+1}X$. Thus a general point $x \in X$ is contained in $T_zS^{k+1}X$ so that
$$S^{k+1}X \subseteq < X > \subseteq T_zS^{k+1}X$$
yields $S^{k+1}X = < X > = \mathbb{P}^{s_{k+1}(X)}$ since $\dim(T_zS^{k+1}X) = s_{k+1}(X)$ by the generality of $z \in S^{k+1}X$.)

(4) if $S^{k+1}X$, $k \geq 0$, is not a linear space, then $S^kX \subseteq \mathrm{Sing}(S^{k+1}X)$. (Hint: Remark that $T_zS^{k+1}X \subseteq < S^{k+1}X > = < S^kX >$. Take $z \in S^kX$ and observe that, via part 1)
$$< S^{k+1}X > = < S^kX > = S(z, < S^k >) \subseteq T_zS^{k+1}X \subseteq < S^{k+1}X >,$$
so that $T_zS^{k+1}X = < S^{k+1}X > \supsetneq S^kX$. By hypothesis the last inclusion is strict, yielding $\dim(T_zS^{k+1}X) > \dim(S^{k+1}X)$. Thus z is a singular point of $S^{k+1}X$ and part 4) follows.)

(5) Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate projective curve. Then $s_k(C) = \min\{2k+1, N\}$.

(Hint: For $k = 0$ it is true and we argue by induction. Suppose $S^kC \subsetneq \mathbb{P}^N$. By proposition 1.2.2 $s_k(C) \geq s_{k-1}(C) + 2$ and the description $S^k(C) = S(C, S^{k-1}C)$ yields $s_k(C) \leq s_{k-1}(C) + 2$ so that $s_k(C) = 2(k-1) + 1 + 2 = 2k + 1$ as claimed.)

(6) Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective of dimension $n \geq 1$. Let $k < k_0$. Prove that $s_k(X) \geq n + 2k$ for every $k < k_0$ and that $s_j(X) = n + 2j$ yields $s_k(X) = n + 2k$ for every $k \leq j$. In particular if $s_k(X) = n + 2k$ for some $k \geq 1$, then $s(X) = n + 2$ and $SX \subseteq \mathbb{P}^N$.

1.6.3. EXERCISE. Let $X \subset \mathbb{P}^N$ be a closed irreducible variety of dimension $\dim(X) = n$. The following holds:

(1)
$$\mathrm{Vert}(X) = \mathbb{P}^l \subseteq \bigcap_{x \in X} T_xX,$$
$l \geq -1$;

(2) if $\mathrm{codim}(\mathrm{Vert}(X), X) \leq 1$, then $\mathrm{Vert}(X) = X = \mathbb{P}^n \subset \mathbb{P}^N$;

(3) if $\dim(S(X,Y)) = \dim(X) + 1$, then $Y \subseteq \mathrm{Vert}(S(X,Y))$;

(4) if char$(K) = 0$,
$$\mathrm{Vert}(X) = \bigcap_{x \in X} T_x X = \mathbb{P}^l \subseteq X,$$
$l \geq -1$;

(5) suppose char$(K) = 0$ and $\emptyset \neq \mathrm{Vert}(X) \subsetneq X$, then $X = S(\mathrm{Vert}(X), X')$ is a cone, where X' is the projection of X from $\mathrm{Vert}(X)$ onto a $\mathbb{P}^{N-l-1}$ skew to $\mathrm{Vert}(X)$ $(\dim(X') = n - l - 1)$.

(Hint:

To prove 1) it is sufficient to show that, given two points $x_1, x_2 \in \mathrm{Vert}(X)$, the line $< x_1, x_2 >$ is contained in $\mathrm{Vert}(X)$, forcing $\mathrm{Vert}(X)$ irreducible and linear by proposition 1.2.2 part 2). Taken $y \in < x_1, x_2 > \setminus \{x_1, x_2\}$ and $x \in X \setminus \mathrm{Vert}(X)$, it is sufficient to prove that $< y, x > \subset X$. By definition the lines $< x_i, x >$ are contained in X and by varying the point $q \in < x_2, x > \subset X$ and by joining it with x_1 we see that the line $< x_1, q >$ is contained in X for every such q, i.e. the plane $\Pi_x = < x_1, x_2, x >$ is contained in X. Since y and x belong to Π_x, the claim follows.

If $\mathrm{Vert}(X) = X$, then $X = \mathbb{P}^n$ by part 1). If there exists $W = \mathbb{P}^{n-1} \subseteq \mathrm{Vert}(X) = \mathbb{P}^l \subseteq X$, i.e. if $l \geq n-1$, we can take $x \in X \setminus W$. Therefore $S(x, W) = \mathbb{P}^n$ and $S(W, x) \subseteq X$ forces $X = \mathbb{P}^n$.

To prove 3) take $y \in Y \setminus \mathrm{Vert}(X)$ and observe that for dimension reasons $S(y, X) = S(Y, X)$ and $S(y, S(X, Y)) = S(y, S(y, X)) = S(y, X) = S(Y, X)$ gives the desired conclusion.

Set $L = \bigcap_{x \in X} T_x X$ and assume char(K)=0. By 1.2.8 $\dim(S(L, X)) = \dim(X)$, yielding $X = S(L, X)$ and $L \subseteq \mathrm{Vert}(X)$, which proves part 4). Part 5) follows in a straightforward way.)

1.6.4. EXERCISE. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension $n = \dim(X)$. Assume char(K)=0, $N \geq n+3$ and $\dim(SX) = n+2$. If through the general point $x \in X$ there passes a line l_x contained in X, then $X \subset \mathbb{P}^N$ is a cone.

(Hint: Let $x \in X$ be a general point. Then $x \notin \mathrm{Vert}(X)$ and $x \notin \mathrm{Vert}(SX)$ since X is non-degenerate, so that $X \subsetneq S(l_x, X) \subseteq SX$. If $\dim(S(l_x, X)) = n+2$, then $S(l_x, X) = SX$. Since $S(l_x, SX) = S(l_x, S(l_x, X)) = S(l_x, X) = SX$, we would deduce $x \in l_x \subseteq \mathrm{Vert}(SX)$. In conclusion l_x is not contained in $\mathrm{Vert}(SX)$ and $\dim(S(l_x, X)) = n+1$. Then the general tangent space to X, $T_y X$, will cut l_x in a point $p_{x,y} := l_x \cap T_y X$. If this point varies with y, then two general tangent spaces $T_{y_1} X$ and $T_{y_2} X$ would contain l_x so that $< l_x, < T_{y_1} X, T_{y_2} X >> = < T_{y_1} X, T_{y_2} X >$ would force $S(l_x, SX) = SX$, i.e. $l_x \subseteq \mathrm{Vert}(SX)$. So the point remain fixed, i.e. $p \in \cap_{y \in X} T_y X = \mathrm{Vert}(X)$ and X is a cone by proposition 1.2.6.)

1.6.5. EXERCISE. Prove Edge's argument from [**Ed**] to the effect that smooth irreducible divisors of type $(0, 2)$, (1,2) and $(2, 1)$ on the Segre varieties $Y = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$, $n \geq 2$, have one apparent double point in the following steps.

(1) Prove first that the only smooth curves, not necessarily irreducible, on a smooth quadric in $\mathbb{P}^3$ having one apparent double point are of the above types.

(2) For $p \notin Y := \mathbb{P}^1 \times \mathbb{P}^n$, the entry locus $\Sigma_p(Y)$ has the form $\mathbb{P}^1 \times \mathbb{P}^1_p$ for some $\mathbb{P}^1_p \subset \mathbb{P}^n$ and spans a linear $\mathbb{P}^3_p$.

(3) If X is a divisor of type (a, b) of Y, the secant lines of X passing through p are exactly the secant lines of $X \cap \mathbb{P}^3_p$ passing through p.

(4) For a general $p \in \mathbb{P}^{2n+1}$, $X \cap \mathbb{P}^3_p$ is a reduced, not necessarily irreducible curve and it is a divisor of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1_p$. Hence X has one apparent double point if and only if $(a, b) \in \{(1, 2),\ (2, 1),\ (2, 0),\ (0, 2)\}$. If $(a, b) = (2, 0)$, then $X = \mathbb{P}^n \amalg \mathbb{P}^n$ is reducible.

The divisors of type $(2, 1)$ are the rational normal scrolls of minimal degree in $\mathbb{P}^{2n+1}$. The divisors of type $(0, 2)$ are isomorphic to $\mathbb{P}^1 \times Q^{n-1}$, where $Q^{n-1} \subset \mathbb{P}^n$ is a quadric hypersurface of maximal rank, so that they admit a structure of *twisted cubic over a split cubic Jordan algebra*, see [**Mk**]; divisors of type $(1, 2)$ are hyperquadric fibrations of special kind. The above varieties are usual called *Edge varieties*. Edge varieties have degree $d = n + 2$, respectively $2n$, $2n + 1$ and in [**AR**] are characterized as the only varieties with one apparent double point of dimension n and degree $d \leq 2n + 1$ for every $n \geq 2$. Moreover in [**AR**] it is shown that for $2n+2 \le d \le 2n+4$ there are only 3 varieties with one apparent double point: for $n = 3$ and $d = 8$ it is the scroll over a surface we cited above; for $n = 4$ and $d = 12$ the linear section of $S^{10} \subset \mathbb{P}^{15}$ and for $n = 6$ and $d = 16$ the variety $\mathbb{G}^{lag}_{\mathbb{R}}(2, 5) \subset \mathbb{P}^{13}$.

1.6.6. EXERCISE. Prove the following facts.

(1) Let $X \subsetneq \mathbb{P}^M \subsetneq \mathbb{P}^N$ be a degenerate variety. Prove that $X^* \subset \mathbb{P}^{N*}$ is a cone of vertex $\mathbb{P}^{M*} = \mathbb{P}^{N-M-1} \subset \mathbb{P}^{N*}$ over the dual variety of X in $\mathbb{P}^M$. Suppose $X = S(L, X')$ is a cone of vertex $L = \mathbb{P}^l$, $l \ge 0$, over a variety $X' \subset M = \mathbb{P}^{N-l-1}$, $M \cap L = \emptyset$. Then $X^* \subset (\mathbb{P}^l)^* = \mathbb{P}^{N-l-1} \subset (\mathbb{P}^N)^*$ is degenerated. Is there any relation between X^* and the dual of X' in M?

Suppose $X \subset \mathbb{P}^N$ is a cone. Prove that $X^* \subset \mathbb{P}^{N*}$ is degenerated. Conclude that $X \subset \mathbb{P}^N$ is degenerated if and only if $X^* \subset \mathbb{P}^{N*}$ is a cone; and, dually, that $X \subset \mathbb{P}^N$ is a cone if and only if $X^* \subset \mathbb{P}^{N*}$ is degenerated.

(2) Let $C \subset \mathbb{P}^N$ be an irreducible non-degenerate projective curve. Then $p_2 : \mathcal{P}_C \to C^* \subset (\mathbb{P}^N)^*$ is a finite morphism so that $\operatorname{def}(C) = 0$.

(3) Let $X \subset \mathbb{P}^N$ be a non-singular variety, then $\mathcal{P}_X \simeq \mathbb{P}(\mathcal{N}_{X/\mathbb{P}^N}(1))$ (Grothendieck's notation), where $\mathcal{N}_{X/\mathbb{P}^N}(1)$ is the the twist of the normal bundle of X in $\mathbb{P}^N$ by $\mathcal{O}_{\mathbb{P}^N}(1)$. Show that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is given by a sublinear system of $|\mathcal{O}_{\mathcal{N}_{X/\mathbb{P}^N}(1)}(1)|$. (Hint: restrict Euler sequence to X and use the standard conormal sequence; interpret these sequences in terms of the associated projective bundles and of the incidence correspondence defining $\mathcal{P}_X$).

(4) Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate complete intersection. Deduce by the previous exercise that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is a finite morphism so that $\dim(X^*) = N - 1$, i.e. $\operatorname{def}(X) = 0$ (Hint: show that $\mathcal{N}_{X/\mathbb{P}^N}(1)$ is a sum of very ample line bundles; deduce that $\mathcal{O}_{\mathcal{N}_{X/\mathbb{P}^N}(1)}(1)$ is very ample and finally that $p_2 : \mathcal{P}_X \to X^* \subset \mathbb{P}^{N*}$ is a finite morphism).

(5) Suppose char(K)=0 and let $C \subset \mathbb{P}^2$ be an irreducible curve, not a line. Show that C^* is an irreducible curve of degree at least 2. Take a tangent line at a point $x \in C$. Show that if T_xC is tangent at another point $y \in C$, $y \ne x$, then the point $(T_xC)^* \in C^*$ is a singular point of C^*. Deduce that if char(K)=0, then a general tangent line is tangent to C only at one point. Deduce that the same is true for an irreducible curve $C \subset \mathbb{P}^N$, $N \ge 3$.

(6) Let $X = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$, $n \ge 1$, be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$. Identify $\mathbb{P}^{2n+1}$ with the projectivization of the vector space of $2 \times n+1$ matrices and show that, due to the fact that there are only two orbits for the action of $GL(2)$ on $\mathbb{P}^N$ and on $(\mathbb{P}^N)^*$, $(\mathbb{P}^1 \times \mathbb{P}^n)^* \simeq \mathbb{P}^1 \times \mathbb{P}^n$ so that $\operatorname{def}((\mathbb{P}^1 \times \mathbb{P}^n)) = n - 1$. Interpret this result geometrically and reverse the construction for $n = 2$ to show directly that $X = X^*$.

(7) Use the same argument as above to show that if $X = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$, or if $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, then $X^* \simeq SX$ and $SX^* \simeq X$.

1.6.7. EXERCISE. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety. Assume char(K)=0 and $SX \subsetneq \mathbb{P}^N$. Then $(SX)^* \subseteq \operatorname{Sing}(X^*) \subsetneq X^*$, i.e. a general bitangent hyperplane represents a singular point of X^*. More generally for a given $k \ge 2$ such that $k < k_0(X)$, we have $(S^kX)^* \subseteq \operatorname{Sing}((S^{k-1}X)^*) \subsetneq (S^{k-1}X)^*$, i.e. a general $(k+1)$-tangent hyperplane represents a singular point of $(S^{k-1}X)^*$.

(Hint: Take $H \in (SX)^*$ general point. Then $H \supseteq T_zSX$, with $z \in SX$ general point. By Corollary 1.3.6, H is tangent to X along $\Sigma_z(X) \setminus (\Sigma_z(X) \cap \operatorname{Sing}(X))$ so that $H \in X^*$. Since X is non-degenerate, then $z \notin X$ implies that the contact locus of H on X is not linear, yielding $H \in \operatorname{Sing}(X^*)$ by Proposition 1.4.5.

Take more a general $H \in (S^kX)^*$ and recall that $S^kX = S(X, S^{k-1}X)$. Then $H \subseteq T_zS^kX$, with $z \in S^kX$ general point. Then there exists $y \in \operatorname{Sm}(S^{k-1}X)$ with $y \in \Sigma^k_z(X)$ and such that $z \in < x, y >$, $x \in X$, $x \ne y$. By Terracini Lemma $T_zS^kX \supseteq T_yS^{k-1}X$ so that $H \in (S^{k-1}X)^*$. Since $x \in X$, $x \in \operatorname{Sing}(H \cap S^{k-1}X)$, so that $p_2^{-1}(H) \subseteq S^{k-1}X$ is not linear since once again $z \in S^kX \setminus S^{k-1}X$ by the non-linearity of S^kX).

1.6.8. EXERCISE. Let $X, Y \subset \mathbb{P}^N$ be closed, irreducible, subvarieties and let $L = \mathbb{P}^l$ be a linear subspace of dimension $l \geq 0$, which does not contain either X or Y. Let $\pi_L : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-l-1}$ be the projection from L. Then:

$$\pi_L(S(X,Y)) = S(\pi_L(X), \pi_L(Y)).$$

In particular, if L does not contain X, then for any non–negative integer k one has:

$$\pi(S^k(X)) = S^k(\pi_L(X)).$$

(Hint: It is clear that $\pi(S(X,Y)) \subseteq S(X', Y')$. Let $x' \in X', y' \in Y'$ be general points. Then there are $x \in X, y \in Y$ such that $\pi(x) = x', \pi(y) = y'$. Thus $\pi(< x, y >) =< x', y' >$, proving that $S(X', Y') \subseteq \pi(S(X,Y))$, i.e. the first assertion. The rest of the statement follows by making induction on k.

1.6.9. EXERCISE. Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety. For all $i = 1, ..., k$ one has:

$$h^{(k-i)}(X_i) = h^{(k)}(X)$$

whereas for all $i \geq 1$ one has:

$$h^{(k)}(X^i) = \max\{0, h^{(k)}(X) - i\}.$$

(Hint: Let $p_0, ..., p_k \in X$ be general points. Terracini's lemma says that $T_{X,p_0,...,p_k}$ is a general tangent space to $S^k(X)$ and that its projection from $T_{X,p_{k-i+1},...,p_k}$ is the general tangent space to $S^{k-i}(X_i)$. This implies the first assertion.

To prove the second assertion, note that it suffices to prove it for $i < h^{(k)}(X)$. Indeed, if $i \geq h^{(k)}(X)$ then, by Lemma 1.5.1 one has $h^{(k)}(X^i) = 0$ since already $h^{(k)}(X^{h^{(k)}}) = 0$. Thus, suppose $i < h^{(k)}(X)$. Let $p_0, ..., p_k \in X$ be general points and take i general points $q_1, ..., q_i$ in $X \setminus (X \cap T_{X,p_0,...,p_k})$. Then the projection of $T_{X,p_0,...,p_k}$ from $< q_0, ..., q_i >$ is the tangent space to $S^k(X^i)$. Furthermore $i < h^{(k)}(X)$ yields $< q_0, ..., q_i > \cap T_{X,p_0,...,p_k} = \emptyset$. This implies the second assertion.

1.6.10. EXERCISE. Fill the details in the above proofs and claims.

Let $0 \leq a_1 \leq a_1 \leq ... \leq a_n$ be integers and set $\mathbb{P}(a_1, ..., a_n) := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$. We will denote by H a divisor in $|\mathcal{O}_{\mathbb{P}(a_1,...,a_n)}(1)|$ and by F a fibre of the structure morphism $\pi : \mathbb{P}(a_1, ..., a_n) \to \mathbb{P}^1$. Notice that the corresponding divisor classes, which we still denote by H and F, freely generate $\mathrm{Pic}(\mathbb{P}(a_1, ..., a_n))$.

Set $r = a_1 + ... + a_n + n - 1$ and consider the morphism:

$$\phi := \phi_{|H|} : \mathbb{P}(a_1, ..., a_n) \to \mathbb{P}^r$$

whose image we denote by $S(a_1, ..., a_n)$. As soon as $a_n > 0$, the morphism ϕ is birational to its image. Then the dimension of $S(a_1, ..., a_n)$ is n and its degree is $a_1 + ... + a_n = r - n + 1$, thus $S(a_1, ..., a_n)$ is a *rational normal scroll*, which is smooth if and only if $a_1 > 0$. Otherwise, if $0 = a_1 = ... = a_i < a_{i+1}$, then $S(a_1, ..., a_n)$ is the cone over $S(a_{i+1}, ..., a_n)$ with vertex a $\mathbb{P}^{i-1}$. One uses the simplified notation $S(a_1^{h_1}, ..., a_m^{h_m})$ if a_i is repeated h_i times, $i = 1, ..., m$.

We will sometimes use the notation H and F to denote the Weil divisors in $S(a_1, ..., a_n)$ corresponding to the ones on $\mathbb{P}(a_1, ..., a_n)$. Of course this is harmless if $a_1 > 0$, since then $\mathbb{P}(a_1, ..., a_n) \simeq S(a_1, ..., a_n)$.

Recall that rational normal scrolls, the Veronese surface in $\mathbb{P}^5$ and the cones on it, and the quadrics, can be characterized as those non–degenerate, irreducible varieties $X \subset \mathbb{P}^r$ in a projective space having minimal degree $\deg(X) = \mathrm{codim}(X) + 1$ (see [**EH**]).

Let $X = S(a_1, \ldots, a_n) \subset \mathbb{P}^r$ be as above. We leave to the reader to see that:

$$X^1 = S(b_1, ..., b_n) \quad \text{where} \quad \{b_1, ..., b_n\} = \{a_1, ..., a_n - 1\}. \tag{1.6.1}$$

One can also consider the projection X' of X from a general $\mathbb{P}^{n-1}$ of the ruling of X. This is not birational to its image if $a_1 = 0$ and one sees that if $a_1 = ... = a_i = 0 < a_{i+1}$, then:

(1.6.2) $$X' = S(c_1, ..., c_{n-i}) \quad \text{where} \quad \{c_1, ..., c_{n-i}\} = \{a_{i+1} - 1, ..., a_n - 1\}.$$

A general tangential projection of $X = S(a_1, \ldots, a_n)$ is the composition of the projection of X from a general $\mathbb{P}^{n-1}$ of the ruling of X and of a general internal projection of X'. Therefore, by putting (1.6.1) and (1.6.2) together, one deduces that if $a_1 = ... = a_i = 0 < a_{i+1}$, then:

(1.6.3) $$X_1 = S(d_1, ..., d_{n-i}) \quad \text{where} \quad \{d_1, ..., d_{n-i}\} = \{a_{i+1} - 1, ..., a_n - 2\}.$$

As a consequence we have:

Let $X = S(a_1, \ldots, a_n) \subset \mathbb{P}^r$ be a rational normal scroll as above. Then:

$$\dim(S^k(X)) = \min\{r, r + k + 1 - \sum_{1 \le j \le n;\ k \le a_j} (a_j - k)\}.$$

In particular, if $r \geq (k+1)n + k$, then $s^{(k)}(X) = (k+1)n + k$ if and only if $a_1 \geq k$.

(Hint: It follows by induction using (1.6.3) and Terracini's lemma. We leave the details to the reader).

A different proof of the same result can be obtained by writing the equations of $S^k(X)$ (see [**Ro**] and [**CJ1**] for this point of view).

1.6.11. EXERCISE. Given positive integers $0 < m_1 \leq ... \leq m_h$ we will denote by $\mathrm{Seg}(\mathbb{P}^{m_1}, ..., \mathbb{P}^{m_h})$, or simply by $\mathrm{Seg}(m_1, ..., m_h)$ the *Segre variety* of type $(m_1, ..., m_h)$, i.e. the image of $\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_h}$ in $\mathbb{P}^r$, $r = (m_1+1) \cdots (m_h+1) - 1$, under the *Segre embedding*. Notice that, if $\mathbb{P}^{m_i} = \mathbb{P}(V_i)$, where V_i is a complex vector space of dimension $m_i + 1$, $i = 1, ..., h$, then $\mathbb{P}^r = \mathbb{P}(V_1 \otimes ... \otimes V_h)$ and $\mathrm{Seg}(m_1, ..., m_h)$ is the set of equivalence classes of indecomposable tensors in $\mathbb{P}^r$. We use the shorter notation $\mathrm{Seg}(m_1^{k_1}, ..., m_s^{k_s})$ if m_i is repeated k_i times, $i = 1, ..., s$.

Recall that $\mathrm{Pic}(\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_h}) \simeq \mathrm{Pic}(\mathrm{Seg}(m_1, ..., m_h)) \simeq \mathbb{Z}^h$, is freely generated by the line bundles $\xi_i = pr_i^*(\mathcal{O}_{\mathbb{P}^{m_i}}(1))$, $i = 1, ..., h$, where $pr_i : \mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_h} \to \mathbb{P}^{m_i}$ is the projection to the i–th factor. A divisor D on $\mathrm{Seg}(m_1, ..., m_h)$ is said to be of *type* $(\ell_1, ..., \ell_h)$ if $\mathcal{O}_{\mathrm{Seg}(m_1,...,m_h)}(D) \simeq \xi_1^{\ell_1} \otimes ... \otimes \xi_h^{\ell_h}$. The line bundle $\xi_1^{\ell_1} \otimes ... \otimes \xi_h^{\ell_h}$ on $\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_h}$ is also denoted by $\mathcal{O}_{\mathbb{P}^{m_1} \times ... \times \mathbb{P}^{m_h}}(\ell_1, ..., \ell_h)$. The hyperplane divisor of $\mathrm{Seg}(m_1, ..., m_h)$ is of type $(1, ..., 1)$.

It is useful to recall what are the defects of the Segre varieties $\mathrm{Seg}(m_1, m_2)$ with $m_1 \leq m_2$. As above, let V_i be complex vector spaces of dimension $m_i + 1$, $i = 1, 2$. We can interpret the points of $\mathbb{P}(V_1 \otimes V_2)$ as the equivalence classes of all $(m_1 + 1) \times (m_2 + 1)$ complex matrices and $\mathrm{Seg}(m_1, m_2) = \mathrm{Seg}(\mathbb{P}(V_1), \mathbb{P}(V_2)))$ as the subscheme of $\mathbb{P}(V_1 \otimes V_2)$ formed by the equivalence classes of all matrices of rank 1. Similarly $S^k(\mathrm{Seg}(m_1, m_2))$ can be interpreted as the subscheme of $\mathbb{P}(V_1 \otimes V_2)$ formed by the equivalence classes of all matrices of rank less than or equal to $k + 1$. Therefore $S^k(\mathrm{Seg}(m_1, m_2)) = \mathbb{P}(V_1 \otimes V_2)$ if and only if $k \geq m_1$. In the case $k < m_1$ one has instead:

$$\mathrm{codim}(S^k(\mathrm{Seg}(m_1, m_2)) = (m_1 - k)(m_2 - k)$$

(see [**ACGH**], pg. 67). As a consequence one has:

$$\delta_k(\mathrm{Seg}(m_1, m_2)) = k(k + 1)$$

if $k < m_1 \leq m_2$.

The degree of $S^k(\mathrm{Seg}(m_1, m_2))$, with $k < m_1 \leq m_2$, are computed by a well known formula by Giambelli [**Gi**], apparently already known to C. Segre (see [**Ro**], pg. 42, and [**Fu1**, 14.4.9] for a modern reference). The case $k = m_1 - 1$, which is the only one we will use later, is not difficult to compute (see [**Ha**], pg. 243) and reads:

$$\deg(S^{m_1-1}(\mathrm{Seg}(m_1, m_2))) = \binom{m_2+1}{m_1}.$$

Given positive integers n, d, we will denote by $V_{n,d}$ the image of $\mathbb{P}^n$ under the d–*Veronese embedding* of $\mathbb{P}^n$ in $\mathbb{P}^{\binom{n+d}{d}-1}$.

If X is a variety of dimension n and Y a subvariety of X, we will denote by $\mathrm{Bl}_Y(X)$ the blow–up of X along Y. If Y is a finite set $\{x_1, ..., , x_n\}$ we denote the blow–up by $\mathrm{Bl}_{x_1,...,x_n}(X)$.

CHAPTER 2

Fulton-Hansen Connectedness Theorem, Scorza Lemma and their applications to projective geometry

2.1. Connectedness principle of Enriques-Zariski-Grothendieck-Fulton-Hansen and some classical theorems in algebraic geometry

In the first chapter we introduced the main definitions of classical projective geometry proving many classical results. Many theorems in classical projective geometry deal with "general" objects. For example the classical Bertini theorem on hyperplane sections, see Theorem 1.4.2. A more refined version of this theorem says that if $f : X \to \mathbb{P}^N$ is morphism, with X proper and such that $\dim(f(X)) \geq 2$, and if $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$ is a general hyperplane, then $f^{-1}(H)$ is irreducible, see [**Ju**, Theorem 6.10] for a modern reference. The *Enriques-Zariski principle* says that *limits of connected varieties remain connected* and it is for example illustrated in the previous example because for an arbitrary $H = \mathbb{P}^{N-1} \subset \mathbb{P}^N$, $f^{-1}(H)$ is connected as we shall prove below.

This result is particularly interesting because, as shown by Deligne and Jouanolou, a small generalization of it proved by Grothendieck, [**Gr2**, XIII 2.3], yields a simplified proof of a beautiful and interesting connectedness theorem of Fulton and Hansen, [**FH**], whose applications are deep and appear in different areas of algebraic geometry and topology. Moreover, Deligne's proof generalizes to deeper statements involving higher homotopy groups when studying complex varieties, see [**D1**], [**D2**], [**Fu2**], [**FL**].

2.1.1. THEOREM. (**Fulton-Hansen Connectedness Theorem**, [**FH**]) *Let X be an irreducible variety, proper over an algebraically closed filed K. Let $f : X \to \mathbb{P}^N \times \mathbb{P}^N$ be a morphism and let $\Delta = \Delta_{\mathbb{P}^N} \subset \mathbb{P}^N \times \mathbb{P}^N$ be the diagonal.*

(1) *If $\dim(f(X)) \geq N$, then $f^{-1}(\Delta) \neq \emptyset$.*
(2) *If $\dim(f(X)) > N$, then $f^{-1}(\Delta)$ is connected.*

We begin by recalling the following "classical" Bertini theorem in a more general form. For a proof we refer to [**Ju**, Theorem 6.10], where the hypothesis $K = \overline{K}$ is relaxed.

2.1.2. THEOREM. (**Bertini Theorem**, see [**Ju**]) *Let X be an irreducible variety and let $f : X \to \mathbb{P}^N$ be a morphism. For a fixed integer $l \geq 1$, let $\mathbb{G}(N-l, N)$ be the Grassmann variety of linear subspaces of $\mathbb{P}^N$ of codimension l. Then*

(1) *if $l \leq \dim(\overline{f(X)})$, then there is a non-empty open subset $U \subseteq \mathbb{G}(N-l, N)$ such that for every $L \in U$,*
$$f^{-1}(L) \neq \emptyset;$$
(2) *if $l < \dim(\overline{f(X)})$, then there is a non-empty open subset $U \subseteq \mathbb{G}(N-l, N)$ such that for every $L \in U$,*
$$f^{-1}(L) \quad \text{is irreducible.}$$

We now show that the Enriques-Zariski principle is valid in this setting by proving the next result, which is the key point towards Theorem 2.1.1. We pass from general linear sections to arbitrary ones and for simplicity we suppose $K = \overline{K}$ as always.

2.1.3. THEOREM. ([**Gr2**], [**FH**], [**Ju**, Theorem 7.1]) *Let X be an irreducible variety and let $f : X \to \mathbb{P}^N$ be a morphism. Let $L = \mathbb{P}^{N-l} \subset \mathbb{P}^N$ be an arbitrary linear space of codimension l.*

(1) *If $l \leq \dim(\overline{f(X)})$ and if X is proper over K, then*
$$f^{-1}(L) \neq \emptyset.$$
(2) *If $l < dim(\overline{f(X)})$ and if X is proper over K, then*
$$f^{-1}(L) \quad \textit{is connected.}$$
More generally for an arbitrary irreducible variety X, if $f : X \to \mathbb{P}^N$ is proper over some open subset $V \subseteq \mathbb{P}^N$, and if $L \subseteq V$, then, when the hypothesis on the dimensions are satisfied, the same conclusions hold for $f^{-1}(L)$.

PROOF. (According to [**Ju**]). We prove the second part of the theorem from which the statements in 1) and 2) follow.

Let $W \subseteq \mathbb{G}(N-l, N)$ be the open subset consisting of linear spaces contained in V and let
$$Z = \{(x, L') \in X \times W \; : \; f(x) \in L'\} \subset \{(x, L') \in X \times \mathbb{G}(N-l, N) \; : \; f(x) \in L'\} = \mathcal{I}.$$
The scheme Z is irreducible since it is an open subset of the Grassmann bundle $p_1 : \mathcal{I} \to X$. Since f is proper over V, the second projection $p_2 : Z \to W$ is a proper morphism. Consider its Stein factorization:
$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "p_2"] & \\ W' \arrow[r, "r"'] & W; \end{tikzcd}$$
the morphism q is proper with connected fibers and surjective, while r is finite. By Theorem 2.1.2 r is dominant and hence surjective if $l \leq \dim(\overline{f(X)})$, respectively generically one-to-one and surjective if $l < \dim(\overline{f(X)})$. In the first case $p_2 : Z \to W$ is surjective so that $f^{-1}(L) \neq \emptyset$ for every $L \in W$. In the second case, since W is smooth, it follows that r is one to one everywhere so that $f^{-1}(L) = q^{-1}(r^{-1}(L))$ is connected for every $L \in W$. $\square$

2.1.4. REMARK. The original proof of Grothendieck used an analogous local theorem proved via local cohomology. His method has been used and extended by Hartshorne, Ogus, Speiser and Faltings. Faltings proved with similar techniques a connectedness theorem for other homogeneous spaces, see [**Fa**], at least in characteristic zero. A different proof of a special case of the above result was also given by Barth in 1969.

Now we are in position to prove the connectedness theorem.

PROOF. (of Theorem 2.1.1, according to Deligne, [**D1**]). The idea is to pass from the diagonal embedding $\Delta \subset \mathbb{P}^N \times \mathbb{P}^N$ to a linear embedding $L = \mathbb{P}^N \subset \mathbb{P}^{2N+1}$, a well known classical trick.

In $\mathbb{P}^{2N+1}$ separate the $2N+2$ coordinates into $[X_0 : \ldots : X_N]$ and $[Y_0 : \ldots : Y_N]$ and think these two sets as coordinates on each factor of $\mathbb{P}^N \times \mathbb{P}^N$. The two N dimensional linear subspaces $H_1 : \; X_0 = \ldots = X_N = 0$ and $H_2 : \; Y_0 = \ldots = Y_N = 0$ of $\mathbb{P}^{2N+1}$ are disjoint. If $V = \mathbb{P}^{2N+1} \setminus (H_1 \cup H_2)$ since there is a unique secant line to $H_1 \cup H_2$ passing through each $p \in V$, there is a morphism
$$\phi : V \to H_1 \times H_2 = \mathbb{P}^N \times \mathbb{P}^N,$$
which to p associates the points $(p_1, p_2) = (< H_2, p > \cap H_1, < H_1, p > \cap H_2)$. In coordinates,
$$\phi([X_0 : \ldots : X_N : Y_0 : \ldots : Y_N] = ([X_0 : \ldots : X_N], [Y_0 : \ldots : Y_N]).$$
Then $\phi^{-1}(\phi(p)) =< p_1, p_2 > \setminus \{p_1, p_2\} \simeq \mathbb{A}^1_K \setminus 0$. Let $L = \mathbb{P}^N \subset V$ be the linear subspace of $\mathbb{P}^{2N+1}$ defined by $X_i = Y_i$, $i = 0, \ldots, N$. Then
$$\phi_{|L} : L \xrightarrow{\simeq} \Delta$$

is an isomorphism. Given $f : X \to \mathbb{P}^N \times \mathbb{P}^N$ we construct the following Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ V \arrow[r, "\phi"] & \mathbb{P}^N \times \mathbb{P}^N, \end{tikzcd}$$

where

$$X' = V \times_{\mathbb{P}^N \times \mathbb{P}^N} X.$$

Clearly ϕ' induces an isomorphism between $f'^{-1}(L)$ and $f^{-1}(\Delta)$. To prove the theorem it is sufficient to verify the corresponding assertion for $f'^{-1}(L)$. To this aim we apply Theorem 2.1.3. Let us verify the hypothesis.

Since $\phi'^{-1}(x) \simeq \phi^{-1}(f(x)) = \mathbb{A}^1_K \setminus 0$ for every $x \in X$, the scheme X' is irreducible and of dimension $\dim(X) + 1$. The morphism f is proper, so that also $f' : X' \to V$ is proper and moreover $\dim(f(X')) = \dim(f(X)) + 1$. If $\dim(f(X)) \geq N$, then $\dim(f(X')) \geq N + 1 = \operatorname{codim}(L, \mathbb{P}^{2N+1})$. If $\dim(f(X')) > N$, then $\dim(f(X')) > N + 1 = \operatorname{codim}(L, \mathbb{P}^{2N+1})$. □

Let us list some immediate consequences of Fulton-Hansen Theorem.

2.1.5. COROLLARY. (**Generalized Bézout Theorem**) *Let X and Y be closed subvarieties of $\mathbb{P}^N$. If $\dim(X) + \dim(Y) \geq N$, then $X \cap Y \neq \emptyset$. If $\dim(X) + \dim(Y) > N$, then $X \cap Y$ is connected (and more precisely $(\dim(X) + \dim(Y) - N)$-connected).*

PROOF. Let $Z = X \times Y$ and let $f = i_X \times i_Y : Z \to \mathbb{P}^N \times \mathbb{P}^N$, where i_X and i_X are the inclusions in $\mathbb{P}^N$. Then $X \cap Y \simeq f^{-1}(\Delta_{\mathbb{P}^N})$ and the conclusions follows from Fulton-Hansen Theorem. □

2.1.6. COROLLARY. (**Generalized Bertini Theorem**) *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate vatiety If $\dim(X) \geq 2$, then every hyperplane section is connected.*

If X is also smooth a general hyperplane section is smooth and irreducible.

PROOF. Let $Z = X \times H$ and let $f = i_X \times i_H : Z \to \mathbb{P}^N \times \mathbb{P}^N$, where i_X and i_X are the inclusions in $\mathbb{P}^N$ and $H = \mathbb{P}^{N-1}$ is a hyperplane. Then $X \cap H \simeq f^{-1}(\Delta_{\mathbb{P}^N})$ and the conclusions follows from Fulton-Hansen Theorem. □

2.2. Zak's applications to Projective Geometry

In this section we come back to projective geometry and apply Fulton-Hansen theorem to prove some interesting and non-classical results in projective geometry. Most of the ideas and the results are due to Fyodor L. Zak, see [**Z2**], [**FL**], [**LV**], and they will be significant improvements of the classical material presented in the first chapter. Other applications to new results in algebraic geometry can be found in [**FH**], [**FL**], [**Fu2**].

We begin with the following key result, which refines a result of Johnson, [**Jo**].

2.2.1. THEOREM. ([**FH**], [**Z2**]) *Let $Y \subseteq X \subset \mathbb{P}^N$ be a closed subvariety of dimension $r = \dim(Y) \leq \dim(X) = n$, with X irreducible and projective. Then either*

(1) $\dim(T^*(Y, X)) = r + n$ *and* $\dim(S(Y, X)) = r + n + 1$*, or*
(2) $T^*(Y, X) = S(Y, X)$.

PROOF. We can suppose Y irreducible and then apply the same argument to each irreducible component of Y. We know that $T^*(Y, X) \subseteq S(Y, X)$ and that $\dim(T^*(Y, X)) \leq r + n$ by construction. Suppose that $\dim(T^*(Y, X)) = r + n$. Since $S(Y, X)$ is irreducible and $\dim(S(Y, X)) \leq r + n + 1$, the conclusion holds.

Suppose now $\dim(T^*(Y, X)) = t < r+n$. We prove that $\dim(S(Y, X)) \leq t$ so that $T^*(Y, X) = S(Y, X)$ follows from the irreducibility of $S(Y, X)$. There exists $L = \mathbb{P}^{N-t-1}$ such that $L \cap T^*(Y, X) = \emptyset = L \cap X$.

The projection $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^t$ restricts to a finite morphism on X and on Y, since $L \cap X = \emptyset$. Then $(\pi_L \times \pi_L)(X \times Y) \subset \mathbb{P}^t \times \mathbb{P}^t$ has dimension $r + n > t$ by hypothesis. By Theorem 2.1.1, the closed set

$$\widetilde{\Delta} = (\pi_L \times \pi_L)^{-1}(\Delta_{\mathbb{P}^t}) \subset Y \times X$$

is connected and contains the closed set $\Delta_Y \subset Y \times X$ so that Δ_Y is closed in $\widetilde{\Delta}$.

We claim that

$$\Delta_Y = \widetilde{\Delta}.$$

This yields $L \cap S(Y, X) = \emptyset$ and hence $\dim(S(Y, X)) \leq N - 1 - \dim(L) = t$.

Suppose $\overline{\widetilde{\Delta} \setminus \Delta_Y} \neq \emptyset$. We shall find $y' \in Y$ such that $\emptyset \neq T^*_{y'}(Y, X) \cap L \subseteq T^*(Y, X) \cap L$ contrary to the assumption. If $\overline{\widetilde{\Delta} \setminus \Delta_Y} \neq \emptyset$, the connectedness of $\widetilde{\Delta}$ implies the existence of $(y', y') \in \overline{\widetilde{\Delta} \setminus \Delta_Y} \cap \Delta_Y$. Let notation be as in definition 1.2.1. i. e. $p_2(p_1^{-1}(y, x)) =< x, y >$ if $x \neq y$ and $p_2(p_1^{-1}(y, x)) = T^*_y(Y, X)$ if $x = y \in Y$. Since for every $(y, x) \in \widetilde{\Delta} \setminus \Delta_Y$ we have $< y, x > \cap L \neq \emptyset$ by definition of π_L (indeed $\pi_L(y) = \pi_L(x)$, $y \neq x$, if and only if $< y, x > \cap L \neq \emptyset$), the same holds for (y', y') so that $p_2(p_1^{-1}(y, x)) \cap L \neq \emptyset$ forces $p_2(p_1^{-1}(y', y')) \cap L \neq \emptyset$. □

2.2.2. COROLLARY. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety of dimension n. Then either*

(1) $\dim(T^*X) = 2n$ *and* $\dim(SX) = 2n + 1$, *or*
(2) $T^*X = SX$.

The following result well illustrates the passage from general to arbitrary linear spaces.

2.2.3. THEOREM. (**Zak's Theorem on Tangencies**) *Let $X \subset \mathbb{P}^N$ be an irreducible projective non-degenerate variety of dimension n. Let $L = \mathbb{P}^m \subset \mathbb{P}^N$ be a linear subspace, $n \leq m \leq N - 1$, which is J-tangent along the closed set $Y \subseteq X$. Then* $\dim(Y) \leq m - n$.

PROOF. Without loss of generality we can suppose that Y is irreducible and then apply the conclusion to each irreducible component. By hypothesis and by definition we get $T^*(Y, X) \subseteq L$. Since $X \subseteq S(Y, X)$ and since X is non-degenerate, $S(Y, X)$ is not contained in L so that $T^*(Y, X) \neq S(Y, X)$. By Theorem 2.2.1 we have $\dim(Y) + n = \dim(T^*(Y, X)) \leq \dim(L) = m$. □

We now come back to the problem of tangency and to contact loci of smooth varieties providing two beautiful applications of the Theorem on Tangencies. We begin with the finiteness of the Gauss map of a smooth variety.

2.2.4. COROLLARY. (**Gauss map is finite for smooth varieties**, Zak) *Let $X \subsetneq \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety of dimension n. Then the Gauss map $\mathcal{G}_X : X \to \mathbb{G}(n, N)$ is finite. If moreover* char(K)=0, *then $\mathcal{G}_X$ is birational onto the image, i.e. X is a normalization of $\mathcal{G}_X(X)$.*

PROOF. As always it is sufficient to prove that $\mathcal{G}_X$ has finite fibers. For every $x \in X$, $\mathcal{G}_X^{-1}(\mathcal{G}_X(x))$ is the locus of points at which the tangent space T_xX is tangent. By Theorem 2.2.3 it has dimension less or equal than $\dim(T_xX) - n = 0$.

If char(K)=0, then every fiber $\mathcal{G}_X^{-1}(\mathcal{G}_X(x))$ is linear by Theorem 1.4.8 and of dimension zero by the first part, so that it reduces to a point as a scheme. □

The next result reveals a special feature of non-singular varieties, since the result is clearly false for cones, see exercise 1.6.6.

2.2.5. COROLLARY. (**Lower bound for the dimension of dual varieties**) *Let $X \subset \mathbb{P}^N$ be a smooth projective non-degenerate variety. Let $X^* \subset \mathbb{P}^{N*}$ be its dual variety. Then $\dim(X^*) \geq \dim(X)$. In particular, if also X^* is smooth, then $\dim(X^*) = \dim(X)$.*

PROOF. By the theorem of the dimension of the fiber, letting notation as in definition 1.4.1, $\dim(X^*) = N - 1 - \dim(p_2^{-1}(H))$, $H \in X^*$ general point. By Theorem 2.2.3, $\dim(p_2^{-1}(H)) \leq N - 1 - \dim(X)$ and the conclusion follows. □

2.2.6. REMARK. In Exercise 1.6.6, we saw that $(\mathbb{P}^1 \times \mathbb{P}^n)^* \simeq \mathbb{P}^1 \times \mathbb{P}^n$ for every $n \geq 1$. In [**E1**], L. Ein shows that if $N \geq 2/3\dim(X)$, if X is smooth, if char(K)=0 and if $\dim(X) = \dim(X^*)$, then $X \subset \mathbb{P}^N$ is either a hypersurface, or $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ Segre embedded, or $\mathbb{G}(1,4) \subset \mathbb{P}^9$ Plücker embedded, or the 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$. In the last three cases $X \simeq X^*$. For a different proof relating duality and secant defectivity, see Corollary 3.3.22.

We apply the Theorem on Tangencies to deduce some strong properties of the hyperplane sections of varieties of small codimension. By Bertini's Theorem proved in the previous section we know that arbitrary hyperplane sections of varieties of dimension at least 2 are connected. When the codimension of the variety is small with respect to the dimension, some further restrictions for the scheme structure appear.

If $X \subset \mathbb{P}^N$ is a non-singular irreducible nondegenerate variety, we recall that for every $H \in X^*$

$$\mathrm{Sing}(H \cap X) = \{x \in X \,:\, T_xX \subset H\},$$

i.e. it is the locus of points at which H is tangent. By Theorem 2.2.3 we get

$$\dim(\mathrm{Sing}(X \cap X) \leq N - 1 - \dim(X),$$

i.e.

$$\mathrm{codim}(\mathrm{Sing}(X \cap H), X \cap H) \geq 2\dim(X) - N.$$

Recall that $H \cap X$ is a Cohen-Macaulay scheme of dimension $\dim(X) - 1$ and that such a scheme is reduced as soon as it is generically reduced ($R_0 + S_1 \Leftrightarrow R_1$).

If $N \leq 2\dim(X) - 1$, then $H \cap X$ is a reduced scheme being non-singular in codimension zero and in particular generically reduced. The condition forces $\dim(X) \geq 2$, so that it is also connected by Bertini Theorem.

If $N \leq 2\dim(X) - 2$, which forces $\dim(X) \geq 3$, then $H \cap X$ is also non-singular in codimension 1, so that it is normal being Cohen-Macaulay. Since it is connected and integral, it is also irreducible. The case of the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ shows that this last result cannot be improved, since an hyperplane containing a $\mathbb{P}^2$ of the ruling yields a reducible, reduced, hyperplane section. Clearly in the same way, if $N \leq 2\dim(X) - k - 1$, $k \geq 0$, then $X \cap H$ is connected, Cohen-Macaulay and non-singular in codimension k. We summarize these result in the following Corollary to the Theorem on Tangencies.

2.2.7. COROLLARY. (Zak) *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. Then*

(1) *if $N \leq 2n - 1$, then every hyperplane section is connected and reduced;*
(2) *if $N \leq 2n - 2$, then every hyperplane section is irreducible and normal;*
(3) *let $k \geq 2$. If $N \leq 2n - k - 1$, then every hyperplane section is irreducible, normal and non-singular in codimension k.*

2.3. Tangential projections, second fundamental form, tangential invariants of algebraic varieties, Scorza Lemma and applications

There are several possible equivalent definitions of the projective second fundamental form $|II_{x,X}| \subseteq \mathbb{P}(S^2(\mathbf{T}_xX))$ of an irreducible projective variety $X \subset \mathbb{P}^N$ at a general point $x \in X$, see for example [**IL**, 3.2 and end of section 3.5]. We shall use the one related to tangential projections, as in [**IL**, remark 3.2.11].

Suppose $X \subset \mathbb{P}^N$ is non-degenerate, as always, let $x \in X$ be a general point and consider the projection from T_xX onto a disjoint $\mathbb{P}^{N-n-1}$:

$$\pi_x : X \dashrightarrow W_x \subseteq \mathbb{P}^{N-n-1}.$$

The map π_x is associated to the linear system of hyperplane sections cut out by hyperplanes containing T_xX, or equivalently by the hyperplane sections of $X \subset \mathbb{P}^N$ singular at x.

Let $\phi : \mathrm{Bl}_x X \to X$ be the blow-up of X at x, let

$$E = \mathbb{P}((\mathbf{T}_x X)^*) = \mathbb{P}^{n-1} \subset \mathrm{Bl}_x X$$

be the exceptional divisor and let H be a hyperplane section of $X \subset \mathbb{P}^N$. The induced rational map $\widetilde{\pi}_x : \mathrm{Bl}_x X \dashrightarrow \mathbb{P}^{N-n-1}$ is defined along E since $X \subset \mathbb{P}^N$ is not a linear space. Indeed, the restriction of $\widetilde{\pi}_x$ to E is given by a linear system in $|\phi^*(H) - 2E|_{|E} \subseteq |-2E_{|E}| = |\mathcal{O}_{\mathbb{P}((\mathbf{T}_xX)^*)}(2)| = \mathbb{P}(S^2(\mathbf{T}_xX))$. This means that to a hyperplane section H tangent to X at x we are associating the projectivization of the affine tangent cone to $H \cap X$ at x. Thus this linear system is empty if and only if the associated quadric hypersurface in $\mathbb{P}(\mathbf{T}_xX)) = \mathbb{P}^{n-1}$ has rank 0. If $\mathrm{def}(X) = k$, then a local calculation of Kleiman, see for example [**E1**, 2.1 (a)], show that the rank of the projectivezed tangent cone to a general tangent hyperplane section at x is a quadric of rank equal to $n - k$. Thus $\widetilde{\pi}$ is not defined along E if and only if $\mathrm{def}(X) = n$, i.e. if and only if $X \subset \mathbb{P}^N$ is a linear subspace.

2.3.1. DEFINITION. The *second fundamental form* $|II_{x,X}| \subseteq \mathbb{P}(S^2(\mathbf{T}_xX))$ *of a non-degenerate irreducible variety* $X \subset \mathbb{P}^N$ *at a general point* $x \in X$ is the non-empty linear system of quadric hypersurfaces in $\mathbb{P}((\mathbf{T}_xX)^*)$ defining the restriction of $\widetilde{\pi}_x$ to E.

Clearly $\dim(|II_{x,X}|) \leq N - n - 1$ and $\widetilde{\pi}_x(E) \subseteq W_x \subseteq \mathbb{P}^{N-n-1}$. From this point of view the base locus on E of the second fundamental form $|II_{x,X}|$ consists of *asymptotic directions*, i.e. of directions associated to lines having a contact to order two with X at x. Thus, when $X \subset \mathbb{P}^N$ is defined by equations of degree at most two, the base locus of the second fundamental form consists of points giving tangent lines contained in X and passing through x. In this case the base locus scheme of $|II_{x,X}|$ in E is exactly the locus of lines through x and contained in X. If X is also smooth, then for general $x \in X$, the variety of lines passign through x can be naturally identified with a smooth not necessarily irreducible subscheme $Y_x \subset \mathbb{P}((\mathbf{T}_xX)^*) = E$.

The following result was classically well known and used repeatedly by Scorza in his papers on secant defective varieties, see [**S1**] and [**S4**].

2.3.2. PROPOSITION. *Let* $X \subset \mathbb{P}^N$ *be a smooth irreducible non-degenerate variety of secant defect* $\delta(X) = \delta \geq 1$ *such that* $SX \subsetneq \mathbb{P}^N$. *Then*

(1) $\dim(|II_{X,x}|) = N - n - 1$ *for* $x \in X$ *general point;*

(2) $N \leq \frac{n(n+3)}{2}$ *and equality holds if and only if* $|II_{X,x}|$ *is the complete linear system of quadrics on* $\mathbb{P}((\mathbf{T}_xX)^*) = \mathbb{P}^{n-1}$.

PROOF. Let notation be as above. To prove part (1) it is sufficient to show that $\dim(\widetilde{\pi}_x(E)) = n - \delta$ because $\widetilde{\pi}_x(E) \subseteq W_x$. Recall that $W_x \subset \mathbb{P}^{N-n-1}$ is a non-degenerate variety of dimension $n - \delta$ by Terracini Lemma.

Let $TX = \cup_{x \in X} T_xX$ be the tangential variety to X. The following formula holds:

$$\dim(TX) = n + 1 + \dim(\widetilde{\pi}_x(E)), \tag{2.3.1}$$

see [**T2**] (or [**GH**, 5.6, 5.7] and [**FP**, Theorem 3.3.1] for a modern reference).

The variety $X \subset \mathbb{P}^N$ is smooth and secant defective, so that $TX = SX$ by Fulton–Hansen Theorem, see Corollary 2.2.2. Therefore $\dim(TX) = 2n + 1 - \delta$ and from (2.3.1) we get $\dim(\widetilde{\pi}_x(E)) = n - \delta$, as claimed.

Since $|II_{X,x}| \subseteq |\mathcal{O}_{\mathbb{P}^{n-1}}(2)|$, we have $N - n - 1 \leq \dim(|\mathcal{O}_{\mathbb{P}^{n-1}}(2)|) = \binom{n+1}{2} - 1$ and the final statements of part (1) and part (2) follow. □

We finally introduce some projective invariants of an irreducible non-degenerate variety $X \subset \mathbb{P}^N$ such that $SX \subsetneq \mathbb{P}^N$. These invariants measure the tangential behaviour of $X \subset \mathbb{P}^N$ and the relative position of two general tangent spaces.

Consider a general point $p \in SX$, $p \in < x, y >$, $x, y \in X$ general points, and take

$$T_pSX =< T_xX, T_yX >$$

.

2.3.3. DEFINITION. The *contact locus of* T_pSX *on* X, $\Gamma_p = \Gamma_p(X) \subset X$, is the closure in X of the locus of smooth points $z \in X$ such that $T_zX \subseteq T_pSX$.

For a general hyperplane $H \subset \mathbb{P}^N$ containing T_pSX, we define the *contact locus of* H *on* $X \subset \mathbb{P}^N$, $\Xi_p(H) = \Xi_p(X, H)$, as the closure of the locus of smooth points $z \in X$ such that $T_zX \subseteq H$.

The contact locus of T_pSX on X was called the *tangential contact locus of* X in [**CC1**].

By Terracini Lemma and by definition we get

$$\Sigma_p \subseteq \Gamma_p \subseteq \Xi_p(H) \tag{2.3.2}$$

for every H containing T_pSX. A monodromy argument shows that the irreducible components of Γ_p, respectively of $\Xi_p(H)$, through x and y are uniquely determined and have the same dimension (and in the second case that this dimension does not depend on the choice of a general $H \supseteq T_pSX$). We define this dimension as $\gamma(X)$, respectively $\xi(X)$. In particular we deduce

$$\delta(X) \leq \gamma(X) \leq \xi(X).$$

Let

$$\pi_x : X \dashrightarrow W_x \subset \mathbb{P}^{N-n-1}$$

be a general tangential projection and let $\widetilde{\gamma}(X)$ be the dimension of the general fiber of the Gauss map

$$\mathcal{G}_{W_x} : W_x \dashrightarrow \mathbb{G}(n - \delta(X), N - n - 1)$$

of the irreducible non-degenerate variety $W_x = \pi_x(X) \subset \mathbb{P}^{N-n-1}$ of dimension $n - \delta(X)$. By definition, for $y \in W_x$ smooth point, $\mathcal{G}_{W_x}(y) = [T_yW_x] \in \mathbb{G}(n - \delta(X), N - n - 1)$, where $\mathbb{G}(m, M)$ is the Grassmanian of projective subspaces of $\mathbb{P}^M$ of dimension $m \geq 0$. Set $\widetilde{\xi}(X) = \operatorname{def}(W_x)$, the dual defect of $W_x \subset \mathbb{P}^{N-n-1}$. Thus

$$\widetilde{\gamma}(X) \leq \widetilde{\xi}(X). \tag{2.3.3}$$

Since $\dim(W_x) = n - \delta(X)$, a general fiber of π_x has pure dimension $\delta(X)$.

The following result generalizes the ideas behind the proof of Scorza Lemma, which we shall describe below.

2.3.4. LEMMA. *Let* $X \subset \mathbb{P}^N$ *be an irreducible non-degenerate variety such that* $SX \subsetneq \mathbb{P}^N$ *and let* $\pi_x : X \dashrightarrow W_x \subset \mathbb{P}^{N-n-1}$ *be a general tangential projection. Then:*

(1) $\xi(X) = \delta(X) + \widetilde{\xi}(X)$;
(2) $\gamma(X) = \delta(X) + \widetilde{\gamma}(X)$;
(3) $0 \leq \gamma(X) - \delta(X) \leq \xi(X) - \delta(X) \leq n - 1 - \delta(X)$.

PROOF. Let us prove (1) and (3), the proof of (2) being similar. Let notation be as above. Consider π_x also as a map from $\mathbb{P}^N \setminus T_xX$ to $\mathbb{P}^{N-n-1}$. Define $\widetilde{H} = \pi_x(H) \subset \mathbb{P}^{N-n-1}$ and let $\widehat{\Xi} = \pi_x(\Xi_p(X))$. For every point $z \in \Xi_p(X) \setminus (T_xX \cap X)$ we get $\pi_x(T_zX) \subseteq T_{\pi_x(z)}W_x$. Thus a smooth point $\pi_x(z) \in W_x$, with $z \in \Xi_p(X) \cap X_{\text{reg}}$, is contained in the contact locus of $\widetilde{H}$ on W_x. Remark that by generic smoothness we can assume that $\pi_x(T_yX) = \pi_x(T_pSX) = T_{\pi_x(y)}W_x$ and that $\pi_x(y)$ is a smooth point of W_x. Therefore $\widehat{\Xi}$ is contained in the contact locus, let us say $\widetilde{\Xi}$, of $\widetilde{H}$ on W_x, yielding $\Xi_p(X) \subseteq \pi_x^{-1}(\widetilde{\Xi})$. On the other hand, by reverting the argument, we immediately see that the irreducible component of $\pi_x^{-1}(\widetilde{\Xi})$ passing through y coincides with the irreducible component of $\Xi_p(X)$ passing through y. By the generality assumptions every irreducible component of $\pi_x^{-1}(\widetilde{\Xi})$ has dimension $\dim(\widetilde{\Xi}) + \delta(X) = \widetilde{\xi}(X) + \delta(X)$, proving part (1). The last inequality in part (3) follows from the fact that $X \subset \mathbb{P}^N$ is non-degenerate. □

The following theorem of Scorza reveals that (smooth) varieties with *good* tangential behaviour have as entry loci quadric hypersurfaces, provided their secant varieties do not fill the ambient space. It easily implies also [**Oh**, Proposition 2.1], where another condition assuring the quadratic entry locus property is introduced.

2.3.5. THEOREM. (**Scorza Lemma**, [**S1**, footnote pg. 170 Opere Scelte Vol. I] and [**S4**]) *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of secant defect $\delta(X) = \delta \geq 1$ such that $SX \subsetneq \mathbb{P}^N$. Suppose that a general tangential projection $\pi_x(X) = W_x \subset \mathbb{P}^{N-n-1}$ is an irreducible variety having birational Gauss map, i.e. $\widetilde{\gamma}(X) = 0$; equivalently suppose that $\gamma(X) = \delta$. Let $y \in X$ be a general point. Then*

- (i) *the irreducible component of the closure of fiber of the rational map $\pi_x : X \dashrightarrow W_x \subset \mathbb{P}^{N-n-1}$ passing through y is either an irreducible quadric hypersurface of dimension δ or a linear space of dimension δ, the last case occurring only for singular varieties.*
- (ii) *There exists on $X \subset \mathbb{P}^N$ a $2(n-\delta)$-dimensional family $\mathcal{Q}$ of quadric hypersurfaces of dimension δ such that through two general points of $x, y \in X$ there passes a unique quadric $Q_{x,y}$ of the family $\mathcal{Q}$. Furthermore, the quadric $Q_{x,y}$ is smooth at the points x and y and it consists of the irreducible components of Σ_p passing through x and y, $p \in < x, y >$ general.*
- (iii) *If X is smooth, then a general member of $\mathcal{Q}$ is smooth.*

PROOF. Let $p \in < x, y >$ be a general point. Then $\pi_x(y) = y' \in W_x$ and $\pi_y(x) = x' \in W_y$. By definition of π_x, respectively π_y,

$$< T_xX, T_yX > = < T_xX, T_{y'}W_x > = < T_{x'}W_y, T_yX > \tag{2.3.4}$$

and these linear spaces have dimension $2n + 1 - \delta = \dim(T_pSX)$ by Terracini Lemma.

The cones $S(T_xX, W_x)$ and $S(T_yX, W_y)$ contains X. Thus, by (2.3.4), we deduce that Γ_p is contained in the the contact locus on $S(T_xX, W_x)$ of $< T_xX, T_{y'}W_x >$, which is $< T_xX, y' > = < T_xX, y >$, and also in the contact locus on $S(T_yX, W_y)$ of $< T_{x'}W_y, T_yX >$, which is $< T_yX, x' > = < T_yX, x >$. This follows from the the hypothesis on W_x, respectively W_y.

In particular, the irreducible component through y of the contact locus on X of $< T_xX, T_{y'}W_x >$, respectively the irreducible component through x of the contact locus of $< T_{x'}W_y, T_yX >$, is contained in $< T_xX, y >$, respectively $< x, T_yX >$. Thus the irreducible components of Γ_p through x and through y have dimension $\gamma(X) = \delta$, coincide with Σ_p^x and Σ_p^y and are contained in $< y, T_xX > \cap < x, T_yX > = \mathbb{P}^{\delta+1}$.

The line $< x, y >$ is a general secant line to X, contained in $\mathbb{P}^{\delta+1}$, so that

$$\{x, y\} \subseteq < x, y > \cap (\Sigma_p^x \cup \Sigma_p^y) \subseteq < x, y > \cap X = \{x, y\},$$

where the last equality is scheme-theoretical by Trisecant Lemma. Thus the hypersurface $\Sigma_p^x \cup \Sigma_p^y \subset \mathbb{P}^{\delta+1}$ has degree two and the points x, y are smooth points of the quadric hypersurface $\Sigma_p^x \cup \Sigma_p^y$. Therefore either $\Sigma_p^x = \Sigma_p^y$ is an irreducible quadric hypersurface, or $\Sigma_p^x \cup \Sigma_p^y$ is a rank 2 quadric hypersurface in $\mathbb{P}^{\delta+1}$. Since Σ_p is equidimensional, Terracini Lemma implies that $\Sigma_p \setminus \mathrm{Sing}(X) = (\Sigma_p^x \cup \Sigma_p^y) \setminus \mathrm{Sing}(X)$.

Let $\mathcal{Q}$ be the family of quadric hypersurfaces generated in this way. A count of parameters shows that the family has dimension $2(n - \delta)$, while the smothness of the entry loci at x and y assures that there is a unique quadric of the family through x and y.

If X is smooth, the arguments of [**FR**, pg. 964–966] yield the smoothness of the general entry locus of X. All the other assertions now easily follow. □

Scorza repeatedly used the above result in [**S1**] and [**S4**] when $\xi(X) = \delta(X) = 1$, even if his argument actually proves Theorem 2.3.5. The fact that varieties with $\xi(X) = \delta(X) \geq 1$ have quadratic entry loci can also be obtained via a strengthening of Terracini Lemma, a result proved by Terracini in [**T3**] and reobtained recently by Chiantini and Ciliberto. Before stating this result we remark that for $k = 1$ it is a consequence of Scorza Lemma since $\nu(X) = \delta(X)$ implies $\gamma(X) = \delta(X)$.

2.3.6. THEOREM. ([**CC2**, Theorem 2.4]) *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety such that $S^k X \subsetneq \mathbb{P}^N$. Then for general $p \in S^k X$ we have*

$$\dim(<\Xi_p(H)>) = (k+1)(\nu_k(X) - n) + s_k(X)$$

and moreover $S^k \Xi_p(H) = <\Xi_p(H)>$.

We now state an important and well known consequence of Scorza Lemma, or of Theorem 2.3.6, whose proofs usually are somehow more technical and less transparent. This has been considered by Severi, [**Sev**], Scorza, [**S1**] and Edwards, [**Ew**], in different statements and formulations.

2.3.7. COROLLARY. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension $n \geq 2$ such that $\dim(S^k X) = n + 2k$ for some $k \geq 1$ and such that $N \geq n + 2k + 1$ (equivalently $S^k X \subsetneq \mathbb{P}^N$). Let $b = \dim(\mathrm{Vert}(X))$. Then either $b = n - 1$ and $X \subset \mathbb{P}^N$ is a cone over a curve or $k = 1$, $N = n + 3$, $b = n - 3$ and $X \subset \mathbb{P}^{n+3}$ is a cone over a Veronese surface in $\mathbb{P}^5$.*

PROOF. Exercises 2.4.1 and 2.4.2. □

Now we are in position to provide a suitable generalization for smooth varieties of the characterization of the Veronese surface, [**Sev**], we have just proved. This result was firstly obtained by Zak during the classification of Scorza varieties of secant defect $\delta = 1$, see [**Z2**, Chapter V], in a different way. The proof presented below, based on tangential projections and its connections with the second fundamental form, has the advantage of revealing a very interesting parallel with the proof of the well known non-embedded characterization of $\mathbb{P}^n$, due to Mori [**Mo2**], as the unique smooth variety with ample tangent bundle. Thus this proof reveals the first instance of the importance of studying rational curves naturally appearing on secant defective varieties, their relations with the second fundamental form and with tangential projections. All the ideas behind this proof are essentially due to Scorza, see [**S4**], who always considered the class of irreducible, not necessarily smooth, secant defective varieties and who firstly realized the importance of the *conic connectedness condition* for embedded varieties.

2.3.8. COROLLARY. *Let $X \subset \mathbb{P}^N$, $N \geq \frac{n(n+3)}{2}$, be a smooth non-degenerate variety of secant defect $\delta(X) \geq 1$. Then $N = \frac{n(n+3)}{2}$ and $X \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ is projectively equivalent to $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$.*

PROOF. Exercise 2.4.4. □

In the following definition, we consider varieties having the *simplest entry locus*.

2.3.9. DEFINITION. A smooth irreducible non-degenerate projective variety $X \subset \mathbb{P}^N$ is said to be a *quadratic entry locus manifold of type* $\delta \geq 0$, briefly a *QEL-manifold of type* δ, if for general $p \in SX$ the entry locus $\Sigma_p(X)$ is a quadric hypersurface of dimension $\delta = \delta(X)$.

Let us remark that the Trisecant Lemma ensures that, as soon as $C_p(X)$ is a linear space and $\mathrm{codim}(X) \geq 2$, the general entry locus $\Sigma_p(X)$, being a hypersurface in $C_p(X)$, is necessarily a quadric hypersurface. Moreover, the smoothness of the general entry locus of a QEL-manifold is a consequence of the smoothness of X (see for example [**FR**, pp. 964–966]).

The next result, which is contained in [**Ve**], shows that the class of QEL-manifolds is sufficiently large and interesting.

2.3.10. PROPOSITION ([**Ve**]). *A smooth non-degenerate variety $X \subset \mathbb{P}^N$, scheme theoretically defined by quadratic equations whose Koszul syzygies are generated by linear ones, is a QEL-manifold.*

The class of QEL-manifolds is not stable under isomorphic projection. So, we extend this notion in the next:

2.3.11. DEFINITION. A smooth irreducible non-degenerate projective variety $X \subset \mathbb{P}^N$ is said to be a *local quadratic entry locus manifold of type* $\delta \geq 0$, briefly an *LQEL-manifold of type* δ, if, for general $x, y \in X$ distinct points, there is a quadric hypersurface of dimension $\delta = \delta(X)$ contained in X and passing through x, y.

Note that, for $\delta = 0$, being an LQEL-manifold imposes no restriction on X.
A further generalization is given by the following class of manifolds.

2.3.12. DEFINITION (cf. also [**KS**]). A smooth irreducible non-degenerate projective variety $X \subset \mathbb{P}^N$ is said to be a *conic-connected manifold*, briefly a *CC-manifold*, if through two general points of X there passes an irreducible conic contained in X.

Smooth cubic hypersurfaces in $\mathbb{P}^4$ are CC-manifolds, which are not $LQEL$-manifolds, so that the classes are all distinct.

2.3.13. LEMMA. *Let X be an LQEL-manifold with $\delta(X) > 0$ and let $x, y \in X$ be general points. There is a unique quadric hypersurface of dimension δ, say $Q_{x,y}$, passing through x, y and contained in X. Moreover, $Q_{x,y}$ is irreducible.*

PROOF. Uniqueness follows from the fact that the general entry locus passing through two general points is smooth at these points (see e.g. [**HR**, Proposition 3.3]). To see that $Q_{x,y}$ is irreducible, we may assume $\delta = 1$ by passing to general hyperplane sections (see Proposition 2.3.14 below). Assume first that X is covered by lines passing through x. Being also smooth, X is a linear space, so it is not an LQEL-manifold. Otherwise, after suitable normalization, the family of conics through x is generically smooth and the result follows. $\square$

A monodromy argument shows that if X is an LQEL-manifold, the general entry locus is a union of quadric hypersurfaces of dimension δ. Let us collect some consequences of the above definitions in the following proposition, whose easy proof is left to the reader.

2.3.14. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate smooth projective variety.*
- (i) *If X is a QEL-manifold and $SX = \mathbb{P}^N$, then X is linearly normal.*
- (ii) *If $X' \subset \mathbb{P}^M$, $M \leq N - 1$, is an isomorphic projection of X, then X' is an LQEL-manifold if and only if X is an LQEL-manifold.*
- (iii) *If X is an (L)QEL-manifold of type $\delta \geq 1$, then a general hyperplane section is an (L)QEL-manifold of type $\delta - 1$.*

All examples of LQEL-manifolds we are aware of are got by isomorphic projections of QEL-manifolds. Therefore, we would like to ask: *Is any linearly normal LQEL-manifold a QEL-manifold?*

2.3.15. LEMMA. ([**Ru2**, Lemma 1.6]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate variety, and assume $\delta > 0$. The irreducible components of the closure of a general fibre of π_x are not linear.*

PROOF. We may assume by passing to linear sections $\delta(X) = 1$. Let l be a line, passing through a general point $y \in X$, which is an irreducible component of the closure of a general fibre of π_x. By Terracini Lemma $T_x(X) \cap T_y(X)$ is a point, say $p_{x,y}$. Since $l \subset \langle T_x(X), y\rangle \cap T_y(X)$, $p_{x,y} \in l$. By symmetry there is also a line l' in $\langle T_y(X), x\rangle \cap X$, $x \in l'$, $p_{x,y} \in l'$. So, $l \cup l'$ is a conic contained in the plane $\langle x, y, p_{x,y}\rangle$ passing through x, y. Reasoning as in the proof of Lemma 2.3.13 we find a contradiction. $\square$

2.4. Exercises

2.4.1. EXERCISE. Let $X \subset \mathbb{P}^N$ be a subvariety such that for some $k \geq 1$ the variety $S^k X \subset \mathbb{P}^N$ has dimension $n + 2k < N$.

(1) Apply Exercise 1.6.2 to conclude that $\dim(SX) = n + 2 < N$;

(2) Prove that for general $x \in X$ the tangential projection is an irreducible curve so that $\widetilde{\gamma}(X) = 0$. Apply Scorza Lemma and Exercise 1.6.4 to deduce that either $X \subset \mathbb{P}^N$ is a cone over a curve or $N = n + 3$ and $X \subset \mathbb{P}^{n+3}$ is a cone of vertex a linear subspace of dimension $n - 3$ over an irreducible non-degenerate surface $S \subset \mathbb{P}^5$ with an irreducible two dimensional family of conics such that through two general points of S there passes a unique conic of this family.

2.4.2. EXERCISE. Show that $S \subset \mathbb{P}^5$ as in the item above is projectively equivalent to the Veronese surface in $\mathbb{P}^5$, concluding the proof of a famous Theorem of Severi, [**Sev**], via the following steps:

(1) Since $W_x \subset \mathbb{P}^2$ is a non-degenerate irreducible curve and since through two general points of S there passes an irreducible conic, then $W_x \subset \mathbb{P}^2$ is an irreducible conic being a linear projection of a conic.

(2) Thus the tangential projection of a conic passing through a general point z is an isomorphism onto W_x. The conic through x and z is contracted by π_x so that the intersection of two general conics of the family is transversal and consists of a unique point, see also the proof of Corollary 3.1.8.

(3) Deduce by the above description that the restriction of $\widetilde{\pi}_x$ to the exceptional divisor of $\mathrm{Bl}_x S \to W_x$ is an isomorphism given by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(2)|$. Conclude that there is no line through x, so that every conic of the family passing through x is irreducible.

(4) Let $\mathcal{C}_x \subset \mathrm{Hilb}_x^{2t+1}(S)$ be an irreducible (rational) curve parametrizing conics of the family passing through a general point $x \in S$. After normalizing and after letting $\widetilde{\mathcal{C}}_x \to \mathcal{C}$ be the normalization morphism, we can suppose that we have the following diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ \widetilde{\mathcal{C}}_x & X, \end{tikzcd}$$

where $\mathcal{F}$ is a smooth surface such that $\pi : \mathcal{F} \to \widetilde{\mathcal{C}}_x$ is a $\mathbb{P}^1$–bundle and ϕ is birational.

(5) The fibers of π are sent into conics through x, while the strict transform on $\mathcal{F}$ of general conics through a general point z are sent into sections of π disjoint from the tautological section $E = \phi^{-1}(x)$, which is contracted to the smooth point x. Deduce that $\mathcal{F} \to \mathbb{P}^1$ is isomorphic to $\mathbb{F}_1 \to \mathbb{P}^1$ as $\mathbb{P}^1$-bundles. If f is a class of a general fiber, then $i \circ \phi : \mathbb{F}_1 \to \mathbb{P}^5$ is given by the linear system $2(E + f)$.

(Hint: if $H \subset \mathbb{P}^5$ is a hyperplane, then $\phi^*(H) = \alpha E + \beta f$; thus $2 = f \cdot \phi^*(H) = \alpha$ and $0 = \phi^*(H) \cdot E = -2e + \beta$; E' section such that $E \cdot E' = 0$, yields $E' = E + ef$ so that $2 = 2(E + ef)^2$ finally yields $e = 1$).

(6) Conclude $S \simeq \mathbb{P}^2$ and $\mathcal{O}_S(1) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$.

2.4.3. EXERCISE. Let $X \subset \mathbb{P}^N$ be an irreducible closed subvariety. Suppose there exists a proper family $\mathcal{C}_x$ of smooth conics passing through a point x and let $V_x \subseteq X$ be the locus described by the conics in $\mathcal{C}_x$. Prove via the following steps that through every point $y \in V_x \setminus x$ there passes at most a finite number of conics in $\mathcal{C}_x$ (if you are interested in, prove directly that there exists a unique conic through every such y!).

(1) Suppose there exists infinitely many smooth conics through x and such a y. Then we can construct the following diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ \widetilde{\mathcal{C}}_{x,y} & X, \end{tikzcd}$$

where $\mathcal{F}$ is a smooth surface, $\widetilde{\mathcal{C}}_{x,y}$ is a smooth curve and $\pi : \mathcal{F} \to \widetilde{\mathcal{C}}_{x,y}$ is a $\mathbb{P}^1$–bundle having two sections: E_x=*conics through* x and E_y=*conics through* y.

(2) Apply Hodge index Theorem to E_x, E_y and the class f of a fiber of π to obtain a contradiction (recall that $E_x^2 < 0$ and $E_y^2 < 0$ since they are contracted by ϕ).

2.4.4. EXERCISE. We shall prove Corollary 2.3.8 in several steps.

(1) Proposition 2.3.2 furnishes $N = \frac{n(n+3)}{2}$, $X \subset \mathbb{P}^N$ linearly normal and that through x there passes no line contained in X (a line contained in X and passing through x clearly furnishes a base point of $|II_{x,X}|$).
(2) Then $\pi_x(X) = W_x \subset \mathbb{P}^{\frac{(n-1)(n+2)}{2}}$ is projectively equivalent to $\nu_2(\mathbb{P}^{n-1}) \subset \mathbb{P}^{\frac{(n-1)(n+2)}{2}}$, being equal to $\widetilde{\pi}_x(E)$.
(3) Scorza Lemma implies that $X \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ has an irreducible family $\mathcal{C}_x$ of dimension $2n-2$ of irreducible conics such that through 2 general points of X there passes a unique conic of the family.
(4) Consider the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ \mathcal{C}_x & X, \end{tikzcd}$$

where $\mathcal{F}$ is the unversal family and ϕ is the tautological morphism. Let $\widetilde{E} = \phi^{-1}(x)$, scheme theoretically.
(5) Every conic in $\mathcal{C}_x$, is smooth by item 1) so that through every point of $X \setminus x$ there passes a unique conic of $\mathcal{C}_x$ by Zariski's Main Theorem and by Exercise 2.4.3, i. e. ϕ is an isomorphism between $\mathcal{F} \setminus \widetilde{E}$ and $X \setminus x$, contracting $\widetilde{E}$ to a point.
(6) Given an arbitrary point $y \in X \setminus x$, there exists a unique smooth conic through x and y, which implies $(T_xX \cap X)_{\text{red}} = x$ and that $\widetilde{\pi}_x : \operatorname{Bl}_x X \to W_x$ is a morphism. Thus given a conic through x and a tangential direction through x there exists a unique smooth conic through x tangent to this direction.
(7) Prove that $\phi : \mathcal{F} \to X$ factors through $\operatorname{Bl}_x X \to X$ inducing an isomorphism between $\widetilde{E}$ and E. Conclude that $\mathcal{F} \simeq \operatorname{Bl}_x X$ and that $\widetilde{E} \simeq \mathbb{P}^{n-1}$.
(8) Conclude that $\operatorname{Bl}_x(X) \to \mathbb{P}^{n-1}$ is isomorphic over $\mathbb{P}^{n-1}$ to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$, see for example [**Ko**, Lemma V.3.7.8]. In conclusion, X is isomorphic to $\mathbb{P}^n$ and the embedding is clearly given by the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(2)|$.

CHAPTER 3

Hartshorne's conjectures, $LQEL$–manifolds, Severi varieties and Scorza varieties

3.1. Hartshorne's conjectures and a refinement of Zak's theorem on linear normality

After the period in which new and solid foundations to the principles of algebraic geometry were rebuilt especially by Zariski, Grothendieck and their schools, at the beginning of the '70 a new trend began. There was a renewed interest in solving concrete problems and in finding applications of the new methods and ideas. One can consult the beautiful book of Robin Hartshorne, [**H1**], to have a picture of that situation. In [**H1**] many outstanding questions, such as the set-theoretic complete intersection of curves in $\mathbb{P}^3$ (still open), the characterization of $\mathbb{P}^N$ among the smooth varieties with ample tangent bundle (solved by Mori in [**Mo1**] and which cleared the path to the foundation of Mori theory, [**Mo2**]) were discussed, or stated, and a lot of other problems solved. In related fields we only mention Deligne proof of the Weil conjectures or later Faltings proof of the Mordell conjecture, which used the new machinery.

The interplay between topology and algebraic geometry returned to flourish. Lefschetz Theorem and Barth-Larsen Theorem, also suggested that smooth varieties, whose codimension is small with respect to their dimension, should have very strong restrictions both topological, both geometrical. To have a feeling we remark that a codimension 2 smooth complex subvariety of $\mathbb{P}^N$, $N \geq 5$, has to be simply connected for example. If $N \geq 6$, there are no known examples of codimension 2 smooth varieties with the exception of the trivial ones, the complete intersection of two hypersurfaces, i.e. the transversal intersection of two hypersurfaces, smooth along the subvariety. In fact, at least for the moment, one is able to construct only these kinds of varieties whose codimension is sufficiently small with respect to dimension. Let us recall the following definition and some notable properties of complete intersections analogous to varieties whose codimension is small with respect to dimension.

3.1.1. DEFINITION. (**Complete intersection**) A variety $X \subset \mathbb{P}^N$ of dimension n is a *complete intersection* if there exist $N-n$ homogeneous polynomials $f_i \in K[X_0, \ldots, X_N]$ of degree $d_i \geq 1$, generating the homogeneous ideal $I(X) \subset K[X_0, \ldots, X_N]$, i.e. $I(X) =< f_1, \ldots, f_{N-n} >$.

Let us recall that since $f_1, \ldots, f_{N-n}$ form a regular sequence in $K[X_0, \ldots, X_N]$, the homogeneous coordinate ring

$$S(X) = \frac{K[X_0, \ldots, X_N]}{I(X)}$$

has depth $n+1$, i.e. $X \subset \mathbb{P}^N$ is an arithmetically Cohen-Macaulay variety. Thus a complete intersection $X \subset \mathbb{P}^N$ is projectively normal, i.e. the restriction morphisms

$$H^0(\mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(\mathcal{O}_X(m))$$

are surjective for every $m \geq 0$, so that X is connected, and $H^i(\mathcal{O}_X(m)) = 0$ for every i such that $0 < i < n$ and for every $m \in \mathbb{Z}$. Moreover, by Grothendieck theorem on complete intersections, $\mathrm{Pic}(X) \simeq \mathbb{Z} < \mathcal{O}_X(1) >$, as soon as $n \geq 3$, see [**H1**]. By Lefschetz theorem complete intersections defined over $K = \mathbb{C}$ are simply connected, as soon as $n \geq 2$ and have the same cohomology $H^i(X, \mathbb{Z})$ of the projective spaces containing them for $i < n$.

Based on some empirical observations, inspired by the Theorem of Barth and Larsen and, according to Fulton and Lazarsfeld, "on the basis of few examples", Hartshorne was led to formulate the following conjectures.

3.1.2. CONJECTURE. (1^{st} **Conjecture of Hartshorne, or Complete Intersection Conjecture**, [**H2**]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

$$\textit{If } N < \frac{3}{2}\dim(X), \textit{i.e. if} \operatorname{codim}(X) < \frac{1}{2}\dim(X), \textit{ then } X \textit{ is a complete intersection.}$$

Let us quote Hartshorne: *While I am not convicted of the truth of this statement, I think it is useful to crystallize one's idea, and to have a particular problem in mind* ([**H2**]).

Hartshorne immediately remarks that the conjecture is sharp, due to the examples of the Grassmann variety of lines in $\mathbb{P}^4$, $\mathbb{G}(1,4) \subset \mathbb{P}^9$, Plücker embedded, and of the spinorial variety of dimension 10, $S^{10} \subset \mathbb{P}^{15}$; moreover, the examples of cones over curves in $\mathbb{P}^3$, not complete intersection, reveals the necessity of the non-singularity assumption. Varieties for which $N = \frac{3}{2}\dim(X)$ and which are not complete intersection are usually called *Hartshorne varieties*. It is not a case that these varieties are homogeneous since a technique for constructing varieties of not too high codimension is exactly via algebraic groups, see for example [**Z2**, Chapter 3] or the appendix to [**LV**].

One of the main difficulties of the problem is a good translation in geometrical terms of the algebraic condition of being a complete intersection and in general of dealing with the equations defining a variety.

It is not here the place to remark how many important results originated and still today arise from this open problem in the areas of vector bundles on projective space, of the study of defining equations of a variety and k-normality and so on. The list of these achievements is too long that we preferred to avoid citations, being confident that everyone has met sometimes a problem or a result related to this conjecture.

Let us recall the following definition.

3.1.3. DEFINITION. (**Linear normality**) A non-degenerate irreducible variety $X \subset \mathbb{P}^N$ is said to be *linearly normal* if the linear system of hyperplane sections is complete, i.e. if the injective, due to non-degenerateness, restriction morphism

$$H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \xrightarrow{r} H^0(\mathcal{O}_X(1))$$

is surjective and hence an isomorphism.

If a variety $X \subset \mathbb{P}^N$ is not linearly normal, then the complete linear system $|\mathcal{O}_X(1)|$ is of dimension greater than N and embeds X as a variety $X' \subset \mathbb{P}^M$, $M > N$. Moreover, there exists a linear space $L = \mathbb{P}^{M-N-1}$ such that $L \cap X' = \emptyset$ and such that $\pi_L : X' \to X \subset \mathbb{P}^N$ is an isomorphism. Indeed, if $V = r(H^0(\mathcal{O}_{\mathbb{P}^N}(1))) \subsetneq H^0(\mathcal{O}_X(1))$ and if $U \subset H^0(\mathcal{O}_{\mathbb{P}^N}(1))$ is a complementary subspace of V in $H^0(\mathcal{O}_{\mathbb{P}^N}(1))$, the one can take $\mathbb{P}^M = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^N}(1)))$, $L = \mathbb{P}(U)$ and the claim follows from the fact that $\pi_L : X' \simeq X \to X \subset \mathbb{P}^N = \mathbb{P}(V)$ is given by the very ample linear system $|V|$. On the contrary, if X is an isomorphic linear projection of a variety $X' \subset \mathbb{P}^M$, $M > N$, then X is not linearly normal.

In the same survey paper Hartshorne posed another conjecture, based on the fact that complete intersections are linearly normal and on some examples in low dimension.

3.1.4. CONJECTURE. (2^{nd} **Conjecture of Hartshorne, or Linear Normality Conjecture**, [**H2**]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.*

$$\textit{If } N < \frac{3}{2}\dim(X)+1, \textit{i.e. if} \operatorname{codim}(X) < \frac{1}{2}\dim(X)+1, \textit{ then } X \textit{ is linearly normal.}$$

Recalling proposition 1.2.7 and the above discussion, we can equivalently reformulate it by means of secant varieties putting "$N = N + 1$".

$$\textit{If } N < \frac{3}{2}\dim(X) + 2, \textit{ then } SX = \mathbb{P}^N.$$

Let us quote once again Hartshorne point of view on this second problem: *Of course in settling this conjecture, it would be nice also to classify all nonlinearly normal varieties with $N = \frac{3n}{2} + 1$, so as to have a satisfactory generalization of Severi's theorem. As noted above, a complete intersection is always linearly normal, so this conjecture would be a consequence of our original conjecture, except for the case $N = \frac{3n}{2}$. My feeling is that this conjecture should be easier to establish than the original one* ([**H2**]). Once again the bound is sharp taking into account the example of the projected Veronese surface in $\mathbb{P}^4$.

The conjecture on linear normality was proved by Zak at the beginning of the '80's and till now it is the major evidence for the possible truth of the complete intersection conjecture. As we shall see Conjecture 3.1.4 is now an immediate consequence of Terracini Lemma and of Theorem 2.2.1. Later we will furnish another proof of this theorem, cfr. Theorem 3.4.5.

3.1.5. THEOREM. (**Zak Theorem on Linear Normality**) *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. If $N < \frac{3}{2}n + 2$, then $SX = \mathbb{P}^N$. Or equivalently if $SX \subsetneq \mathbb{P}^N$, then $\dim(SX) \geq \frac{3}{2}n + 1$ and hence $N \geq \frac{3}{2}n + 2$.*

PROOF. Suppose that $SX \subsetneq \mathbb{P}^N$, then there exists a hyperplane H containing the general tangent space to SX, let us say T_zSX. Then by Corollary 1.3.6, the hyperplane H is tangent to X along $\Sigma_z(X)$, which by the generality of z has pure dimension $\delta(X) = 2n+1-\dim(SX)$. Since $T(\Sigma_z(X), X) \subseteq H$, the non-degenerate variety $S(\Sigma_z(X), X) \supseteq X$ is not contained in H, yielding $T(\Sigma_z(X), X) \neq S(\Sigma_z(X), X)$. By Theorem 2.2.1 we get

$$2n + 1 - \dim(SX) + n + 1 = \dim(S(\Sigma_z(X), X)) \leq \dim(SX),$$

i.e.

$$3n + 2 \leq 2\dim(SX)$$

implying

$$N - 1 \geq \dim(SX) \geq \frac{3}{2}n + 1.$$

□

Now we are in position to provide a slight refinement of Zak's Linear Normality Theorem, [**Z2**, Theorem 2.8]. The proof is essentially identical to Zak's one but it reveals the importance of the projective invariants defined above. This new bound also strengthens the bound for smooth varieties obtained by Landsberg involving $\widetilde{\gamma}(X) = \gamma(X) - \delta(X)$, which equals $\dim(F_v)$ in Landsberg notation, see [**L1**] and also [**IL**, 3.15].

3.1.6. THEOREM. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety such that $SX \subsetneq \mathbb{P}^N$. Let $b = \dim(\mathrm{Sing}(X))$, $\xi = \xi(X)$ and $\delta = \delta(X)$. Then:*

$$\dim(SX) \geq \frac{3}{2}n + \frac{1-b}{2} + \frac{\xi - \delta}{2}; \tag{3.1.1}$$

$$N \geq \frac{3}{2}n + 1 + \frac{1-b}{2} + \frac{\xi - \delta}{2}; \tag{3.1.2}$$

$$n \leq \frac{1}{3}(2N + b - (\xi - \delta)) - 1. \tag{3.1.3}$$

In particular if $X \subset \mathbb{P}^N$ is also smooth, then

$$\dim(SX) \geq \frac{3}{2}n + 1 + \frac{\xi - \delta}{2}; \tag{3.1.4}$$

$$N \geq \frac{3}{2}n + 2 + \frac{\xi - \delta}{2} \tag{3.1.5}$$

and

$$n \leq \frac{2(N-2)}{3} - \frac{\xi - \delta}{3}. \tag{3.1.6}$$

PROOF. If $\xi \leq b+1$, then
$$\dim(SX) > \frac{3}{2}n + \frac{1-b}{2} + \frac{\xi - \delta}{2},$$
so that we can assume $\xi \geq b+2$ and hence $n \geq b+3$.

Fix a general $p \in SX$ and consider a general hyperplane $H \subset \mathbb{P}^N$ containing T_pSX. By definition H is tangent to X along $\Xi_p \setminus \mathrm{Sing}(X)$. Consider a general $L = \mathbb{P}^{N-b-1}$ and set $\widehat{X} = X \cap L$, $\widehat{\Xi}_p = \Xi_p \cap L$ and $\widehat{H} = H \cap L$. The variety $\widehat{X} \subset \mathbb{P}^{N-b-1} = L$ is smooth, irreducible and non-degenerate of dimension $n - b - 1 \geq 2$, while the hyperplane $\widehat{H} = \mathbb{P}^{N-b-2}$ is tangent to $\widehat{X}$ along the variety $\widehat{\Xi}_p$, whose dimension is $\xi - b - 1 \geq 1$.

Since $\widehat{X} \subset L$ is non-degenerate and contained in $S(\widehat{\Xi}_p, \widehat{X})$, we get $S(\widehat{\Xi}_p, \widehat{X}) \neq T(\widehat{\Xi}_p, \widehat{X})$, where $T(\widehat{\Xi}_p, \widehat{X}) := \cup_{y \in \widehat{\Xi}_p} T_y\widehat{X} \subseteq \widehat{H}$. Therefore by applying Theorem 2.2.1 to $S(\widehat{\Xi}_p, \widehat{X})$ we deduce that

$$\begin{aligned}\dim(SX) - b - 1 = \dim(SX \cap L) \geq \dim(S\widehat{X}) \geq \dim(S(\widehat{\Xi}_p, \widehat{X})) = \\ = \dim(\widehat{\Xi}_p) + \dim(\widehat{X}) + 1 = (\xi - b - 1) + (n - b - 1) + 1.\end{aligned}$$

Hence

$$2n + 1 - \delta \geq n + \xi - b = n + \delta + \widetilde{\xi} - b,$$

where $\widetilde{\xi} = \xi - \delta$, that is
$$\delta \leq \frac{n + b + 1 - \widetilde{\xi}}{2}.$$

Thus
$$\dim(SX) = 2n + 1 - \delta \geq \frac{3}{2}n + \frac{1-b}{2} + \frac{\widetilde{\xi}}{2} = \frac{3}{2}n + \frac{1-b}{2} + \frac{\xi - \delta}{2}.$$

Since $SX \subsetneq \mathbb{P}^N$, we deduce $N \geq \dim(SX) + 1$, which combined with the above estimates yields $n \leq \frac{1}{3}(2N + b - (\xi - \delta)) - 1$. The other assertions are now obvious. $\square$

We recall the notion of Severi variety, [**Z2**].

3.1.7. DEFINITION. A smooth non-degenerate irreducible variety $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ of dimension n such that $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$ is called *a Severi variety*.

Theorem 3.1.6 implies that a Severi variety $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ has $\xi(X) = \gamma(X) = \delta(X) = \frac{n}{2}$ and that $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$ is a hypersurface.

With these powerful instruments at hand we can immediately prove, via Scorza Lemma, the following interesting corollary, in a way slightly different from [**Z2**, IV.2.1, IV.3.1, IV.2.2].

3.1.8. COROLLARY. *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety. Then*

(i) *$X \subset \mathbb{P}^{\frac{3}{2}n+2}$ is a $LQEL$-variety of type $\delta = \frac{n}{2}$.*

(ii) *The image of a general tangential projection of X, $\pi_x(X) = W_x \subset \mathbb{P}^{\frac{n}{2}+1}$, is a smooth quadric hypersurface.*

(iii) *Given three general points $x, y, z \in X$, let $Q_{x,z}$, respectively $Q_{y,z}$, be the smooth quadrics passing through x and z, respectively y and z. Then $Q_{x,z} \cap Q_{y,z} = z$, the intersection being transversal.*

PROOF. As we observed above, for a Severi variety we have $\xi(X) = \gamma(X) = \delta(X) = \frac{n}{2}$ due to Theorem 3.1.6. The conclusion of the first part follows from Scorza Lemma.

The proof of Theorem 3.1.6 yields $SX = S(\Sigma_q, X)$ for a general $q \in SX$. From Terracini Lemma we get $T_xX \cap T_w\Sigma_q = \emptyset$ for a general $x \in X$ and for a general $w \in \Sigma_q$. Thus $\dim(\pi_x(\Sigma_q)) = \frac{n}{2}$ for $q \in SX$ general. Since $\pi_x(X) = W_x$ has dimension $\frac{n}{2}$, we deduce that $\pi_x(\Sigma_q) = W_x$ for $q \in SX$ general. Therefore the variety $W_x \subset \mathbb{P}^{\frac{n}{2}+1}$ is a quadric hypersurface, being a hypersurface and also a non-degenerate linear projection of a quadric hypersurface. The smoothness of W_x follows from $0 = \widetilde{\xi}(X) = \operatorname{def}(W_x)$. In particular the restriction of π_x to Σ_q is an isomorphism. Scorza Lemma also yields $\Sigma_p = \pi_x^{-1}(\pi_x(z))$. Take $q \in < y, z >$ general and consider Σ_q. From the previous analysis $\pi_{x|\Sigma_q} : \Sigma_q \to W_x$ is an isomorphism so that Σ_q intersects Σ_p only at z, the intersection being transversal. $\square$

Clearly the dimension n of a Severi variety $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ is even so that the first case to be considered is $n = 2$. These are smooth surfaces in $\mathbb{P}^5$ such that $SX \subsetneq \mathbb{P}^5$. They were completely classified in the classical and well known theorem of Severi, [**Sev**], which is Corollary 2.3.7 here, saying that X is projectively equivalent to the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. This justifies the name given by Zak to such varieties. By Theorem 3.1.5, it follows that $SX \subset \mathbb{P}^{\frac{3}{2}n+2}$ is necessarily an hypersurface, i.e. $\dim(SX) = \frac{3}{2}n + 1$.

In Exercise 1.6.1 we showed that the Segre variety $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is an example of Severi variety of dimension 4. Indeed $N = 8 = \frac{3}{2} \cdot 4 + 2$ and SX is a cubic hyersurface, see *loc. cit.*. By the classical work of Scorza, last page of [**S1**], it turns out that $\mathbb{P}^2 \times \mathbb{P}^2$ is the only Severi variety of dimension 4. We shall provide a short, geometrical and elementary proof of this fact later, see Theorem 3.3.12.

The realization of the Grassmann variety of lines in $\mathbb{P}^5$ Plücker embedded, $X = \mathbb{G}(1, 5) \subset \mathbb{P}^{14}$, as the variety given by the pfaffians of the general antisymmetric 6×6 matrix, yields that $\mathbb{G}(1, 5)$ is a Severi variety of dimension 8 such that its secant variety is a degree 3 hypersurface, see for example [**Ha**, pg. 112 and pg. 145] for the last assertion.

A less trivial examples is a variety studied by Elie Cartan and also by Room. It is a homogeneous complex variety of dimension 16, $X \subset \mathbb{P}^{26}$, associated to the representation of E_6 and for this reason called E_6*-variety*, or *Cartan variety* by Zak. It has been shown by Lazarsfeld and Zak that its secant variety is a degree 3 hypersurface, see for example [**LV**] and [**Z2**, Chapter 3].

There is a unitary way to look at these 4 examples, by realizing them as "Veronese surfaces over the composition algebras over K", $K = \overline{K}$ and char(K)=0, [**Z2**, Chapter 3]. Let $\mathcal{U}_0 = K, \mathcal{U}_1 = K[t]/(t^2 + 1)$, $\mathcal{U}_2 =$ quaternion algebra over $K, \mathcal{U}_3 =$ Cayley algebra over K. For $K = \mathbb{C}$, we get $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and the octonions numbers $\mathbb{O}$. Let $\mathcal{I}_i, i = 0, \ldots, 3$, denote the Jordan algebra of Hermitian (3×3)-matrices over $\mathcal{U}_i, i = 0, \ldots, 3$. A matrix $A \in \mathcal{I}_i$ is called *Hermitian* if $\overline{A}^t = A$, where the bar denotes the involution in $\mathcal{U}_i$. Let

$$X_i = \{[A] \in \mathbb{P}(\mathcal{I}_i) \,:\, \operatorname{rk}(A) = 1\} \subset \mathbb{P}(\mathcal{I}_i).$$

Then

$$N_i = \dim(\mathbb{P}(\mathcal{I})) = 3 \cdot 2^i + 2, \;\; n_i = \dim(X_i) = 2^{i+1} = 2\dim_K(\mathcal{U}_i),$$

and

$$SX = \{[A] \in \mathbb{P}(\mathcal{I}_i) \,:\, \operatorname{rk}(A) \leq 2\} = V(\det(A)) \subset \mathbb{P}(\mathcal{I}_i)$$

is a degree 3 hypersurface. By definition $X_i \subset \mathbb{P}(\mathcal{I}_i)$ is a Severi variety of dimension 2^{i+1}, which is seen to be one of the above examples.

A Theorem of Jacobson states that over a fixed algebraically closed field K there are only four Jordan algebras, the algebras $\mathcal{U}_i$'s, and hence these are the only examples which can be constructed in this way.

The highly non-trivial and very beautiful result, which is essentially equivalent to Jacobson Classification Theorem, is the following beautiful result firstly proved by Zak in [**Z1**].

3.1.9. THEOREM. (**Zak's Classification of Severi varieties**, [**Z1**], [**Z2**], [**LV**], [**L1**], [**Ru2**, Corollary 3.2] or Theorem 3.3.12 here) *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety of dimension n, defined over an algebraically closed field K of characteristic 0. Then X is projectively equivalent to one of the following:*

(1) *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$;*
(2) *the Segre 4-fold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
(3) *the Grassmann variety $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$;*
(4) *the E_6-variety $X \subset \mathbb{P}^{26}$.*

We shall obtain this classification result as part of the classification of $LQEL$-manifolds of type $\delta = \frac{n}{2}$ proved in Theorem 3.3.12 below.

3.2. Basics of deformations theory of (smooth) rational curves on smooth projective varieties

In this section we recall some definitions and results concerning the parameter spaces of rational curves on smooth algebraic varieties. These notions will be immediately applied to some explicit problems in the next sections. The standard reference for the most technical results is the book [**Ko**]. A more geometrical and simpler introduction to the subject (and containing almost all the results needed here) is contained in [**De1**] (see also [**De2**]). Another recent and very interesting source for the theory of Hilbert Schemes and for Deformation Theory of more general varieties is the book [**Sr**].

Let us begin with the basic definitions.

3.2.1. DEFINITION. Let $P(t) = \sum_{i=0}^{d} a_i \binom{t+d}{i} \in \mathbb{Q}[t]$ be a numerical polynomial of degree $d \geq 0$, i. e. $a_i \in \mathbb{Z}$ for every $i = 0, \dots d$ and $a_d \neq 0$. Let T be a fixed scheme, which in many applications will simply be $\mathrm{Spec}(K)$ with K an algebraically closed field. Let X be a scheme projective over T and let $\mathcal{O}(1)$ be a ϕ-ample line bundle, where $\phi : X \to T$ is the structural morphism. Let

$$Hilb^P_T(X) : \{T - Schemes\}^o \to Sets$$

be the contravariant functor defined for every T-scheme S by

$$Hilb^P_T(X)(S) = \{\chi \subset S \times_T X\},$$

the set of subchemes of $S \times_T X$, proper and flat over S and such that χ_s has Hilbert polynomial $P = P(t)$ relative to $\mathcal{O}(1)$ for every $s \in S$. When $T = \mathrm{Spec}(K)$ we set $\mathrm{Hilb}^P(X) = \mathrm{Hilb}^P_{\mathrm{Spec}(K)}(X)$.

If the functor $Hilb^P_T(X)$ is representable, we indicate by $\mathrm{Hilb}^P_T(X)$ the T-scheme representing it and called the *Hilbert scheme of closed T-subvarieties of X with Hilbert polynomial $P = P(t)$*.

If $\mathrm{Hilb}^P_T(X)$ exists, there is a *universal family* $\chi^P \subset X \times_T \mathrm{Hilb}^P_T(X)$ proper and flat over $\mathrm{Hilb}^P_T(X)$ such that for every T-scheme S and for every $\chi \in Hilb^P_T(X)(S)$ there exists a unique T-morphism $f : S \to \mathrm{Hilb}^P(X)$ such that

$$\chi = S \times_{\mathrm{Hilb}^P(X)} \chi^P \subset S \times_T X,$$

via the base change induced by f.

Finally we can define *the Hilbert scheme of X* as

$$\mathrm{Hilb}_T(X) = \amalg_P \mathrm{Hilb}^P_T(X).$$

For every $W \in \mathrm{Hilb}^P_T(X)(T)$, we have a unique morphism $f : T \to \mathrm{Hilb}^P(X)$ such that

$$W = T \times_{\mathrm{Hilb}^P(X)} \chi^P \subset T \times_T X \simeq X.$$

If $T = \mathrm{Spec}(K)$, let us indicate by $[W] = f(\mathrm{Spec}(K)) \in \mathrm{Hilb}^P(X)(K)$. In this case, the above tautological non-sense means exactly that the restrictions of the obvious projections to the factors define a tautological diagram

$$\begin{tikzcd} \chi^P \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ \mathrm{Hilb}^P(X) & X, \end{tikzcd} \tag{3.2.1}$$

such that $\phi(\pi^{-1}([W])) = W$ as schemes.

The first important and rather difficult result in the theory of Hilbert scheme is the following Existence Theorem, see for example [**Ko**, I.1.4] or [**Sr**, Theorem 4.3.4].

3.2.2. THEOREM. *For every scheme X projective over T and for every numerical polynomial $P(t)$, the Hilbert scheme $\mathrm{Hilb}_T^P(X)$ exists and it is projective over T. In particular $\mathrm{Hilb}^P(X)$ is projective over K for every scheme X projective over K, so that under these hypothesis it has a finite number of irreducible components.*

It is well known that in general $\mathrm{Hilb}^P(X)$ is neither irreducible nor reduced so that it is important to analyze its scheme structure, locally and globally.

Let $g : \mathrm{Spec}(K[\epsilon]/(\epsilon^2)) \to \mathrm{Hilb}^P(X)$ such that $g(\mathrm{Spec}(K)) = [W]$, where K is the residual field at the unique closed point of $\mathrm{Spec}(K[\epsilon]/(\epsilon^2))$. Then it is well known that such g's correspond to tangent vectors to $\mathrm{Hilb}^P(X)$ at $[W]$, i.e.

$$T_{[W]}\,\mathrm{Hilb}^P(X) = Hilb^P(X)(\mathrm{Spec}(K[\epsilon]/(\epsilon^2))).$$

Let $\mathcal{I}_W/\mathcal{I}_W^2$ be the *conormal bundle of W in X* and let $N_{W/X} = (\mathcal{I}_W/\mathcal{I}_W^2)^*$ be *the normal bundle of W in X*. A fundamental result in the infinitesimal study of Hilbert schemes is the following, see [**Ko**, I.2.15] and [**Sr**, Theorem 4.3.5].

3.2.3. THEOREM. *Let X be a scheme projective over K. Then:*

(1)
$$T_{[W]}\,\mathrm{Hilb}^P(X) = H^0(N_{W/X}).$$
In particular $\dim_{[W]}(\mathrm{Hilb}^P(X)) \leq h^0(N_{W/X})$.

(2)
$$\dim_{[W]}(\mathrm{Hilb}^P(X)) \geq h^0(N_{W/X}) - h^1(N_{W/X}).$$
In particular if $h^1(N_{W/X}) = 0$, then
$$\dim_{[W]}(\mathrm{Hilb}^P(X)) = h^0(N_{W/X}),$$
$\mathrm{Hilb}^P(X)$ *is smooth at $[W]$ so that under this hypothesis there exists an unique irreducible component of* $\mathrm{Hilb}^P(X)$ *containing* $[W]$.

3.2.4. DEFINITION. For a closed subscheme $Z \subset X$ of a scheme X projective over K, we can define the contravariant functor

$$Hilb^{P,Z}(X) : \{K - Schemes\}^o \to Sets$$

which to every K-scheme S associates the set

$$Hilb^{P,Z}(X)(S) = \{\chi \subset S \times_K X\},$$

of flat families of closed subschemes of X parametrized by S, having Hilbert polynomial $P = P(t)$ and such that $Z \cap \chi_s \neq \emptyset$ for every $s \in S(K)$.

The functor $Hilb^{P,Z}(X)$ is representable by a closed subscheme $\mathrm{Hilb}^{P,Z}(X) \subseteq \mathrm{Hilb}^P(X)$ called the *Hilbert scheme of subvarieties of X with Hilbert polynomial $P = P(t)$ and intersecting Z*. If Z consists of a finite number of points (this will always be the case in which we shall use this scheme), then we shall call it the *Hilbert scheme of subvarieties of X with Hilbert polynomial $P = P(t)$ and passing through Z* and for $Z = x \in X$ a point we set $\mathrm{Hilb}_x^P(X) = \mathrm{Hilb}^{P,x}(X)$, respectively $\mathrm{Hilb}_x(X) = \mathrm{Hilb}^x(X)$, being sure that no confusion will arise for this abuse of notation and that everyone will agree that this is not the case $T = x$!

As before, there is a *universal family* $\chi_Z^P \subset X \times \mathrm{Hilb}^{P,Z}(X)$, flat over $\mathrm{Hilb}^{P,Z}(X)$, of closed subschemes of X such that for every scheme S and for every $\chi \in Hilb^{P,Z}(X)(S)$ there exists a unique morphism $f : S \to \mathrm{Hilb}^{P,Z}(X)$ such that this base change induces the equality

$$\chi = S \times_{\mathrm{Hilb}^{P,Z}(X)} \chi_Z^P \subset S \times X.$$

For $\mathrm{Hilb}^{P,Z}(X)$ there are results analogous to those of Theorem 3.2.2 and of Theorem 3.2.3, which will not be recalled here since we shall describe them for rational curves on a smooth variety from a different point of view we now introduce.

3.2.5. DEFINITION. Let Y be a projective scheme over the fixed algebraically closed field K and let X be a scheme quasi-projective over K. Fix an ample line bundle $H = \mathcal{O}(1)$ on X, a numerical polynomial $P(t) \in \mathbb{Q}[t]$ and for every morphism $f : S \times_K Y \to S \times_K X$ let $f_s : Y \to X$ be the morphism induced by base extension by a point $s \in S(K)$. Let

$$Hom^P(Y, X) : \{K - Schemes\}^o \to Sets$$

be the contravariant functor defined for every K-scheme S by

$$Hom^P(Y, X)(S) = \{f : S \times_K Y \to S \times_K X\}$$

such that $\chi(f_s^*(\mathcal{O}(m)) = P(m)$ for m sufficiently large and for every $s \in S(K)$. Then $Hom^P(Y, X)$ is called the *functor of morphism from Y to X with Hilbert polynomial P.*

If the functor $Hom^P(Y, X)$ is representable, we indicate by $\mathrm{Hom}^P(Y, X)$ the scheme representing it and called the *scheme of morphism from Y to X with Hilbert polynomial $P = P(t)$.*

If $\mathrm{Hom}^P(Y, X)$ exists, there is a *universal morphism*

$$\mathrm{ev} : Y \times \mathrm{Hom}^P(Y, X) \to X$$

such that for every K-scheme S and for every $g : Y \times_K S \to X \times_K S \in Hom^P(Y, X)(S)$ there exists a unique K-morphism $\widetilde{g} : S \to \mathrm{Hom}^P(Y, X)$ such that

$$g = \mathrm{ev} \circ (\mathbb{I}_Y \times \widetilde{g}).$$

Then we define the *scheme of morphisms from Y to X*:

$$\mathrm{Hom}(Y, X) = \amalg_P \mathrm{Hom}^P(Y, X).$$

For every morphism $f : Y \to X \in \mathrm{Hom}^P(Y, X)(K)$ having Hilbert polynomial P, we have a unique morphism $\widetilde{f} : \mathrm{Spec}(K) \to \mathrm{Hom}^P(Y, X)$ such that $f = \mathrm{ev} \circ (\mathbb{I}_Y \times \widetilde{f})$. Let us indicate by $[f] = \widetilde{f}(\mathrm{Spec}(K)) \in \mathrm{Hom}^P(Y, X)(K)$. The above tautological non-sense means exactly that the restrictions of ev to $Y \times_K [f]$ is exactly $f : Y \to X$, i. e. that

$$\mathrm{ev}(y, [f]) = f(y).$$

The Existence of Hilbert Schemes implies the existence of $\mathrm{Hom}(Y, X)$ under the above hypothesis, see expecially [**Gr1**] and also [**Ko**, I.1.10] , [**Sr**, §4.6.6].

3.2.6. THEOREM. *For every scheme Y projective over a K, for every scheme X quasi-projective over K and for every numerical polynomial $P(t)$, the scheme* $\mathrm{Hom}^P(Y, X)$ *exists and it is quasi-projective over K.*

PROOF. (Sketch) Suppose first that X is projective over K. Let S be a K-scheme, let $f : Y \times_K S \to X \times_K S \in Hom(Y, X)(S)$. If we consider f as a S-morphism, $Y \times_K S$ and $X \times_K S$ as S-schemes, we can define

$$\Gamma = (\mathbb{I}_Y, f) : Y \times_K S \to (Y \times_K S) \times_S (X \times_K S) \simeq Y \times_K X \times_K S.$$

Then Γ is a closed immersion so that $\Gamma(Y \times_K S) \simeq Y \times_K S$ is proper and flat over S and $\Gamma(Y \times_K S) \in \mathrm{Hilb}(Y \times_K X)(S)$. This gives a morphism

$$\Gamma : Hom(Y, X) \to Hilb(Y \times_k X),$$

and hence a morphism $\Gamma : \mathrm{Hom}(Y, X) \to \mathrm{Hilb}(Y \times_K X)$, which is an open immersion of schemes. From the projectivity over K of every $\mathrm{Hilb}^P_X(Y \times_K X)$, see Theorem 3.2.2 observing that $Y \times_K X$ is projective over K, one deduces that $\mathrm{Hom}^P(Y, X)$ is quasi-projective over K for every numerical polynomial $P(t)$.

If X is quasi-projective over K, then there exists an open immersion $i : X \to \overline{X}$ with $\overline{X}$ projective over K. For every K-scheme S there is a morphism $Hom(Y, X)(S) \to Hom(Y, \overline{X})$ defined by sending $f : Y \times_K S \to X \times_K S$ to $(i, \mathbb{I}_S) \circ f : Y \times_K S \to \overline{X} \times_K S$. One then deduces the existence of a morphism $j : \mathrm{Hom}^P(Y, X) \to \mathrm{Hom}(Y, \overline{X})$ and proves that j is an open immersion. Finally one deduces the quasi-projectivity over K of $\mathrm{Hom}^P(Y, X)$. For more details consult [**Gr1**]. ☐

Let $g : \mathrm{Spec}(K[\epsilon]/(\epsilon^2)) \to \mathrm{Hom}^P(Y, X)$ such that $g(\mathrm{Spec}(K)) = [f]$, where K is the residual field at the unique closed point of $\mathrm{Spec}(K[\epsilon]/(\epsilon^2))$. Then such g's correspond to tangent vectors to $\mathrm{Hom}^P(Y, X)$ at $[f]$, i.e.

$$T_{[f]} \mathrm{Hom}^P(Y, X) = Hom^P(Y, X)(\mathrm{Spec}(K[\epsilon]/(\epsilon^2)).$$

Let $\Omega_{X/k}$ be the *sheaf of Kähler differentials of* X. We have the following infinitesimal results for $\mathrm{Hom}(Y, X)$, see [**De1**, §2.2] and [**Ko**, I.2.16].

3.2.7. THEOREM. *Let notation be as above. Then:*

(1)
$$T_{[f]} \mathrm{Hom}(Y, X) = H^0(\mathcal{H}om(f^*(\Omega_{X/K}), \mathcal{O}_Y)).$$
In particular if X is smooth along $f(Y)$, then $\dim_{[f]}(\mathrm{Hom}(Y, X)) \leq h^0(f^*(T_X))$, *where* $T_X = \mathcal{H}om(\Omega_{X/K}, \mathcal{O}_X)$ *is the tangent bundle of* X.

(2) *If X is projective and non-singular along $f(Y)$, then*
$$\dim_{[f]}(\mathrm{Hom}(Y, X)) \geq h^0(f^*(T_X)) - h^1(f^*(T_X)).$$
In particular if $h^1(f^*(T_X)) = 0$, *then* $\dim_{[f]}(\mathrm{Hom}(Y, X)) = h^0(f^*(T_X))$, $\mathrm{Hom}(Y, X)$ *is smooth at* $[f]$ *so that there exists an unique irreducible component of* $\mathrm{Hom}(Y, X)$ *containing* $[f]$.

Suppose that $C \subset X$ is a smooth rational and that $X \subset \mathbb{P}^N$ is a smooth projective variety. Then $N_{C/X} \simeq \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(a_i)$ being a locally free sheaf of rank $n - 1 = \mathrm{codim}(C, X)$. If $f : C \to X$ is the embedding of C into X, then $f^*(T_X) = T_{X|C} = \oplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(b_i)$ and everything is easily computable. Moreover we have the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(2) \simeq T_{\mathbb{P}^1} \to T_{X|C} \to N_{C/X} \to 0,$$

which is very useful for computations. By Theorem 3.2.7 the exact sequence at level of H^0 can be interpreted as saying that the tangent space to $\mathrm{Hilb}(X)$ at $[C]$ is given by factoring out the automorphism of $C \simeq \mathbb{P}^1$ from the tangent space to $\mathrm{Hom}(\mathbb{P}^1, X)$ at $f : C \to X$. Indeed, $\mathrm{Hom}(\mathbb{P}^1, \mathbb{P}^1)$ contains the group subscheme of automorphism of $\mathbb{P}^1$, whose tangent space at the identity is exactly $H^0(T_{\mathbb{P}^1})$.

3.2.8. DEFINITION. Let Y be a projective scheme over the fixed algebraically closed field K, let X be a scheme quasi-projective over K, let $B \subset Y$ be a closed subscheme and let $g : B \to X$ be a fixed morphism, which for every K-scheme S induces a morphism $g \times \mathbb{I}_S : B \times_K S \to X \times_K S$. Let $Hom(Y, X; g)(S)$ be the set of morphism from $Y \times_K S \to X \times_K S$ whose restriction to $B \times_K S \subset Y \times_K S$ coincide with $g \times \mathbb{I}_S$. This defines a contravariant functor $Hom(Y, X; g) : \{K - schemes\}^0 \to Sets$ which is a subfunctor of $Hom(Y, X)$. The natural restriction morphisms

$$\rho(S) : Hom(Y, X)(S) \to Hom(B, X)$$

induce a morphism

$$\rho : \mathrm{Hom}(Y, X) \to \mathrm{Hom}(B, X)$$

such that $\rho^{-1}([g])$ is exactly $\mathrm{Hom}(Y, X : g)$, proving the representability of $Hom(Y, X; g)$ by a projective scheme over K which will be indicated by $\mathrm{Hom}(Y, X; g)$ and which is naturally a closed subscheme of $\mathrm{Hom}(Y, X)$.

Fix an ample line bundle $H = \mathcal{O}(1)$ on X, a numerical polynomial $P(t) \in \mathbb{Q}[t]$ and Let

$$Hom^P(Y, X; g) : \{K - Schemes\}^o \to Sets$$

be the contravariant functor defined for every K-scheme S by

$$Hom^P(Y, X; g)(S) = \{f \in Hom(Y, X; g(S) \ : \ \chi(f^*(\mathcal{O}(m)) = P(m) \text{ for } m >> 0\}.$$

The functor $Hom^P(Y, X; g)$ is representable and we indicate by $\mathrm{Hom}^P(Y, X; g)$ the scheme representing it, which is called the *scheme of morphism from Y to X with Hilbert polynomial $P = P(t)$ whose restriction to B is X.*

There is a *universal morphism*

$$\mathrm{ev} : Y \times \mathrm{Hom}(Y, X; g) \to X$$

such that for every scheme S and for every $\phi : Y \times_K S \to X \times_K S \in Hom(Y, X; g)(S)$ there exists a unique morphism $\widetilde{\phi} : S \to \mathrm{Hom}(Y, X; g)$ such that

$$\phi = \mathrm{ev} \circ (\mathbb{I}_Y \times \widetilde{\phi}).$$

Then we have as always

$$\mathrm{Hom}(Y, X; g) = \amalg_P \mathrm{Hom}^P(Y, X; g).$$

Suppose for semplicity that X is smooth along $f(Y)$, where $f : Y \to X$ is a morphism which restricts to g on B. The restriction $\rho : \mathrm{Hom}(Y, X) \to \mathrm{Hom}(B, X)$ naturally induces a morphism of K-vector spaces

$$\rho : H^0(f^*(T_X)) \to H^0(g^*(T_X)),$$

so that

$$T_{[f]} \mathrm{Hom}(Y, X; g) = \ker(\rho : H^0(f^*(T_X)) \to H^0(g^*(T_X))) = H^0(f^*(T_X) \otimes \mathcal{I}_B), \tag{3.2.2}$$

where $\mathcal{I}_B$ is the ideal sheaf of B in Y.

We have finally the following infinitesimal results for $\mathrm{Hom}(Y, X; g)$, see [**De1**, §2.3].

3.2.9. THEOREM. *Let Y and X be schemes projective over K, let B be a closed subscheme of Y and let $f : Y \to X$ be a morphism whose restriction to B is $g : B \to X$. Suppose moreover that X is smooth along $f(Y)$. Then:*

(1)
$$T_{[f]} \mathrm{Hom}(Y, X; g) = H^0(f^*(T_X) \otimes \mathcal{I}_B),$$
yielding $\dim(\mathrm{Hom}(Y, X; g) \leq h^0(f^*(T_X) \otimes \mathcal{I}_B)$.

(2)
$$\dim_{[f]}(\mathrm{Hom}(Y,X;g)) \geq h^0(f^*(T_X)\otimes\mathcal{I}_B) - h^1(f^*(T_X)\otimes\mathcal{I}_B).$$
In particular if $h^1(f^*(T_X)\otimes\mathcal{I}_B)=0$, *then* $\dim_{[f]}(\mathrm{Hom}(Y,X;g)) = h^0(f^*(T_X)\otimes\mathcal{I}_B)$, *the scheme* $\mathrm{Hom}(Y,X;g)$ *is smooth at* $[f]$ *so that there exists a unique irreducible component of* $\mathrm{Hom}(Y,X;g)$ *containing* $[f]$.

3.2.10. DEFINITION. Let $0\in\mathbb{P}^1$, let $x\in X$ and let $g:0\to X$ be a morphism such that $g(0)=x$. In this case $\mathrm{Hom}(\mathbb{P}^1,X;g)$ will be indicated by $\mathrm{Hom}(\mathbb{P}^1,X;0\to x)$.

Let $X\subset\mathbb{P}^N$ be a fixed embedding. Then $\mathrm{Hom}_d(\mathbb{P}^1,X)=\mathrm{Hom}^{dt+1}(\mathbb{P}^1,X)$ parametrizes morphism $f:\mathbb{P}^1\to X$ such that the cycle $f_*(\mathbb{P}^1)$ has degree d. Let us remember that, if $C=f(\mathbb{P}^1)$, then $\deg(f_*(\mathbb{P}^1)) = \deg(f)\cdot\deg(C)$. In the same way one can define $\mathrm{Hom}_d(\mathbb{P}^1,X;g)=\mathrm{Hom}^{dt+1}(\mathbb{P}^1,X;g)$.

Before passing to the applications of our interest we need an explicit description of the differential of $\mathrm{ev}: Y\times\mathrm{Hom}(Y,X;g)\to X$.

3.2.11. THEOREM. ([**Ko**, Proposition II.3.10]) *Let* $B\subset\mathbb{P}^1$ *a scheme of lenght* $|B|\leq 2$ *and let* $g:B\to X$ *be a morphism to a smooth projective variety* X. *Let*
$$\mathrm{ev}:\mathbb{P}^1\times\mathrm{Hom}(\mathbb{P}^1,X;g)\to X$$
be the evaluation morphism and let $f:\mathbb{P}^1\to X$ *such that* $[f]\in\mathrm{Hom}(\mathbb{P}^1,X;g)$.

If
$$f^*(T_X)\otimes\mathcal{I}_B=\oplus_{i=1}^n\mathcal{O}_{\mathbb{P}^1}(\alpha_i),$$
then for every $p\in\mathbb{P}^1\setminus B$ *we have*
$$\mathrm{rk}(d(\mathrm{ev})(p,[f]))=\#\{i\mid\alpha_i\geq 0\}.$$

Now we can finally state and prove two fundamental tools for the classification of $LQEL$-manifolds in the next sections.

3.2.12. COROLLARY. ([**Ko**, Corollary II.3.11, Theorem II.3.12]) *Let notation be as in Theorem 3.2.11 and suppose char(K)=0. Then*

i) ($B=\emptyset$) *Suppose* $V\subset\mathrm{Hom}(\mathbb{P}^1,X)$ *is an irreducible component such that* $\mathrm{ev}:\mathbb{P}^1\times V\to X$ *is dominant. Then for* $[f]\in V$ *general* $f^*(T_X)$ *is generated by global sections.*

ii) ($|B|=1$) *Suppose* $V\subset\mathrm{Hom}(\mathbb{P}^1,X;0\to x)$ *is an irreducible component such that* $\mathrm{ev}:\mathbb{P}^1\times V\to X$ *is dominant. Then for* $[f]\in V$ *general* $f^*(T_X)$ *is ample.*

PROOF. Since $\mathrm{ev}:\mathbb{P}^1\times V\to X$ is dominant there exists an open subset $U\subseteq X$ such that for every $(p,[f])\in\mathrm{ev}^{-1}(U)$ we have $n=\mathrm{rk}(d(\mathrm{ev})(p,[f]))=\#\{i\mid\alpha_i\geq 0\}$, where $f^*(T_X)=\oplus_{i=1}^n\mathcal{O}_{\mathbb{P}^1}(\alpha_i)$ so that $\alpha_i\geq 0$ for every $i=1,\ldots,n$ and $f^*(T_X)$ is generated by global sections.

Reasoning as above and recalling that for $B=x$ we have $\mathcal{I}_B\simeq\mathcal{O}_{\mathbb{P}^1}(-1)$, letting $f^*(T_X)\otimes\mathcal{I}_B = \oplus_{i=1}^n\mathcal{O}_{\mathbb{P}^1}(\alpha_i)$ we deduce $\alpha_i\geq 0$ for every $i=1,\ldots,n$ so that $f^*(T_X)=\oplus_{i=1}^n\mathcal{O}_{\mathbb{P}^1}(\beta_i)$ with $\beta_i>0$ for every $i=1,\ldots,n$, as claimed. □

The next result will permit to count parameters without worring about pathologies.

3.2.13. COROLLARY. *Let notation be as in Theorem 3.2.11 and assume char(K)=0. Then*

i) *if $X \subset \mathbb{P}^N$ and if through a general point of X there passes a rational curve of degree $d \geq 1$ in the given embedding, then there exists an open dense subset $U \subseteq X$ and an irreducible component $V \subseteq \mathrm{Hom}_d(\mathbb{P}^1, X)$ such that for every $[f] \in V$ with $f(\mathbb{P}^1) \cap U \neq \emptyset$ we have $f^*(T_X)$ generated by global sections.*

   Let $C = f(\mathbb{P}^1) \subset X$ be such a curve and suppose that f is an isomorphism so that C is a smooth rational curve. Then $h^0(N_{C/X}) = -K_X \cdot C + n - 3$, the Hilbert scheme of rational curves of degree d is smooth at $[C]$ of dimension $-K_X \cdot C - 2$. There exists a unique irreducible component $\mathcal{C} \subseteq \mathrm{Hilb}^{dt+1}(X)$ containing $[C]$ and dominating X.

ii) *Suppose $X \subset \mathbb{P}^N$ and let $x \in X$ be a point. If through x and a general point of X there passes a rational curve of degree $d \geq 1$ in the given embedding, then there exists an open dense subset $U \subseteq X$ and an irreducible component $V \subseteq \mathrm{Hom}_d(\mathbb{P}^1, X; 0 \to x)$ such that for every $[f] \in V$ with $f(\mathbb{P}^1) \cap U \neq \emptyset$ we have $f^*(T_X)$ ample.*

   Let $C = f(\mathbb{P}^1) \subset X$ be such a curve and suppose that f is an isomorphism so that C is a smooth rational curve. Then $H^0(N_{C/X}(-1)) = -K_X \cdot C - 2$, the Hilbert scheme of rational curves of degree d passing through x is smooth at $[C]$ of dimension $-K_X \cdot C - 2$ so that there exists a unique irreducible component $\mathcal{C}_x \subseteq \mathrm{Hilb}^{dt+1}_x(X)$ containing $[C]$ and dominating X.

PROOF. The hypothesis in i), respectively ii), assure that $\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Hom}_d(\mathbb{P}^1, X) \to X$ is dominant, respectively $\mathrm{ev} : \mathbb{P}^1 \times \mathrm{Hom}(\mathbb{P}^1, X; 0 \to x) \to X$ is dominant. Thus there exists an irreducible component $V \subseteq \mathrm{Hom}(\mathbb{P}^1, X)$, respectively $V \subseteq \mathrm{Hom}(\mathbb{P}^1, X; 0 \to x)$, such that $\mathrm{ev} : \mathbb{P}^1 \times V \to X$ is dominant. Then we can take as $U \subseteq X$ the open non-empty subset constructed in the proof of the two parts of Corollary 3.2.12.

Let $N_{C/X} = \oplus_{j=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(\gamma_i)$ From the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(2) \simeq T_{\mathbb{P}^1} \to T_{X|C} \to N_{C/X} \to 0,$$

we deduce that $N_{C/X}$ is generated by global sections, respectively ample, and that

$$\deg(N_{C/X}) = \deg(T_{X|C}) - 2 = -K_X \cdot C - 2.$$

By Riemann-Roch and by the fact that $\gamma_j \geq 0$ for every j, we deduce

$$h^0(N_{C/X}) = \sum_{j=1}^{n-1} h^0(\mathcal{O}_{\mathbb{P}^1}(\gamma_j)) = \sum_{j=1}^{n-1} (\gamma_j + 1) = -K_X \cdot C + n - 3,$$

as claimed in i).

In case ii), we have $\gamma_j > 0$ for every j so that

$$h^0(N_{C/X}(-1)) = \sum_{j=1}^{n-1} h^0(\mathcal{O}_{\mathbb{P}^1}(\gamma_j - 1)) = \sum_{j=1}^{n-1} \gamma_j = -K_X \cdot C - 2,$$

as claimed. The other conclusions follow directly from Theorem 3.2.3 since $h^1(N_{C/X}) = 0$, respectively $h^1(N_{C/X}(-1)) = 0$. $\square$

3.3. Classification of $LQEL$-varieties, of Severi varieties, of Conic-connected manifolds and of varieties with small duals

3.3.1. Qualitative properties of CC-manifolds and of $LQEL$-manifolds. We describe the conics naturally appearing on CC-manifolds and on $LQEL$-manifolds of type $\delta > 0$ and relate them to intrinsic invariants, using the tools introduced in Theorem 3.2.13.

3.3.1. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a CC-manifold of secant defect δ. Let $C = C_{x,y}$ be a general conic through the general points $x, y \in X$ and let $c = [C]$ be the point representing C in the Hilbert scheme of X. Let $\mathcal{C}_x$ be the unique irreducible component of the Hilbert scheme of conics passing through x which contains the point c.*

(i) *We have $n + \delta \geq -K_X \cdot C = \dim(\mathcal{C}_x) + 2 \geq n + 1$ and the locus of conics through x and y is contained in the linear space*

$$\mathbb{P}^{\delta+1} = \langle T_x X, y\rangle \cap \langle x, T_y X\rangle.$$

(ii) *The equality $-K_X \cdot C = n + \delta$ holds if and only if $X \subset \mathbb{P}^N$ is an LQEL-manifold.*

(iii) *If $\delta \geq 3$, then $X \subset \mathbb{P}^N$ is a Fano manifold with $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ and index $i(X) = \frac{\dim(\mathcal{C}_x)}{2} + 1$.*

PROOF. We have the universal family $g : \mathcal{F}_x \to \mathcal{C}_x$ and the tautological morphism $f : \mathcal{F}_x \to X$, which is surjective. Since $C \in \mathcal{C}_x$ is a general conic and since $x \in X$ and y are general points, we get $\dim(\mathcal{C}_x) = -K_X \cdot C - 2$.

Indeed, by Theorem 3.2.13,

$$T_{X|C} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

is ample. Hence $a_i > 0$ for every $i = 1, \ldots, n$, and $N_{C/X}$ is ample being a quotient of $TX_{|C}$. Thus $\mathcal{C}_x$ is smooth at $C \in \mathcal{C}_x$ and of dimension

$$H^0(N_{C/X}(-1)) = H^0(T_{X|C}(-1)) - 2 = -2 + \sum_{i=1}^{n} a_i = -K_X \cdot C - 2.$$

Take a general point $y \in X$ and a general $p \in \langle x, y\rangle$. The conics passing through x and y are parameterized by $g(f^{-1}(y))$, which has pure dimension

$$\dim(\mathcal{F}_x) - n = \dim(\mathcal{C}_x) + 1 - n = -K_X \cdot C - 1 - n.$$

We claim that the locus of conics through x and y, denoted by $\mathcal{L}_{x,y}$, has dimension $-K_X \cdot C - n$ and is clearly contained in the irreducible component of the entry locus (with respect to p) through x and y. Indeed, conics through x, y and another general point $z \in \mathcal{L}_{x,y}$ have to be finitely many. Otherwise, their locus would fill up the plane $\langle x, y, z\rangle$ and this would imply that the line $\langle x, y\rangle$ is contained in X. But we have excluded linear spaces from the definition of CC and LQEL-manifolds. Therefore $\delta \geq -K_X \cdot C - n$, that is $-K_X \cdot C \leq n + \delta$. The locus of conics is contained in $\langle T_x X, y\rangle \cap \langle x, T_y X\rangle$, which is a linear space of dimension $\delta + 1$ by Terracini Lemma. This proves (i).

If $-K_X \cdot C = n + \delta$, then, for $p \in \langle x, y\rangle$ general, the irreducible component $\Sigma^p_{x,y}$ of the entry locus passing through x and y coincides with the locus of conics through x and y, so that it is contained in $\langle T_x X, y\rangle \cap \langle x, T_y X\rangle = \mathbb{P}^{\delta+1}$. Thus $\Sigma^p_{x,y}$ is a quadric hypersurface by the Trisecant Lemma and by the generality of x and y (if $\delta = n$, $X \subset \mathbb{P}^{n+1}$ is a quadric hypersurface). So, (ii) is proved.

Finally, (iii) follows from the Barth–Larsen Theorem, the fact that X contains moving conics and (i). □

3.3.2. COROLLARY. *Let $X \subset \mathbb{P}^N$ be an $LQEL$-manifold of type $\delta \geq 1$. Then:*

(1) *X is a simply connected manifold such that $\mathrm{H}^0(\Omega_X^{\otimes m}) = 0$ for every $m \geq 1$ and $\mathrm{H}^i(\mathcal{O}_X) = 0$ for every $i > 0$.*

(2) *There exists on X an irreducible family of conics $\mathcal{C}$ of dimension $2n + \delta - 3$, whose general member is smooth. This family describes an open subset of an irreducible component of the Hilbert scheme of conics on X.*

(3) *Given a general point $x \in X$, let $\mathcal{C}_x$ be the family of conics in $\mathcal{C}$ passing through x. Then $\mathcal{C}_x$ has dimension $n + \delta - 2$, equal to the dimension of the irreducible components of $\mathcal{C}_x$ describing dense subsets of X.*

(4) *Given two general points $x, y \in X$, the locus $Q_{x,y}$ of the family $\mathcal{C}_{x,y}$ of smooth conics in $\mathcal{C}$ passing through x and y is a smooth quadric hypersurface of dimension δ. The family $\mathcal{C}_{x,y}$ is irreducible and of dimension $\delta - 1$.*

(5) *A general conic $C \in \mathcal{C}_x$ intersects $\mathrm{T}_x X$ only at x. Moreover the tangent lines to smooth conics contained in X and passing through $x \in X$ describe an open subset of $\mathbb{P}((\mathbf{T}_x X)^*)$.*

PROOF. The variety X is clearly rationally connected. The conclusions of part (1) are contained in [**KMM**, Proposition 2.5] and also in [**De1**, Corollary 4.18].

Part (4) is the definition of an $LQEL$-variety. Indeed, the plane spanned by every conic through x and y contains the line $\langle x, y\rangle$, which is a general secant line to X. Thus a conic through x and y is contained in the entry locus of every $p \in \langle x, y\rangle$ not on X, so that for $p \in \langle x, y\rangle$ general it is contained in the smooth quadric hypersurface $Q_{x,y}$, the unique irreducible component of Σ_p passing through x and y.

Let us prove parts (3) and (5). Fixing two general points $x, y \in X$ there exists a smooth quadric hypersurface $Q_{x,y} \subset X \subset \mathbb{P}^N$ of dimension $\delta \geq 1$ through x and y. Thus there exists a smooth conic passing through x and a general point $y \in X$. In particular, there exists an irreducible family of smooth conics passing through x, let us say $\mathcal{C}_x^1$, whose members describe a dense subset of X.

Let $\widetilde{\mathcal{C}_x}$ be the universal family over $\mathcal{C}_x$ and let $\pi : \widetilde{\mathcal{C}_x} \to X$ be the tautological morphism. By part (4) and the Theorem on the dimension of the fibers we get that $\dim(\widetilde{\mathcal{C}_x^1}) = n + \delta - 1$. Thus $n + \delta - 2 = \dim(\mathcal{C}_x^1)$ and part (3) and (5) are proved. Part (2) now easily follows in the same way counting dimensions.

Let $C \subset X$ be a general smooth conic passing through $x \in X$. If we fix a direction through x, there exists a conic in $\mathcal{C}_x^1$ tangent to the fixed direction. If the direction does not correspond to a line contained in X, then the conic is irreducible. and hence smooth at x.

Consider the map $\tau_x : \mathcal{C}_x \dashrightarrow \mathbb{P}((\mathbf{T}_x X)^*)$, which associates to a conic in $\mathcal{C}_x$ its tangent line at x. The closure of the image of τ_x in $\mathbb{P}((\mathbf{T}_x X)^*)$ has dimension $n - 1$ by the above analysis, containing the open set parametrizing tangent direction not corresponding to lines through x and contained in X. □

On an $LQEL$-manifold of type $\delta \geq 2$ there are also lines coming from the entry loci and we proceed to investigate them. The following result is essentially well known, see also [**Hw1**, Proposition 1.5]; we recall its proof for reader's convenience.

3.3.3. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety. Then:*

(1) *The Hilbert scheme of lines passing through a general point $x \in X$, if not empty, is smooth and can be identified with a smooth not necessarily irreducible variety $Y_x \subseteq \mathbb{P}^{n-1} = \mathbb{P}((\mathbf{T}_x X)^*)$.*

(2) *If Y_x^j, $j = 1, \ldots, m$, are the irreducible components of Y_x, then we have*

$$\dim(Y_x^l) + \dim(Y_x^p) \leq n - 2 \quad \textit{for every } l \neq p.$$

PROOF. We argue essentially as in the proof of Theorem 3.3.2. By Theorem 3.2.13 for every line L contained in $X \subset \mathbb{P}^N$ and passing through the general point $x \in X$ we have $TX_{|L} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i(L))$, with $a_n(L) \geq a_{n-1}(L) \geq \cdots \geq a_1(L) \geq 0$. Moreover $a_n(L) \geq 2$ because $T\mathbb{P}^1$ is a subbundle of $TX_{|L}$. On the other hand, $T\mathbb{P}^N_{|L} = \mathcal{O}(2) \oplus \mathcal{O}(1)^{N-1}$ contains $TX_{|L}$ as a subbundle so that $a_n(L) = 2$ and $1 \geq a_{n-1}(L) \geq \cdots \geq a_1(L) \geq 0$ (i.e. the arbitrary line L is a *standard* (or *minimal*) curve in the sense of Mori Theory. It follows that the map which associates to each line through x its tangent direction is a closed embedding by Theorem 3.2.13 so that we can identify the Hilbert scheme of lines through x with a variety $Y_x \subset \mathbb{P}^{n-1} = \mathbb{P}((\mathbf{T}_x X)^*)$, which is smooth. Indeed, $N_{L/X} = \bigoplus_{j=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(b_j(L))$ with $b_j(L) \geq 0$ for every $j = 1, \ldots, n-1$, being the quotient of a locally free sheaf generated by global sections. Therefore $h^1(N_{L/X}(-1)) = 0$ and Y_x is smooth at the point corresponding to L. Since L was an arbitrary line through x, Y_x is smooth. The conditions on the dimension of two irreducible components simply say that these components cannot intersect in $\mathbb{P}^{n-1}$. □

Now we prove a fundamental result on the geometry of lines on an $LQEL$-manifold, which, via the study of the projective geometry of $Y_x \subset \mathbb{P}^{n-1}$ and of its dimension, will yield significant obstructions for the existence of $LQEL$-manifolds of type $\delta \geq 3$. The most relevant part for future applications is part (4), d). Part (1) holds more generally for every smooth secant defective variety.

3.3.4. THEOREM. *Suppose that $X \subset \mathbb{P}^N$ is an $LQEL$-manifold of type δ.*

(1) ([**S4**, p. 282, Opere Complete, vol. I]) *If $\delta \geq 1$, then*
$$\widetilde{\pi}_x : \mathbb{P}((\mathbf{T}_x X)^*) \dashrightarrow W_x \subseteq \mathbb{P}^{N-n-1}$$
is dominant, so that $\dim(|II_{x,X}|) = N - n - 1$ *and* $N \leq \frac{n(n+3)}{2}$.

(2) *If $\delta \geq 2$, the smooth, not necessarily irreducible, variety $Y_x \subset \mathbb{P}^{n-1}$ is non-degenerate and it consists of irreducible components of the base locus scheme of $|II_{x,X}|$. Moreover, the closure of the irreducible component of a general fiber of $\widetilde{\pi}_x$ passing through a general point $p \in \mathbb{P}((\mathbf{T}_x X)^*)$ is a linear space $\mathbb{P}^{\delta-1}_p$, cutting scheme-theoretically Y_x in a quadric hypersurface of dimension $\delta - 2$.*

(3) *If $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*)$ is irreducible and if $\delta \geq 2$, then $SY_x = \mathbb{P}((\mathbf{T}_x X)^*)$ and $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*)$ is a QEL-manifold of type $\delta(X) - 2$.*

(4) *If $\delta \geq 3$, then*

a) $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$.

b) *For any line $L \subset X$, $-K_X \cdot L = \frac{n+\delta}{2}$, so that $i(X) = \frac{n+\delta}{2}$, where $i(X)$ is the index of the Fano manifold X. In particular $n + \delta \equiv 0 \pmod 2$, that is $n \equiv \delta \pmod 2$.*

c) *There exists on X an irreducible family of lines of dimension $\frac{3n+\delta}{2} - 3$ such that for a general L in this family*
$$TX_{|L} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\frac{n+\delta}{2}-2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\frac{n-\delta}{2}+1}.$$

d) *If $x \in X$ is general, then $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*)$ is a QEL-manifold of dimension $\frac{n+\delta}{2} - 2$, of type $\delta(X) - 2$ and such that $SY_x = \mathbb{P}((\mathbf{T}_x X)^*)$;*

PROOF. Part (1) is classical and as we said above holds for every smooth secant defective variety. Since its proof is self-contained and elementary for $LQEL$-varieties, we include it for the reader's convenience. It suffices to show that, via the restriction of $\widetilde{\pi}_x$, the exceptional divisor $E = \mathbb{P}((\mathbf{T}_x X)^*)$ dominates $W_x \subseteq \mathbb{P}^{N-n-1}$. Take a general point $y \in X$. By part (6) of Theorem 3.3.2, there exists a conic $C_{x,y}$ through x and y, cutting $\mathrm{T}_x X$ only at x. Thus $\pi_x(C_{x,y}) = \pi_x(y) \in W_x$ is a general point and clearly $\widetilde{\pi}_x(\mathbb{P}(\mathbf{T}_x C_{x,y})) = \pi_x(C_{x,y}) = \pi_x(y)$. Therefore the restriction of $\widetilde{\pi}_x$ to E is dominant as a map to $W_x \subseteq \mathbb{P}^{N-n-1}$, yielding $\dim(|II_{x,X}|) = N - n - 1$. In particular $N - n - 1 = \dim(|II_{x,X}|) \leq \dim(|\mathcal{O}_{\mathbb{P}^{n-1}}(2)|) = \frac{n(n+1)}{2} - 1$ and $N \leq \frac{n(n+3)}{2}$.

Suppose from now on $\delta \geq 2$. If $y \in X$ is a general point and if $C_{x,y}$ is a smooth conic through x and y the point $\mathbb{P}(\mathbf{T}_x C_{x,y})$ is a general point of $\mathbb{P}((\mathbf{T}_x X)^*)$, by Theorem 3.3.2 part (6). Consider the unique quadric hypersurface $Q_{x,y}$ of dimension $\delta \geq 2$ through x and y, the irreducible component through x and y of the entry locus of a general $p \in \langle x, y\rangle$. Then $C_{x,y} \subset Q_{x,y}$ and $\mathrm{T}_x C_{x,y} \subset \mathrm{T}_x Q_{x,y}$. Take a line L_x through x and contained in $Q_{x,y}$, which can be thought of as a point of $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*)$. The plane $\langle L_x, \mathrm{T}_x C_{x,y}\rangle$ is contained in $\mathrm{T}_x Q_{x,y}$ so that it cuts $Q_{x,y}$ at least in another line L'_x, clearly different from $\mathrm{T}_x C_{x,y}$. Thus $\mathrm{T}_x C_{x,y}$ belongs to the pencil generated by L_x and L'_x, which projectivezed in $\mathbb{P}((\mathbf{T}_x X)^*)$ means that through the general point $\mathbb{P}(\mathbf{T}_x C_{x,y}) \in \mathbb{P}((\mathbf{T}_x X)^*)$ there passes the secant line $\langle \mathbb{P}(\mathbf{T}_x L_x), \mathbb{P}(\mathbf{T}_x L'_x)\rangle$ to Y_x. Therefore $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*)$ is non-degenerate and the join of Y_x with itself equals $\mathbb{P}((\mathbf{T}_x X)^*)$. For an irreducible $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*)$ this means exactly $SY_x = \mathbb{P}((\mathbf{T}_x X)^*)$. The scheme $\mathrm{T}_x Q_{x,y} \cap Q_{x,y}$ is a quadric cone with vertex x and base a smooth quadric hypersurface of dimension $\delta - 2$. The lines in $\mathrm{T}_x Q_{x,y} \cap Q_{x,y}$ describe a smooth quadric hypersurface of dimension $\delta - 2$, $\widetilde{Q}_{x,y} \subset Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*)$, whose linear span $\langle \widetilde{Q}_{x,y}\rangle = \mathbb{P}^{\delta-1}$ passes through $r = \mathbb{P}(\mathbf{T}_x C_{x,y})$. Since $\widetilde{\pi}_x : \mathbb{P}((\mathbf{T}_x X)^*) \dashrightarrow W_x \subseteq \mathbb{P}^{N-n-1}$ is given by a linear system of quadrics vanishing on Y_x, the whole $\mathbb{P}^{\delta-1}$ is contracted by $\widetilde{\pi}_x$ to $\widetilde{\pi}_x(r)$. The closure of the irreducible component of $\widetilde{\pi}_x^{-1}(\widetilde{\pi}_x(r))$ passing through r has dimension $n-1-\dim(W_x)=\delta-1$ so that it coincides with $\langle\widetilde{Q}_{x,y}\rangle=\mathbb{P}^{\delta-1}$. This also shows that Y_x is an irreducible component of the support of the base locus scheme of $|II_{x,X}|$ and also that, when irreducible, $Y_x\subset\mathbb{P}((\mathbf{T}_xX)^*)$ is a QEL-manifold of type $\delta-2$. Indeed in this case $r\in SY_x$ is a general point and every secant or tangent line to the smooth irreducible variety Y_x passing through r is contracted by $\widetilde{\pi}_x$, since the quadrics in $|II_{x,X}|$ vanish on Y_x. Thus every secant line through r is contained in $\langle\widetilde{Q}_{x,y}\rangle$ and the entry locus with respect to r is exactly $\widetilde{Q}_{x,y}$. This concludes the proof of parts (2) and (3).

Suppose $\delta\geq 3$ and let us concentrate on part (4). Item (a) follows directly from Barth–Larsen Theorems, see [**BL**], but we provide a direct proof using the geometry of $LQEL$-varieties. There are lines through a general point $x\in X$, for example the ones constructed from the family of entry loci. Reasoning as in Proposition 3.3.3, we get

$$TX_{|L}=\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}(1)^{m(L)}\oplus\mathcal{O}_{\mathbb{P}^1}^{n-m(L)-1}$$

for every line L through x. Thus if such a line comes from a general entry locus, we get

$$2+m(L)=-K_X\cdot L=\frac{-K_X\cdot C}{2}=\frac{n+\delta}{2},$$

yielding $m(L)=\frac{n+\delta}{2}-2$.

We define R_x to be the locus of points on X which can be joined to x by a connected chain of lines whose numerical class is $\frac{1}{2}[C]$, $C\in\mathcal{C}$ a general conic. By construction we get $R_x=X$, so that the Picard number of X is one by [**Ko**, IV.3.13.3]. Since the variety X is simply connected being rationally connected, see Theorem 3.3.2, we deduce $\mathrm{Num}(X)=\mathrm{NS}(X)=\mathrm{Pic}(X)\simeq\mathbb{Z}\langle\mathcal{O}(1)\rangle$. Thus $X\subset\mathbb{P}^N$ is a Fano variety, $Y_x\subset\mathbb{P}^{n-1}$ is equidimensional of dimension $\frac{n+\delta}{2}-2$ and $m(L)=\frac{n+\delta}{2}-2$ for every line L through x. We claim that Y_x is irreducible.

Indeed, if there were two irreducible components $Y_x^1,Y_x^2\subset Y_x\subset\mathbb{P}((\mathbf{T}_xX)^*)$, then $\dim(Y_x^1)+\dim(Y_x^2)=n+\delta-4\geq n-1$, in contrast to Proposition 3.3.3.

The fact that $Y_x\subset\mathbb{P}((\mathbf{T}_xX)^*)$ is a QEL-manifold of type $\delta(X)-2$ such that $SY_x=\mathbb{P}((\mathbf{T}_xX)^*)$ follows from part (3) above. Therefore all the assertions are now proved. □

3.3.5. EXAMPLE. (*Segre varieties $X=\mathbb{P}^l\times\mathbb{P}^m\subset\mathbb{P}^{lm+l+m}$, $l\geq 1$, $m\geq 1$, are QEL-manifolds of type $\delta=2$*) By Proposition 2.3.10, we know that $X=\mathbb{P}^l\times\mathbb{P}^m\subset\mathbb{P}^{lm+l+m}$ is a QEL-manifold, clearly with $\delta\geq 2$, and we calculate its type, that is we determine $\delta(X)$.

The locus of lines through a point $x\in X$ is easily described, being the union of the two linear spaces of the rulings through x, that is $Y_x=\mathbb{P}^{l-1}\amalg\mathbb{P}^{m-1}\subset\mathbb{P}^{l+m-1}$. Letting notation be as in Theorem 3.3.2, we have $C\equiv L_1+L_2$, where the lines L_1 and L_2 belongs to different rulings. Then $n+\delta=-K_X\cdot C=(-K_X\cdot L_1)+(-K_X\cdot L_2)=(l-1)+2+(m-1)+2=n+2$, so that $\delta(\mathbb{P}^l\times\mathbb{P}^m)=2$ for every $l,m\geq 1$.

3.3.6. EXAMPLE. (*Grassmann varieties of lines $\mathbb{G}(1,r)\subset\mathbb{P}^{\binom{r+1}{2}-1}$ are QEL-manifolds of type $\delta=4$*) It is well known that $Y_x\subset\mathbb{P}((\mathbf{T}_x\mathbb{G}(1,r))^*)\simeq\mathbb{P}^{2r-3}$ is projectively equivalent to the Segre variety $\mathbb{P}^1\times\mathbb{P}^{r-2}\subset\mathbb{P}^{2r-3}$. Moreover, $\mathbb{G}(1,r)\subset\mathbb{P}^{\binom{r+1}{m+1}-1}$ is a QEL-manifold, for example by Proposition 2.3.10, and we determine its type δ. Take $x,y\in\mathbb{G}(1,r)$ general. They represent two lines $l_x,l_y\subset\mathbb{P}^r$, $r\geq 3$, which are skew so that $\langle l_x,l_y\rangle=\mathbb{P}^3_{x,y}\subseteq\mathbb{P}^r$. The Plücker embedding of the lines in $\mathbb{P}^3_{x,y}$ is a $\mathbb{G}(1,3)_{x,y}\subseteq\mathbb{G}(1,r)$ passing through x and y. Therefore $\delta(\mathbb{G}(1,r))\geq 4$. Thus $r-1=\dim(Y_x)=-K_X\cdot L-2$, where $L\subset\mathbb{G}(m,r)$ is an arbitrary line, yielding $-K_X=(r+1)H$, H an hyperplane section. Finally $r+1=\frac{2(r-1)+\delta}{2}$ by Theorem 3.3.4, that is $\delta=4$.

3.3.7. EXAMPLE. (*Spinor variety $S^{10}\subset\mathbb{P}^{15}$ and E_6-variety $X\subset\mathbb{P}^{26}$ as QEL-manifolds*) Let us analyze the 10-dimensional spinor variety $S^{10}\subset\mathbb{P}^{15}$. It is scheme theoretically defined by 10 quadratic forms defining a map $\phi:\mathbb{P}^{15}\dashrightarrow\phi(\mathbb{P}^{15})\subset\mathbb{P}^9$. The image $Q=\phi(\mathbb{P}^{15})\subset\mathbb{P}^9$ is a smooth 8-dimensional quadric hypersurface and the closure of every fiber is a $\mathbb{P}^7$ cutting X along a smooth quadric hypersurface, see for example [**ESB**].

In particular $\delta(X) = 6$ and $X \subset \mathbb{P}^{15}$ is a Fano manifold of index $i(X) = 8 = n-2$ such that $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$. It is a so called Mukai variety with $b_2(X) = 1$ and by the above description it is a QEL-manifold of type $\delta = 6$. For every $x \in X$ the variety $X^1 := Y_x(X) \subset \mathbb{P}^9$ is a variety of dimension $\frac{n+\delta(X)}{2} - 2 = 6$, defined by $\mathrm{codim}(X) = 5$ quadratic equations yielding a dominant map $\widetilde{\pi}_x : \mathbb{P}^9 \dashrightarrow \mathbb{P}^4$. The general fiber of $\widetilde{\pi}_x$ is a linear $\mathbb{P}^5$ cutting $Y_x \subset \mathbb{P}^9$ along a smooth quadric hypersurface of dimension 4; see Theorem 3.3.4. It is almost clear (and well known) that $Y_x \simeq \mathbb{G}(1,4) \subset \mathbb{P}^9$ Plücker embedded. From $Y_x \subset \mathbb{P}^9$ we can construct the locus of tangent lines and obtain $X^2 := Y_x(X^1) \subset \mathbb{P}^5$, the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$; see Example 3.3.6.

We can begin the process with the 16-dimensional variety $X = E_6 \subset \mathbb{P}^{26}$, a Fano manifold of index $i(X) = 12$ with $b_2(X) = 1$ and with $\delta(X) = 8$. This is a QEL-manifold of type $\delta = 8$, being the center of a $(2,2)$ special Cremona transformation, see [**ESB**]. By applying the above constructions one obtains $X^1 = Y_x(X) = S^{10} \subset \mathbb{P}^{15}$, see [**Z2**, IV]. One could also apply [**Mk**], since $X^1 \subset \mathbb{P}^{15}$ has dimension 10 and type $\delta = 6$ so that it is a Fano manifold of index $i(X) = (n+\delta)/2 = 8 = n-2$. Hence $X^2 = Y_x(X^1) = \mathbb{G}(1,4) \subset \mathbb{P}^9$ and finally $X^3 = Y_x(X^2) = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$.

The examples discussed above and the results of Theorem 3.3.4 suggest to iterate the process, whenever possible, of attaching to an $LQEL$-manifold of type $\delta \geq 3$ a non-degenerate QEL-manifold $Y_x \subset \mathbb{P}^{n-1}$ of type $\delta - 2$ such that $SY_x = \mathbb{P}^{n-1}$. If $r \geq 1$ is the largest integer such that $\delta - 2r \geq 1$, and if $X \subset \mathbb{P}^N$ is a $LQEL$-manifold of type δ, then the process can be iterated r times, obtaining QEL-manifolds of type $\delta - 2k \geq 3$ for every $k = 1, \ldots, r-1$.

3.3.8. DEFINITION. Let $X \subset \mathbb{P}^N$ be an $LQEL$-manifold of type $\delta \geq 3$. Let

$$r_X = \sup\{r \in \mathbb{N} : \delta \geq 2r+1\}.$$

For every $k = 1, \ldots, r_X - 1$, we define inductively

$$X^k = X^k(z_0, \ldots, z_{k-1}) = Y_{z_{k-1}}(X^{k-1}(z_0, \ldots, z_{k-2})),$$

where $z_i \in X^i$, $i = 0, \ldots, k-1$, is a general point and where $X^0 = X$.

The process is well defined by Theorem 3.3.4 since for every $k = 1, \ldots, r_X - 1$, the variety X^k is a QEL-manifold of type $\delta(X^k) = \delta - 2k \geq 3$. The QEL-manifold X^k depends on the choices of the general points $z_0, \ldots, z_{k-1}$ used to define it. The type and dimensions of the X^k's are well defined and we are interested in the determination of these invariants.

The following result is crucial for the rest of the paper. Its proof is a direct consequence of part (4), d) of Theorem 3.3.4.

3.3.9. THEOREM. *Let $X \subset \mathbb{P}^N$ be an $LQEL$-manifold of type $\delta \geq 3$. Then:*

(1) *For every $k = 1, \ldots, r_X$, the variety $X^k \subset \mathbb{P}^{\frac{n+(2^{k-1}-1)\delta}{2^{k-1}} - 2k+1}$ is a QEL-manifold of type $\delta(X^k) = \delta - 2k$, of dimension $\dim(X^k) = \frac{n+(2^k-1)\delta}{2^k} - 2k$, such that $SX^k = \mathbb{P}^{\frac{n+(2^{k-1}-1)\delta}{2^{k-1}} - 2k+1}$; in particular, $\mathrm{codim}(X^k) = \frac{n-\delta}{2^k} + 1$.*

(2) *2^{r_X} divides $n - \delta$, that is $n \equiv \delta \pmod{2^{r_X}}$.*

3.3.10. REMARK. Much weaker forms of the Divisibility Theorem were proposed in [**Oh**, Theorem 0.2] after long computations with Chern classes.

The hypothesis $\delta \geq 3$ is clearly sharp for the congruence established in part (2) of Theorem 3.3.9, or for its weaker form proved in part (4) of Theorem 3.3.4. Indeed for the Segre varieties $X_{l,m} = \mathbb{P}^l \times \mathbb{P}^m \subset \mathbb{P}^{lm+l+m}$, $1 \leq l \leq m$, of odd dimension $n = l + m$ we have $\delta(X_{l,m}) = 2$; see Example 3.3.5.

It is worthwhile remarking that the above result is not true for arbitrary smooth secant defective varieties having $\delta(X) \geq 3$ neither in the weaker form of a parity result. One can consider smooth non-degenerate complete intersections $X \subset \mathbb{P}^N$ with $N \leq 2n-2$ and such that $n \not\equiv N - 1 \pmod 2$. It is easy to see that for an arbitrary non-degenerate smooth complete intersection $X \subset \mathbb{P}^N$ with $N \leq 2n+1$, we have $SX = \mathbb{P}^N$. If $N \leq 2n-2$, then $\delta(X) = 2n+1-N \geq 3$ and $\delta \equiv N - 1 \pmod 2$.

Infinite series of secant defective smooth varieties $X \subset \mathbb{P}^N$ of dimension n with $SX \subsetneq \mathbb{P}^N$, $\delta(X) \geq 3$ and such that $n \not\equiv \delta(X) \pmod{2^{r_X}}$ can be constructed in the following way. Take $Z \subset \mathbb{P}^N$ a smooth QEL-manifold of type $\delta \geq 4$ and dimension n such that $SZ \subsetneq \mathbb{P}^N$. Consider a $\mathbb{P}^{N+1}$ containing the previous $\mathbb{P}^N$ as a hyperplane, take $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$ and let $Y = S(p, Z) \subset \mathbb{P}^{N+1}$ be the cone over Z of vertex p. If $W \subset \mathbb{P}^{N+1}$ is a general hypersurface of degree $d > 1$, not passing through p, then $X = W \cap Y \subset \mathbb{P}^{N+1}$ is a smooth non-degenerate variety of dimension n such that $SX = S(p, SZ) \subsetneq \mathbb{P}^{N+1}$. Thus $\delta(X) = \delta(Z) - 1 = \delta - 1 \geq 3$ and $n \not\equiv \delta(X) \pmod{2^{r_X}}$ since $n \equiv \delta \pmod{2^{r_X}}$. Clearly also $n \not\equiv \delta(X) \pmod 2$.

One can take, for example, $Z_n = \mathbb{G}(1, \frac{n}{2}+1) \subset \mathbb{P}^{\frac{n(n+6)}{8}}$, $n \geq 8$, which are QEL-manifolds of dimension $n \geq 8$ and type $\delta = 4$ such that $SZ \subsetneq \mathbb{P}^{\frac{n(n+6)}{8}}$.

3.3.2. Some classification results. In this section we classify various important classes of $LQEL$-manifolds. [**IR1**] contains the complete classification of CC-manifolds with $\delta(X) \leq 2$ and hence that of $LQEL$-manifolds of type $\delta = 1, 2$, see Theorem 3.3.16 below.

Let us recall that a non-degenerate smooth projective variety $X \subset \mathbb{P}^{\frac{3}{2}n}$ is said to be a *Hartshorne variety* if it is not a complete intersection. It is worth remarking that there exist Hartshorne varieties different from surfaces in $\mathbb{P}^3$ and from the ones described in item ii) and iii) of Corollary 3.3.11 below, see for example [**E2**, Proposition 1.9]. This last fact was kindly pointed out to me by Giorgio Ottaviani.

The first relevant application of the Divisibility Property is the following classification of $LQEL$-manifolds of type $\delta > \frac{n}{2}$, which answers a problem posed in [**KS**, 0.12.6].

3.3.11. COROLLARY. *Let $X \subset \mathbb{P}^N$ be an $LQEL$-manifold of type δ with $\frac{n}{2} < \delta < n$. Then $X \subset \mathbb{P}^N$ is projectively equivalent to one of the following:*

i) *the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;*
ii) *the Plücker embedding $\mathbb{G}(1,4) \subset \mathbb{P}^9$;*
iii) *the 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$;*
iv) *a general hyperplane section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$;*
v) *a general hyperplane section of $S^{10} \subset \mathbb{P}^{15}$.*

In particular, smooth quadric surfaces in $\mathbb{P}^3$, $\mathbb{G}(1,4) \subset \mathbb{P}^9$ and $S^{10} \subset \mathbb{P}^{15}$ are the only $LQEL$-manifolds, modulo projective equivalence, which are also Hartshorne varieties.

PROOF. By assumption $\delta > 0$. If $\delta \leq 2$, then $n = 3$ and $\delta = 2 = n - 1$. Therefore $N = 5$ and X is projectively equivalent to the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ by Proposition 3.3.13 below.

From now on we can assume $\delta \geq 3$ and that $X \subset \mathbb{P}^N$ is a Fano manifold with $\mathrm{Pic}(X) = \mathbb{Z}\langle\mathcal{O}(1)\rangle$. By Theorem 3.3.9 there exists an integer $m \geq 1$ such that $2\delta > n = \delta + m2^{r_X}$, so that

$$\delta > m2^{r_X}. \tag{3.3.1}$$

Suppose $\delta = 2r_X + 2$. From $2r_X + 2 > m2^{r_X}$ it follows $m = 1$ and $r_X \leq 2$. Hence either $\delta = 4$ and $n = 6$ and $X \subset \mathbb{P}^N$ is a Fano manifold of index $i(X) = (n + \delta)/2 = 5 = n - 1$ or $\delta = 6$ and $n = 10$ and $X \subset \mathbb{P}^N$ is a Fano manifold as above and of index $i(X) = (n + \delta)/2 = 8 = n - 2$. In the first case by [**Fj**, Theorem 8.11] we get case ii). In the second case we apply [**Mk**], obtaining case iii).

Suppose $\delta = 2r_X + 1$. From (3.3.1) we get $2r_X + 1 > m2^{r_X}$ forcing $m = 1$ and $r_X = 1, 2$. Therefore either $\delta = 3$ and $n = \delta + m2^{r_X} = 5$; or $\delta = 5$ and $n = \delta + m2^{r_X} = 9$. Reasoning as above, we get cases iv) and v).

To prove the last part let us recall that for a non-degenerate smooth variety $X \subset \mathbb{P}^{\frac{3}{2}n}$ necessarily $SX = \mathbb{P}^{\frac{3}{2}n}$; see Theorem 3.1.5. Thus $\delta(X) = \frac{n}{2} + 1 > \frac{n}{2}$ and applying the first part we deduce that we are either in case ii) or iii) or that $\frac{n}{2} + 1 = \delta = n$, i.e. $n = 2$, concluding the proof. □

Another interesting application of Theorem 3.3.9 is the classification of $LQEL$-manifolds of type $\delta = \frac{n}{2}$. For such varieties we get immediately that $n = 2, 4, 8$ or 16 and among them we find Severi varieties. Indeed, by [**Z2**, IV.2.1, IV.3.1, IV.2.2], see Corollary 3.1.8, Severi varieties are $LQEL$-manifolds of type $\delta = \frac{n}{2}$. Once we know that $n = 2, 4, 8$ or 16, it is rather simple to classify Severi varieties, as we shall show in Proposition 3.1.9 below, see also [**Z2**, IV.4] and [**L1**]. For $n = 2, 4$ the result is classical and well known while in our approach the $n = 8$ case follows from the classification of Mukai manifolds, [**Mk**]. The less obvious case is $n = 16$. What is notable, in our opinion, is not the fact that this proof is short, easy, natural, immediate and almost self-contained but the perfect parallel between our argument based on the Divisibility Theorem and some proofs of Hurwitz Theorem on the dimension of composition algebras over a field such as the one contained in [**La**, V.5.10], see also [**Cu**, Chap. 10. Sec. 36]. Surely this connection is well known today, see [**Z2**, pg. 89–91], but the other proofs of the classification of Severi varieties did not make this parallel so transparent.

About this result and the word "generalization" we would like to quote Herman Weyl: "*Before you can generalize, formalize and axiomatize, there must be a mathematical substance*", [**We**]. There is no doubt that the mathematical substance in this problem is entirely due to Fyodor Zak, who firstly brilliantly solved it in [**Z1**].

3.3.12. COROLLARY. *Let $X \subset \mathbb{P}^N$ be an $LQEL$-manifold of type $\delta = \frac{n}{2}$. Then $n = 2, 4, 8$ or 16 and $X \subset \mathbb{P}^N$ is projectively equivalent to one of the following:*

i) *the cubic scroll $S(1, 2) \subset \mathbb{P}^4$;*
ii) *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or one of its isomorphic projection in $\mathbb{P}^4$;*
iii) *the Segre 4-fold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;*
iv) *a general 4-dimensional linear section $X \subset \mathbb{P}^7$ of $\mathbb{G}(1, 4) \subset \mathbb{P}^9$;*
v) *the Segre 4-fold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ or one of its isomorphic projections in $\mathbb{P}^7$;*
vi) *a general 8-dimensional linear section $X \subset \mathbb{P}^{13}$ of $S^{10} \subset \mathbb{P}^{15}$;*
vii) *the Plücker embedding $\mathbb{G}(1, 5) \subset \mathbb{P}^{14}$ or one of its isomorphic projection in $\mathbb{P}^{13}$;*
viii) *the E_6-variety $X \subset \mathbb{P}^{26}$ or one of its isomorphic projection in $\mathbb{P}^{25}$;*
ix) *a 16-dimensional linearly normal rational variety $X \subset \mathbb{P}^{25}$, which is a Fano variety of index 12 with $SX = \mathbb{P}^{25}$, $\operatorname{def}(X) = 0$ and such that the base locus of $|II_{x,X}|$, $Z_x \subset \mathbb{P}^{15}$, is the union of a 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$ with $C_pS^{10} \simeq \mathbb{P}^7$, $p \in \mathbb{P}^{15} \setminus S^{10}$.*

In particular, a Severi variety $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ is projectively equivalent to a linearly normal variety as in ii), v), vii) *or* viii).

PROOF. By assumption n is even. If $n < 6$, then $n = 2$ or $n = 4$. If $n = 2$, the conclusion is well known, see [**Sev**] or Proposition 3.3.13. If $n = 4$, then $\delta = 2 = n - 2$. If H is a hyperplane section and if $C \in \mathcal{C}$ is a general conic, then $(K_X + 3H) \cdot C = -n - \delta + 2n - 2 = 0$ by part (5) of Theorem 3.3.2. Suppose $X \subset \mathbb{P}^N$ is a scroll over a curve, which is rational by Theorem 3.3.2. Since for a rational normal scroll either $SX = \mathbb{P}^N$ or $\dim(SX) = 2n + 1$, we get $N = \dim(SX) = 2n + 1 - \delta = 7$ so that $X \subset \mathbb{P}^7$ is a rational normal scroll of degree 4, which is the case described in iii). If $X \subset \mathbb{P}^N$ is not a scroll over a curve, $|K_X + 3H|$ is generated by global sections, see [**Io**, Theorem 1.4], and since through two general points of X there passes such a conic, we deduce $-K_X = 3H$. Thus $X \subset \mathbb{P}^N$ is a del Pezzo manifold, getting cases iii), iv) or v) by [**Fj**, Theorem 8.11].

Suppose from now on $n \geq 6$, $\delta = n/2 \geq 3$ and hence that $X \subset \mathbb{P}^N$ is a Fano manifold with $\operatorname{Pic}(X) = \mathbb{Z}\langle\mathcal{O}(1)\rangle$. By Theorem 3.3.9, 2^{r_X} divides $n - \delta = \frac{n}{2} = \delta$ so that 2^{r_X+1} divides n and $\delta = \frac{n}{2}$ is even. By definition of r_X, $\frac{n}{2} = 2r_X + 2$, so that, for some integer $m \geq 1$,

$$m2^{r_X+1} = n = 4(r_X + 1).$$

Therefore either $r_X = 1$ and $n = 8$, or $r_X = 3$ and $n = 16$. In the first case we get that $X \subset \mathbb{P}^N$ is a Fano manifold as above and of index $i(X) = (n+\delta)/2 = 6 = n-2$ and we are in cases vi) and vii) by [**Mk**]. In the remaining cases $Y_x \subset \mathbb{P}^{15}$ is a 10-dimensional QEL-manifold of type $\delta = 6$ so that $Y_x \subset \mathbb{P}^{15}$ is projectively equivalent to $S^{10} \subset \mathbb{P}^{15}$ by Corollary 3.3.11. Furthermore

$$N - 16 = \dim(|II_{X,x}|) + 1 \leq \mathrm{h}^0(\mathcal{I}_{S^{10}}(2)) = 10.$$

Thus either $N = 26$ and $Y_x \simeq S^{10} \subset \mathbb{P}^{15}$ is the base locus of the second fundamental form or one easily sees that we are in case ix). If $|II_{X,x}| \simeq |H^0(\mathcal{I}_{S^{10}}(2))|$, we are in case viii) by [**L2**], see also [**L3**] and also Proposition 3.3.15 or Proposition 3.4.22 below. □

We collect below some classification results used previously in this section, whose proof is straightforward.

3.3.13. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be an $LQEL$-manifold of type $\delta = n-1 \geq 1$. Then $n = 2$ or $n = 3$, $N \leq 5$ and $X \subset \mathbb{P}^N$ is projectively equivalent to one of the following:*

(1) *$\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ Segre embedded, or one of its hyperplane sections;*
(2) *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or one of its isomorphic projections in $\mathbb{P}^4$.*

PROOF. Theorem 3.3.4 part(4) yields $n-1=\delta\leq 2$. Thus either $n=2$ and $\delta=1$, or $n=3$ and $\delta=2$.

Suppose first $n = 2$. Let $C \subset X$ be a general entry locus. Since $-K_X \cdot C = 3$, Theorem 3.3.2, we get $C^2 = 1$ via adjunction formula. Moreover, $h^1(\mathcal{O}_X) = 0$ by Theorem 3.3.2, so that $h^0(\mathcal{O}_X(C)) = h^0(\mathcal{O}_{\mathbb{P}^1}(1)) + 1 = 3$ and $|C|$ is base point free. The birational morphism $\phi = \phi_{|C|} : X \to \mathbb{P}^2$ sends a conic C into a line. Thus $\phi^{-1} : \mathbb{P}^2 \dashrightarrow X \subset \mathbb{P}^N$ is given by a sublinear system of $|\mathcal{O}_{\mathbb{P}^2}(2)|$ of dimension at least four and the conclusion for $n = 2$ is now immediate. If $n = 3$, we get the conclusion by passing to a hyperplane section, taking into account Proposition 2.3.14. □

3.3.3. Reconstruction Severi varieties of dimension 2, 4, 8 and 16. We propose an elementary approach to the classification of Severi varieties. Arguing as in Corollary 3.3.12, we get immediately via the Divisibility Theorem that the dimension of a Severi variety is 2, 4, 8 or 16. At this point it should be clear that the classification of Severi varieties in dimension 2, 4, 8 and 16 is a straightforward consequence of the above results. As we said above for dimension 2 and 4 it is classical and elementary. Due to the beautifulness of this classification results, we shall reproduce here a short argument also as an interesting tour in higher dimensional projective geometry.

Let us recall the following picture of the known Severi varieties in dimension $n = 2$, 4, 8 and 16. Let $Y \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be either $\emptyset$, respectively $\mathbb{P}^1 \amalg \mathbb{P}^1 \subset \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ Segre embedded, $S^{10} \subset \mathbb{P}^{15}$ the 10 dimensional spinor variety. On $\mathbb{P}^{n-1}$ we take coordinates $x_0, \ldots, x_{n-1}$ and on $\mathbb{P}^n$ coordinates $x_0, \ldots, x_n$. Let $Q_1, \ldots, Q_{\frac{n}{2}+2}$ be the quadratic forms in the variables $x_0, \ldots, x_{n-1}$ defining $Y \subset \mathbb{P}^{n-1}$. The subvariety $Y \subset \mathbb{P}^n$ is scheme-theoretically defined by the $\frac{3n}{2} + 3$ quadratric forms: Q_i, $x_n x_j$, $i = 1, \ldots, \frac{n}{2} + 2$, $j = 0, \ldots, n$. More precisely these quadric hypersurfaces form the linear system of quadrics on $\mathbb{P}^n$ vanishing along Y, that is $|H^0(\mathcal{I}_{Y,\mathbb{P}^n}(2))|$. Let

$$\phi_{|H^0(\mathcal{I}_{Y,\mathbb{P}^n}(2))|} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\frac{3n}{2}+2}.$$

For $Y = \emptyset$, clearly $\phi(\mathbb{P}^2) = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. For $Y = \mathbb{P}^1 \amalg \mathbb{P}^1 \subset \mathbb{P}^3$ we get $\phi(\mathbb{P}^4) = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, a particular form of a result known to C. Segre, see [**Sg**], found when studying for the first time the nowadays called *Segre varieties*. For $Y = \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, one obtains $\phi(\mathbb{P}^8) = \mathbb{G}(1,5) \subset \mathbb{P}^{14}$ Plücker embedded, a particular case of a general result of Semple, see [**Sm**] and [**RS**]. For $Y = S^{10} \subset \mathbb{P}^{15}$, Zak has shown in [**LV**] and [**Z2**], chapter III, that $\phi(\mathbb{P}^{16}) = E_6 \subset \mathbb{P}^{26}$ is the Cartan, or E_6, variety.

The birational inverse of ϕ, $\phi^{-1} : X \dashrightarrow \mathbb{P}^n$, with $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ one of the Severi variety described above, is given by the linear projection from the linear space $\mathbb{P}^{\frac{n}{2}+1}_p =< \Sigma_p >$, $p \in SX$ a general point. We prove that, more generally and a priori, a Severi variety of dimension n can be birationally projected from $\mathbb{P}^{\frac{n}{2}+1}_p =< \Sigma_p >$, $p \in SX$ general, onto $\mathbb{P}^n$. This was originally proved in [**Z2**, IV.2.4 f)] and we furnish a proof for the reader convenience.

3.3.14. PROPOSITION. *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety. Let $p \in SX$ be a general point, let $\Sigma_p \subset \mathbb{P}^{\frac{n}{2}+1}_p$ be its entry locus and let*

$$\pi = \pi_{L_p} : X \dashrightarrow \mathbb{P}^n$$

be the projection from $L_p =<\Sigma_p>= \mathbb{P}_p^{\frac{n}{2}+1} = L_p$. *Let* $\widetilde{X} = Bl_{\Sigma_p} X \xrightarrow{\alpha} X$, *let* E *be the exceptional divisor and let* F *be the strict transform of* $H_p = T_pSX \cap X$ *on* $\widetilde{X}$. *Let* $\widetilde{\pi} : \widetilde{X} \to \mathbb{P}^n$ *be the resolution of* π_L *and let* $\widetilde{\pi}(F) = T_pSX \cap \mathbb{P}^n = \mathbb{P}^{n-1} \subset \mathbb{P}^n$. *By definition of* $\widetilde{\pi}$ *we have* $\widetilde{\pi}^{-1}(\mathbb{P}^{n-1}) = E \cup F$. *Then:*

i) *if* $\dim(\widetilde{\pi}^{-1}(z)) > 0$, $z \in \mathbb{P}^n$, *then* $\widetilde{\pi}^{-1}(z) \subseteq E \cup F$;
ii) *the morphism* $\widetilde{\pi}$ *is birational and defines an isomorphism between* $\widetilde{X} \setminus (E \cup F)$ *and* $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$. *In particular, the locus of indetermination of* $\widetilde{\pi}^{-1}$ *is a subscheme* $Y \subset \mathbb{P}^{n-1}$.

PROOF. Let $z \in \mathbb{P}^n \setminus \mathbb{P}^n$, let $Z = \widetilde{\pi}^{-1}(z))$ and suppose $\dim(Z) > 0$. Then $Z \cap (E \cup F) = \emptyset$, so that $\alpha(Z) = W$ is positive dimensional and it does not cut L_p. Thus W contains a positive dimensional variety $M \subset X$ such that $L_p \cap M = \emptyset$ and such that $\pi(M) = z$. This contradicts the fact that a linear projection, when it is defined everywhere, is a finite morphism. The first part is proved.

To prove part (ii) let us remark that if $p \in< x, y >$, $x, y \in X$ general points, then

(3.3.2) $$L_p =<\Sigma_p>=< T_xX, y > \cap < T_yX, x >,$$

by Terracini Lemma (see also the proof of Scorza Lemma). The projection from the linear space $< T_xX, y >$ can be thought as the composition of the tangential projection $\pi_x : X \dashrightarrow W_x \subset \mathbb{P}^{\frac{n}{2}+1}$ and the projection of the smooth quadric hypersurface W_x from the point $\pi_x(y)$. Thus the projection from $< T_xX, y >$, $\pi_{x,y} : X \dashrightarrow \mathbb{P}^{\frac{n}{2}}$ is dominant and for a general point $z \in X$ we get

(3.3.3) $$< T_xX, y, z > \cap X \setminus (< T_xX, y > \cap X) = \pi_{x,y}^{-1}(\pi_{x,y}(z)) = Q_{x,z} \setminus (Q_{x,z} \cap < T_xX, y >),$$

where as always $Q_{x,z}$ is the entry locus of a general point on $< x, z >$. Similarly

(3.3.4) $$< T_yX, x, z > \cap X \setminus (< T_yX, x > \cap X) = \pi_{y,x}^{-1}(\pi_{y,x}(z)) = Q_{y,z} \setminus (Q_{y,z} \cap < T_xX, y >).$$

By definition of projection we have that

(3.3.5) $$\pi^{-1}(\pi(z)) =< L_p, z > \cap (X \setminus \Sigma_p).$$

By the generality of z, we get

(3.3.6) $$\pi^{-1}(\pi(z)) =< L_p, z > \cap (X \setminus H_p).$$

By Terracini Lemma the linear spaces $< T_xX, y >$ and $< T_yX, x >$ are contained in T_pSX, so that $< T_xX, y > \cap X$ and $< T_yX, x > \cap X$ are contained in H_p. By combining (3.3.2), (3.3.3), (3.3.4) and (3.3.6) we finally get

$$z \subseteq \pi^{-1}(\pi(z)) \subseteq Q_{x,z} \cap Q_{y,z} = z,$$

where the last equality is scheme theoretical by Corollary 3.1.8 and by the generality of $x, y, z \in X$. $\square$

3.3.15. PROPOSITION. *Let* $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ *be a Severi variety of dimension* n. *Then* $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ *is projectively equivalent to one of the following:*

(1) *the Veronese surface* $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$;
(2) *the Segre 4-fold* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;
(3) *the Grassmann variety* $\mathbb{G}(1, 5) \subset \mathbb{P}^{14}$;
(4) *the Cartan (or* E_6*) variety* $X \subset \mathbb{P}^{26}$.

PROOF. By Proposition 3.1.8 a Severi variety is a QEL-variety of type $\delta = \frac{n}{2}$. The Divisibility Theorem easily implies $n = 2, 4, 8$ or 16. For $n = 2$ one can apply Proposition 3.3.13 (or Corollary 2.3.7) to get case 1).

Assume $n = 4$, so that $\delta = 2$. The base locus scheme of $|II_{x,X}|$, which is a linear system of dimension 3, is a smooth not necessarily irreducible curve in $\mathbb{P}^3$ with one apparent double point by Theorem 3.3.4. It immediately follows that $Y_x \subset \mathbb{P}^3$ is the union of two skew lines and that it coincides with the base locus scheme of $|II_{x,X}|$.

Suppose $n = 8$ and $\delta = 4$. By Theorem 3.3.4, the variety $Y_x \subset \mathbb{P}^7$ is a smooth, irreducible, non-degenerate, QEL-variety of dimension 4 and such that $SY_x = \mathbb{P}^7$. Furthermore, by Theorem 3.3.4 part 1), there are at least six quadric hypersurfaces vanishing on $Y_x \subset \mathbb{P}^7$. By restricting to a general $\mathbb{P}^3 \subset \mathbb{P}^7$, the usual Castelnuovo Lemma yields $\deg(Y_x) \leq 4$ and hence $\deg(Y_x) = 4$ since $Y_x \subset \mathbb{P}^7$ is non-degenerate. Thus $Y_x \subset \mathbb{P}^7$ is projectively equivalent to the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ and clearly it is the base locus scheme of $|II_{x,X}|$.

Suppose $n = 16$. Then by Theorem 3.3.4, $Y_x \subset \mathbb{P}^{15}$ is a Mukai variety of dimension 10 and type $\delta = 6 > n/2 = 5$. Thus by Corollary 3.3.11, $Y_x \subset \mathbb{P}^{15}$ is projectively equivalent to $S^{10} \subset \mathbb{P}^{15}$.

From now on we suppose $n \geq 4$ so that a general entry locus is not a divisor on X. Let $p \in SX \setminus X$ be a general point and let $\mathbb{P}_p^{\frac{n}{2}+1}$ be the locus of secant lines through p. Take a $\mathbb{P}^n$ disjoint from $\mathbb{P}_p^{\frac{n}{2}+1}$ and let $\varphi : X \dashrightarrow \mathbb{P}^n$ be the projection from $\mathbb{P}_p^{\frac{n}{2}+1}$. By Proposition 3.3.14 the map φ is birational and an isomorphism on $X \setminus (T_pSX \cap X)$. Let $Y \subset T_pSX \cap \mathbb{P}^n = \mathbb{P}^{n-1}$ be the base locus scheme of $\varphi^{-1} : \mathbb{P}^n \dashrightarrow X \subset \mathbb{P}^{\frac{3n}{2}+2}$, that is of φ^{-1} composed with the inclusion $i : X \to \mathbb{P}^{\frac{3n}{2}+2}$.

Take a general point $y \in \Sigma_p$. A general smooth conic through y cuts Σ_p transversally so that it is mapped onto a line by φ. By Theorem 3.3.2 there is an irreducible family of such conics through y of dimension $\frac{3n}{2} - 2$. By varying $y \in \Sigma_p$ the projected lines form a $(2n-2)$-dimensional family of lines on $\mathbb{P}^n$, which is then a part of the whole family of lines in $\mathbb{P}^n$. This means that φ^{-1} is given by a linear system of quadrics hypersurfaces vanishing on the subscheme $Y \subset \mathbb{P}^{n-1}$. Moreover since $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ is linearly normal, φ^{-1} is given by $|H^0(\mathcal{I}_{Y,\mathbb{P}^n}(2))|$.

Consider a general point $q \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$, which we can write as $\varphi(x)$ with $x \in X$ general. Consider the family of lines through $\varphi(x)$ and parametrized by the not necessarily irreducible variety $Y_{\text{red}} \subset \mathbb{P}^{n-1}$. The image via φ^{-1} of these lines are lines passing through x. Indeed, these lines cannot be contracted by Proposition 3.3.14, they cut Y and the restriction of φ^{-1} to such a line is given by a sublinear system of $|\mathcal{O}_{\mathbb{P}^1}(2)|$ with a base point. Thus we get a morphism $\alpha_x : Y_{\text{red}} \to Y_x$, since Y_x is isomorphic to the Hilbert scheme of lines through x. Moreover, the birational map φ^{-1} is an isomorphism near $\varphi(x)$, so that the morphism α_x is one-to-one.

A general line through a general point $x \in X$ is sent into a line passing through $\varphi(x)$, because it does not cut the center of projection. Since φ^{-1} is given by a linear system of quadric hypersurfaces, a general line through $\varphi(x)$ cuts Y in one point, proving that $\alpha_x : Y_{\text{red}} \to Y_x$ is dominant and hence surjective. Thus $\alpha_x : Y_{\text{red}} \to Y_x$ is an isomorphism by Zariski Main Theorem. Moreover, the variety $Y_{\text{red}} \subset \mathbb{P}^{n-1}$ is projectively equivalent to Y_x. Thus $Y_{\text{red}} \subset \mathbb{P}^{n-1}$ has homogeneous ideal generated by $\frac{n}{2} + 2$ quadratic equations and therefore it coincides with $Y \subset \mathbb{P}^{n-1}$.

Therefore the previous analysis furnishes that $Y \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \amalg \mathbb{P}^1 \subset \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, respectively $S^{10} \subset \mathbb{P}^{15}$. The conclusion follows from the birational representation of the known Severi varieties recalled above. □

3.3.4. Classification of conic-connected manifolds. The following Classification Theorem is one of the main result of [**IR1**]. As CC-manifolds are stable under isomorphic projection, we may assume X to be linearly normal.

3.3.16. THEOREM. ([**IR1**, Theorem 2.1]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible linearly normal non-degenerate CC-manifold of dimension n. Then either $X \subset \mathbb{P}^N$ is a Fano manifold with $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1) \rangle$ and of index $i(X) \geq \frac{n+1}{2}$, or it is projectively equivalent to one of the following:*

(i) $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$.
(ii) *The projection of $\nu_2(\mathbb{P}^n)$ from the linear space $\langle \nu_2(\mathbb{P}^s) \rangle$, where $\mathbb{P}^s \subset \mathbb{P}^n$ is a linear subspace; equivalently $X \simeq \operatorname{Bl}_{\mathbb{P}^s}(\mathbb{P}^n)$ embedded in $\mathbb{P}^N$ by the linear system of quadric hypersurfaces of $\mathbb{P}^n$ passing through $\mathbb{P}^s$; alternatively $X \simeq \mathbb{P}_{\mathbb{P}^r}(\mathcal{E})$ with $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus n-r} \oplus \mathcal{O}_{\mathbb{P}^r}(2)$, $r = 1, 2, \ldots, n-1$, embedded by $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Here $N = \frac{n(n+3)}{2} - \binom{s+2}{2}$ and s is an integer such that $0 \leq s \leq n-2$.*

(iii) *A hyperplane section of the Segre embedding $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{N+1}$. Here $n \geq 3$ and $N = ab + a + b - 1$, where $a \geq 2$ and $b \geq 2$ are such that $a + b = n + 1$.*
(iv) *$\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$ Segre embedded, where a, b are positive integers such that $a + b = n$.*

3.3.17. COROLLARY. *A CC-manifold is a Fano manifold with second Betti number $b_2 \leq 2$; for $b_2 = 2$ it is also rational.*

The preceding Theorem reduces the classification of CC-manifolds to the study of Fano manifolds having large index and Picard group $\mathbb{Z}$. The next result, essentially due to Hwang–Kebekus [**HK**, Theorem 3.14], shows that, conversely, such Fano manifolds are conic-connected. Note that we slightly improve the bound on the index given in [**HK**]. The first part of the Proposition is well known.

3.3.18. PROPOSITION. ([**IR2**, Proposition 2.4]; cf. also [**HK**]) *Let $X \subset \mathbb{P}^N$ be a Fano manifold with $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$ and $-K_X = i(X)H$, H being the hyperplane section and $i(X)$ the index of X. Let as always $Y_x \subset \mathbb{P}(T_x^*(X)) = \mathbb{P}^{n-1}$ be the Hilbert scheme of lines through a general point $x \in X$.*

(i) *If $i(X) > \frac{n+1}{2}$, then $X \subset \mathbb{P}^N$ is ruled by lines so that Y_x is nonempty and smooth. If $i(X) \geq \frac{n+3}{2}$, $Y_x \subset \mathbb{P}^{n-1}$ is also irreducible.*
(ii) *If $i(X) \geq \frac{n+3}{2}$ and $SY_x = \mathbb{P}^{n-1}$, then $X \subset \mathbb{P}^N$ is a CC-manifold.*
(iii) *If $i(X) > \frac{2n}{3}$, then $X \subset \mathbb{P}^N$ is a CC-manifold.*

PROOF. By a Theorem of Mori [**Mo1**] the variety $X \subset \mathbb{P}^N$ is ruled by a family of rational curves $\mathcal{L}$ such that for $L \in \mathcal{L}$ we have $-K_X \cdot L \leq n + 1$. It follows that $X \subset \mathbb{P}^N$ is ruled by lines, that is $H \cdot L = 1$, because

$$n + 1 \geq -K_X \cdot L = i(X)(H \cdot L) > H \cdot L \frac{n+1}{2}.$$

By Proposition 3.3.3 the Hilbert scheme of lines passing through x is smooth equidimensional and it can be identified with a subscheme $Y_x \subset \mathbb{P}(T_x^*(X)) = \mathbb{P}^{n-1}$ of dimension $i(X) - 2$. Thus, if $i(X) - 2 \geq \frac{n-1}{2}$, $Y_x \subset \mathbb{P}^{n-1}$ is irreducible.

Part (ii) follows from [**HK**, Theorem 3.14].

To prove part (iii), first observe that $i(X) \geq \frac{n+3}{2}$, unless $n \leq 6$. If $n \leq 6$, $i(X) > \frac{2n}{3}$ gives $i(X) \geq n - 1$, so the conclusion follows by the classification of del Pezzo manifolds (see [**Fj**]). To conclude, using part (ii), it is enough to see that $SY_x = \mathbb{P}^{n-1}$.

By [**Hw1**, Theorem 2.5], the variety $Y_x \subset \mathbb{P}^{n-1}$ is non-degenerate and by hypothesis

$$n - 1 < \frac{3i(X) - 6}{2} + 2 = \frac{3\dim(Y_x)}{2} + 2,$$

so that $SY_x = \mathbb{P}^{n-1}$ by Zak Linear Normality Theorem; see Theorem 3.1.5. □

3.3.19. COROLLARY. *If $X \subset \mathbb{P}^{n+r}$ is a smooth non-degenerate complete intersection of multi-degree $(d_1, d_2, \ldots, d_r)$ with $n > 3\big(\sum_1^r d_i - r - 1\big)$, then X is a CC-manifold.*

3.3.5. Classification of varieties with small dual. For an irreducible variety $Z \subset \mathbb{P}^N$, we defined $\mathrm{def}(Z) = N - 1 - \dim(Z^*)$ as the dual defect of $Z \subset \mathbb{P}^N$, where $Z^* \subset \mathbb{P}^{N*}$ is the dual variety of $Z \subset \mathbb{P}^N$. In [**E1**, Theorem 2.4] it is proved that if $\mathrm{def}(X) > 0$, then $\mathrm{def}(X) \equiv n(\mathrm{mod}\, 2)$, a result usually attributed to Landman. Moreover, Zak Theorem on Tangencies implies that $\dim(X^*) \geq \dim(X)$ for a smooth non-degenerate variety $X \subset \mathbb{P}^N$; see Corollary 2.2.5.

We combine the geometry of CC and LQEL-manifolds to give a new proof of [**E1**, Theorem 4.5]. Our approach avoids the use of Beilinson spectral sequences and more sophisticated computations as in [**E1**, 4.2, 4.3, 4.4].

We begin by recalling some basic facts from [**E1**].

3.3.20. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate variety and assume that $\mathrm{def}(X) > 0$. Then*

(i) ([**E1**, Theorem 2.4]) *through a general point $x \in X$ there passes a line $L_x \subset X$ such that $-K_X \cdot L_x = \frac{n+\mathrm{def}(X)+2}{2}$, so that* $\mathrm{def}(X) \equiv n \,(\mathrm{mod}\, 2)$*;*

(ii) ([**E1**, Theorem 3.2]) $\mathrm{def}(X) = n-2$ *if and only if $X \subset \mathbb{P}^N$ is a scroll over a smooth curve, i.e. it is a $\mathbb{P}^{n-1}$-bundle over a smooth curve, whose fibers are linearly embedded.*

The following Proposition reinterprets the result of Hwang and Kebekus [**HK**, Theorem 3.14] on Fano manifolds with large index.

3.3.21. PROPOSITION. ([**IR2**, Proposition 4.3]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate variety. Assume that X is a Fano manifold with* $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$ *and let $x \in X$ be a general point.*

(i) *If* $\mathrm{def}(X) > 0$ *and* $\mathrm{def}(X) > \frac{n-6}{3}$*, then X is a CC-manifold with* $\delta \geq \mathrm{def}(X)+2$*. Moreover, if* $\delta = \mathrm{def}(X)+2$*, then X is an LQEL-manifold of type* $\delta = \mathrm{def}(X)+2$*.*

(ii) *If X is an LQEL-manifold of type δ and* $\mathrm{def}(X) > 0$*, then* $\delta = \mathrm{def}(X)+2$*.*

PROOF. In the hypothesis of (i), [**E1**, Theorem 2.4] yields $i(X) = \frac{n+\mathrm{def}(X)+2}{2} > \frac{2n}{3}$ so that X is a CC-manifold by Proposition 3.3.18. Proposition 3.3.1 yields $\delta \geq \mathrm{def}(X)+2$ and also the remaining assertions of (ii) and (ii). $\square$

We recall that according to Hartshorne Conjecture, if $n > \frac{2}{3}N$, then $X \subset \mathbb{P}^N$ should be a complete intersection and that complete intersections have no dual defect, see part (4) of Exercise refexecisedual. Thus, assuming Hartshorne Conjecture, the following result yields the complete list of manifolds $X \subset \mathbb{P}^N$ such that $\dim(X^*) = \dim(X)$. The second part says that under the LQEL hypothesis the same results hold without any restriction.

3.3.22. THEOREM. *Let $X \subset \mathbb{P}^N$ be a a smooth irreducible non-degenerate variety such that* $\dim(X) = \dim(X^*)$.

(i) ([**E1**, Theorem 4.5]) *If $N \geq \frac{3n}{2}$, then X is projectively equivalent to one of the following:*
   (a) *a smooth hypersurface $X \subset \mathbb{P}^{n+1}$, $n = 1, 2$;*
   (b) *a Segre variety $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$;*
   (c) *the Plücker embedding $\mathbb{G}(1,4) \subset \mathbb{P}^9$;*
   (d) *the* 10*-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$.*

(ii) *If X is an LQEL-manifold, then it is projectively equivalent either to a smooth quadric hypersurface $Q \subset \mathbb{P}^{n+1}$ or to a variety as in* (b), (c), (d) *above.*

PROOF. Clearly $\mathrm{def}(X) = 0$ if and only if $X \subset \mathbb{P}^{n+1}$ is a hypersurface, giving case (a), respectively that of quadric hypersurfaces. From now on we suppose $\mathrm{def}(X) > 0$ and hence $n \geq 3$. By parts (i) and (ii) of Proposition 3.3.20, $\mathrm{def}(X) = n-2$ and $N = 2n-1$ if and only if we are in case (b); see also [**E1**, Theorem 3.3, c)].

Thus, we may assume $0 < \mathrm{def}(X) \leq n-4$, that is $N \leq 2n-3$. Therefore $\delta \geq 4$ and X is a Fano manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$. Moreover, in case (i), $\mathrm{def}(X) = N-n-1 > \frac{n-6}{3}$ by hypothesis. Thus Proposition 3.3.21 yields that X is also a CC-manifold with $\delta \geq \mathrm{def}(X)+2$. Taking into account also the last part of Proposition 3.3.21, from now on we can suppose that X is a CC-manifold with $\delta \geq \mathrm{def}(X)+2 \geq 3$.

We have $n - \delta \leq N-1-n = \mathrm{def}(X) \leq \delta - 2$, that is $\delta \geq \frac{n}{2}+1$. Zak Linear Normality Theorem implies $SX = \mathbb{P}^N$, so that

$$N = \dim(SX) = 2n+1-\delta \leq \frac{3n}{2}.$$

Since $N \geq \frac{3n}{2}$, we get $N = \frac{3n}{2}$, $\delta = \frac{n}{2}+1 = \mathrm{def}(X)+2$ and n even. Therefore X is an LQEL-manifold of type $\delta = \frac{n}{2}+1$ by Proposition 3.3.21. Corollary 3.3.11 concludes the proof, yielding cases (c) and (d). $\square$

3.3.6. A refined linear normality bound for $LQEL$–varieties. We defined inductively in Definition 3.3.8 some varieties naturally attached to a $LQEL$-manifold of type $\delta \geq 3$. Let $i_j = i(X^j)$, $j = 1, \ldots, r_X - 1$, be the index of the Fano manifold X^j. Computing as in Theorem 3.3.9 we get:

$$i_j = \frac{n-\delta}{2^{j+1}} + \delta - 2j, \; 0 \leq j \leq r_X - 1. \tag{3.3.7}$$

3.3.23. COROLLARY. ([**Fu1**, Theorem 3]) *Let $X \subset \mathbb{P}^N$ be an n-dimensional $LQEL$-manifold of type δ. If*

$$\delta > 2[\log_2 n] + 2 \text{ or } \delta > \min_{k \in \mathbb{N}} \{\frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1}\},$$

then $N = n + 1$ and $X \subset \mathbb{P}^{n+1}$ is a quadric hypersurface.

PROOF. If $\delta > 2[\log_2 n] + 2$, then $n < 2^r$, where $r = [(\delta - 1)/2]$. By Theorem 3.3.9, 2^r divides $n - \delta$. This is possible only if $\delta = n$. Thus X is a hyperquadric. Now assume we have the second inequality. Note that for a fixed n, the minimum $\min_{k \in \mathbb{N}}\{\frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1}\}$ is achieved for some $k \leq n/2$, so we may assume that for some $k \leq n/2$, we have $\delta > \frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1} = 2k + \frac{n-2k}{2^{k-1}+1} \geq 2k$, so that $\delta \geq 2k+1$. Now we can consider the variety $Y_k \subset \mathbb{P}^{\dim Y_{k-1}-1}$. Note that $\dim Y_k = i(Y_{k-1}) - 2$ and

$$\dim Y_{k-1} = 2i_{k-1} - \delta(Y_{k-1}) = \frac{n-\delta}{2^{k-1}} + \delta - 2k + 2.$$

On the other hand, $Y_k \subset \mathbb{P}^{\dim(Y_{k-1})-1}$ is non-degenerate and it contains a hyperquadric of dimension $\delta - 2k$, which is strictly bigger than $(\dim Y_{k-1} - 2)/2$ under our assumption on δ. Now [**Z2**, Corollary I.2.20] implies that $Y_k \subset \mathbb{P}^{\dim(Y_{k-1})-1}$ is a hypersurface. Since it is a non-degenerate hypersurface by Theorem 3.3.9 and a $LQEL$-manifold, Y_{k-1} is a quadric hypersurface. This easily implies yields the conclusion. □

We now state a sharper Linearly Normality Bound for $LQEL$-manifolds, see Theorem 3.1.5.

3.3.24. COROLLARY. *Let $X \subset \mathbb{P}^N$ be a $LQEL$-manifold of type δ, not a quadric hypersurface. Then*

$$\delta \leq \min_{k \in \mathbb{N}}\{\frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1}\} \leq \frac{n+8}{3}$$

and

$$N \geq \dim(SX) \geq 2n + 1 - \min_{k \in \mathbb{N}}\{\frac{n}{2^{k-1}+1} + \frac{2^k k}{2^{k-1}+1}\} \geq \frac{5}{3}(n-1).$$

Furthermore $\delta = \frac{n+8}{3}$ if and only if $X \subset \mathbb{P}^N$ is projectively equivalent to one of the following:

i) *a smooth 4-dimensional quadric hypersurface $X \subset \mathbb{P}^5$;*
ii) *the 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$;*
iii) *the E_6-variety $X \subset \mathbb{P}^{26}$ or one of its isomorphic projection in $\mathbb{P}^{25}$;*
iv) *a 16-dimensional linearly normal rational variety $X \subset \mathbb{P}^{25}$, which is a Fano variety of index 12 with $SX = \mathbb{P}^{25}$, dual defect $\operatorname{def}(X) = 0$ and such that the base locus scheme $C_x \subset \mathbb{P}^{15}$ of $|II_{x,X}|$ is the union of 10-dimensional spinor variety $S^{10} \subset \mathbb{P}^{15}$ with $C_p S^{10} \simeq \mathbb{P}^7$, $p \in \mathbb{P}^{15} \setminus S^{10}$.*

PROOF. We shall prove only the second part. If $\delta = \frac{n+8}{3}$, then $n - \delta = \frac{2n-8}{3}$. Suppose $\delta = 2r_X + 1$, so that $n - \delta = \frac{12r_X - 18}{3}$. By Theorem 3.3.9 we deduce that 2^{r_X} should divide $4r_X - 6$, which is not possible.

Suppose now $\delta = 2r_X + 2$, so that $n - \delta = \frac{12r_X - 12}{3} = 4(r_X - 1)$. Since 2^{r_X} has to divide $4(r_X - 1)$, we get $r_X = 1, 2, 3$ and, respectively, $n = 4, 10, 16$ with $\delta = 4, 6$, respectively 8. The conclusion follows from Theorems 3.3.11 and 3.3.12. □

Let us observe that Lazarsfeld and Van de Ven posed the question if for an irreducible smooth projective non-degenerate n-dimensional variety $X \subset \mathbb{P}^N$ with $SX \subsetneq \mathbb{P}^N$ the secant defect is bounded, see [**LV**]. This question was motivated by the fact that for the known examples we have $\delta(X) \leq 8$, the bound being attained for the sixteen dimensional Cartan variety $E_6 \subset \mathbb{P}^{26}$, which is a $LQEL$-variety of type $\delta = 8$. Based on these remarks and on the above results one could naturally formulate the following problem.

Question: Is a $LQEL$-manifold $X \subset \mathbb{P}^N$ with $\delta > 8$ a smooth quadric hypersurface?

3.4. Additivity of higher secant defects, maximal embeddings and Scorza varieties

In this section we study the behaviour of the higher secant defects $\delta_k = \delta_k(X)$, $k \geq 1$, of an irreducible smooth non-degenerate variety of dimension n, $X \subset \mathbb{P}^N$.

Let us recall that, for $z \in S^k X$ general point,

$$\delta_k = \dim(\Sigma_z^k(X)) = \dim(S^{k-1}X) + n + 1 - \dim(S^k X) = s_{k-1} + n + 1 - s_k.$$

So higher the defect, smaller the dimension of $S^k X$. As we shall see below, if X is secant defective, i.e. if $\delta_1 = \delta > 0$, then its k-secant defect has to be at least $k\delta$, so that a secant defective variety has a minimum k-secant defect determined a priori. Of special interest will be secant defective varieties for which each δ_k will attain the minimal value $k\delta$. What is not at all clear at this point is the fact that these varieties can be completely classified in every dimension, at least in characteristic 0, and that they are suitable generalizations of Severi varieties.

First we need a further application of Terracini Lemma to the description of the tangent space to the entry locus of SX at a general point of it. As a minimal generalization we can define the projections onto the i-factor $\phi_i : X_1 \times X_2 \to X_i$ and for $z \in S(X_1, X_2)$, define $\Sigma_z(X_i) = \phi_i(p_1(p_2^{-1}(z)))$, where the morphism p_i's are the map used for the definition of the join. We remark that $\dim(\Sigma_z(X_1)) = \dim(\Sigma_z(X_2)) = \dim(X_1) + \dim(X_2) + 1 - \dim(S(X_1, X_2))$. With this notation we get the following result.

3.4.1. PROPOSITION. *Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties and assume* char(K)=*0. Suppose $S(X,Y) \supsetneq X$ and $S(X,Y) \supsetneq Y$ to avoid trivialities. If $z \in S(X,Y)$ is a general point, if $x \in \Sigma_z(X)$ is a general point and if $< z, x > \cap Y = y \in \Sigma_z(Y)$, then y is a smooth point of $\Sigma_z(Y)$,*

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(Y) >= T_xX \cap < x, T_yY >,$$

$$T_y\Sigma_z(Y) = T_yY \cap < y, T_x\Sigma_z(X) >= T_yY \cap < y, T_xX >$$

and

$$T_xX \cap T_yY = T_x\Sigma_z(X) \cap T_y\Sigma_z(Y).$$

In particular for $z \in SX$ general point, X not linear, and for $x \in \Sigma_z(X)$ general point, we have that, if $< x, z > \cap X = y \in \Sigma_z(X)$, then y is a smooth point of $\Sigma_z(X)$,

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(X) >= T_xX \cap < x, T_yY >$$

and

$$T_xX \cap T_yX = T_x\Sigma_z(X) \cap T_y\Sigma_z(X).$$

PROOF. Exercise 3.5.1. □

Let us start with a general property of varieties defined over a field of characteristic 0.

3.4.2. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety. Suppose* char(K)=*0. Let $k \geq 1$ be such that $S^k X \subsetneq \mathbb{P}^N$, let $x, y \in X$ and $u \in S^{k-1}X$ be general points. Then*

$$T_xX \cap T_yX \cap T_uS^{k-1}X = \emptyset.$$

PROOF. Let $z \in S^kX$ be a general point and let $S^kX = S(X, S^{k-1}X)$. Then by Corollary 1.3.6 the linear space $T_zS^k \neq \mathbb{P}^N$ is tangent along $\Sigma_z^k(X)$ so that it contains $T(\Sigma_z^k(X), X)$. Since X is non-degenerate, $S(\Sigma_z^k(X), X)$ is not contained in T_zS^kX. By Theorem 2.2.1 we get $\dim(S(\Sigma_z^k(X), X)) = \dim(\Sigma_z^k(X)) + \dim(X) + 1$ so that, for $x, y \in X$ general points,

$$T_xX \cap T_y\Sigma_z^k(X) = \emptyset. \tag{3.4.1}$$

By Proposition 3.4.1, if $u \in S^{k-1}X$ is general, then

$$T_y\Sigma_z^k(X) = T_yX \cap < y, T_uS^{k-1}X > . \tag{3.4.2}$$

By equations (3.4.1) and (3.4.2), we finally obtain

$$T_xX \cap T_yX \cap T_uS^{k-1}X \subseteq T_xX \cap T_yX \cap < y, T_uS^{k-1}X >= T_xX \cap T_y\Sigma_z^k(X) = \emptyset.$$

□

By combining Terracini Lemma with the above proposition, we immediately obtain a proof in characteristic zero, the case we treat in the whole chapter, of the next theorem, which is true for arbitrary fields. For a proof valid in arbitrary characteristic one can consult [**Z2**], pg. 109.

3.4.3. THEOREM. (**Additivity of higher secant defects**, Zak) *Let $X \subset \mathbb{P}^N$ be an irreducible smooth non-degenerate projective variety. Let $k \in \mathbb{N}$, $1 \leq k \leq k_0$. Then*

$$\delta_k \geq \delta_{k-1} + \delta \geq k\delta.$$

PROOF. Fix k, $2 \leq k \leq k_0$, the result being trivial for $k = 1$. By definition $S^{k-1}X \subsetneq \mathbb{P}^N$, so that if $x, y \in X$ and $u \in S^{k-2}X$ are general points, by Proposition 3.4.2, we get $T_xX \cap T_yX \cap T_uS^{k-2}X = \emptyset$.
Let

$$L_1 = T_xX \cap T_uS^{k-2}X \text{ and } L_2 = T_xX \cap T_yX.$$

By Terracini Lemma, $\dim(L_1) = \delta_{k-1} - 1$ and $\dim(L_2) = \delta - 1$ since $x, y \in X$ and $u \in S^{k-2}X$ are general points. Let $S^kX = S(X, S^{k-1}X)$ and set

$$L = T_xX \cap < T_yX, T_uS^{k-2}X > .$$

Once again by Terracini Lemma,

$$\dim(L) = \delta_k - 1.$$

Since $L_i \subseteq L$ and $L_1 \cap L_2 = T_xX \cap T_yX \cap T_uS^{k-2}X = \emptyset$, then

$$\delta_k - 1 = \dim(L) \geq \dim(< L_1, L_2 >) = \delta_{k-1} - 1 + \delta - 1 - (-1) = \delta_{k-1} + \delta - 1,$$

yielding the conclusions. □

We deduce some interesting corollaries of the above result. For a real number $r \in \mathbb{R}$, $[r]$ denotes the largest integer not exceeding r.

3.4.4. COROLLARY. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate variety of dimension n. Suppose $\delta > 0$. Then $k_0 \leq [\frac{n}{\delta}]$, i.e.*

$$S^{[\frac{n}{\delta}]}X = \mathbb{P}^N.$$

PROOF. Recall that $\delta_{k_0} \leq n$ by its definition so that

$$n \geq \delta_{k_0} \geq k_0\delta,$$

i.e.

$$\frac{n}{\delta} \geq k_0.$$

□

The second application is a different proof of Hartshorne conjecture on linear normality, cfr. theorem 3.1.5.

3.4.5. COROLLARY. (**Zak Theorem on Linear Normality**) *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n. If $N < \frac{3}{2}n+2$, then $SX = \mathbb{P}^N$. Or equivalently if $SX \subsetneq \mathbb{P}^N$, then $\dim(SX) \geq \frac{3}{2}n+1$ and hence $N \geq \frac{3}{2}n+2$.*

PROOF. Let us prove the last part. If $\delta > \frac{n}{2}$, then $1 = [\frac{n}{\delta}] \geq k_0 \geq 1$ yields $SX = \mathbb{P}^N$. So $\delta \leq \frac{n}{2}$ as soon as $SX \subsetneq \mathbb{P}^N$. This yields

$$N \geq \dim(SX) + 1 = 2n + 2 - \delta \geq \frac{3n}{2} + 2.$$

□

We begin a systematic study of smooth secant defective varieties in characteristic zero and try to determine the restrictions in terms of the embedding. We saw that if $\delta = 0$ and $N \geq 2n+1$, there always exist smooth non-degenerate varieties $X \subset \mathbb{P}^N$ of dimension n and with $\delta = 0$. For example one takes as X a smooth complete intersection of $N - n$ general hypersurfaces. Moreover, by Corollary 1.4.4, for such varieties, if $k < k_0(X)$, then $s_k(X) = (k+1)n + k$, $\delta_k(X) = 0$ and $s_{k_0} = N$, so that N and k_0 are not determined or at least bounded by a function of n and δ and both can grow arbitrarily.

On the other hand, Corollary 3.4.4 and Theorem 3.4.3 say that for non-degenerate varieties of fixed dimension n and with fixed $\delta > 0$, k_0 and N are bounded from above by a function depending on n and δ. Indeed, $k_0 \leq [\frac{n}{\delta}]$, so that, by Corollary 1.3.6 part 4 and by Theorem 3.4.3,

$$N = s_{k_0} = (k_0+1)(n+1) - 1 - \sum_{i=1}^{k_0} \delta_i \leq (k_0+1)(n+1) - 1 - \delta \sum_{i=1}^{k_0} i, \tag{3.4.3}$$

is bounded by a function depending only on n, δ and k_0.

So a secant defective smooth non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension n can be isomorphically projected in $\mathbb{P}^M$, $M \leq 2n$, but due to the secant deficiency it cannot be the isomorphic projection of a variety living in a projective space of arbitrary large dimension. The result of theorem 3.4.3 and the definition of S^kX and of k_0 say that linearly normal secant defective varieties with higher $N+1 = h^0(\mathcal{O}_X(1))$ are those for which δ_k is the minimum possible, i.e. varieties such that $\delta_k = k\delta$.

On the base of the previous discussion let us introduce some definitions and collect the above argument in a more systematic statement. We can think linear projection as a partial order in the set of the embeddings of a variety X in projective space. Of particular interest will be maximal and minimal elements with respect to this partial order.

3.4.6. DEFINITION. (**Functions** $M(n,\delta)$**,** $m(n,\delta)$ **and** $f(n,\delta,k)$) All varieties $X \subset \mathbb{P}^N$ are supposed to be smooth, non-degenerate and projective.

Let us define, it it exists (otherwise we put it equal to ∞), for $n \geq 1$ and for $\delta \geq 0$,

$$M(n,\delta) := \max\{N \,:\, \exists X \subset \mathbb{P}^N : \dim(X) = n,\, \delta(X) = \delta\}.$$

In the same way we define

$$m(n,\delta) := \min\{N \,:\, \exists X \subset \mathbb{P}^N : \dim(X) = n,\, \delta(X) = \delta\}.$$

Inspired by (3.4.3) we define, following Zak, for $k \geq 0$, for $n \geq 1$ and for $\delta \geq 0$:

$$f(k,n,\delta) := (k+1)(n+1) - \frac{k(k+1)\delta}{2} - 1.$$

We saw that $M(n, 0) = \infty$. Clearly $m(n, \delta) = 2n + 1 - \delta$. Indeed general complete intersection of dimension n in $\mathbb{P}^{2n+1-\delta}$ are smooth non-degenerate varieties with $SX = \mathbb{P}^{2n+1-\delta}$ so that $\delta(X) = \delta$ and $m(n, \delta) \leq 2n + 1 - \delta$. On the other hand every variety $X \subset \mathbb{P}^N$ with $\delta(X) = \delta$ and of dimension n has $2n + 1 - \delta = \dim(SX) \leq N$, yielding $m(n, \delta) \geq 2n + 1 - \delta$.

The equation (3.4.3) can be read as, if $\delta > 0$, then $N \leq f(k_0, n, \delta)$.

Let us reinterpret Corollary 3.4.4 in terms of these functions and study their first properties.

3.4.7. PROPOSITION. *Let the pairs (n, δ) in the statement be such that the functions $M(n, \delta)$ and $m(n, \delta)$ are defined. Then*

(1) *if $\delta > \frac{n}{2}$, then $M(n, \delta) = m(n, \delta) = 2n + 1 - \delta$;*
(2) $M(n, \delta - 1) \geq M(n, \delta) + 1$;
(3) $M(n - 1, \delta - 1) \geq M(n, \delta) - 1$.

PROOF. By Corollary 3.4.4, $\delta > \frac{n}{2}$ gives $k_0 = 1$ and hence $SX = \mathbb{P}^N$ so that $N = \dim(SX) = 2n + 1 - \delta = m(n, \delta)$ is determined by n and δ, yielding part 1).

Suppose given $X \subset \mathbb{P}^N$, $\dim(X) = n$ and $\delta(X) = \delta \geq 1$. Let $p \in \mathbb{P}^{N+1} \setminus \mathbb{P}^N$, set $Y = S(p, X)$ and take $X' = Y \cap H \subset \mathbb{P}^{N+1}$, $H \subset \mathbb{P}^{M(n,N+1}$ a general hypersurface of degree $d > 1$. The variety X' is smooth, non-degenerate, irreducible and of dimension n with $\delta(X') = \delta(X) - 1$ and $SX' = S(p, SX)$. Indeed, $SX' \subseteq S(p, SX)$ so that it will be sufficient to prove the first part of the claim. Let $\pi_p : X' \to X$ be the projection from p onto $\mathbb{P}^N$. By Terracini lemma, if $p'1, p'2 \in X'$ are general points, then

$$\begin{aligned}
\delta(X') + 1 &= \dim(< p, T_{p'_1}X > \cap < p, T_{p'_2}X >) \\
&= \dim(< p, T_{p'_1}X > \cap < p, T_{p'_2}X > \cap \mathbb{P}^N) + 1 \\
&= \dim((< p, T_{p'_1}X > \cap \mathbb{P}^N) \cap (< p, T_{p'_2}X > \cap \mathbb{P}^N)) + 1 \\
&= \dim(T_{\pi_p(p'_1)}X \cap T_{\pi_p(p'_2)}X) + 1 = \delta.
\end{aligned}$$

Suppose given $X \subset \mathbb{P}^N$, $\dim(X) = n$ and $\delta(X) = \delta \geq 1$. Let $X' = X \cap H \subset H = \mathbb{P}^{N-1}$ be a general hyperplane section. By Terracini Lemma and by the generality of H, if one takes $p_1, p_2 \in X' = X \cap H$ general, then $\delta(X') - 1 = \dim(T_{p_1}X' \cap T_{p_2}X') = \dim(T_{p_1}X \cap T_{p_2}X \cap H) = \delta - 2$ so that $\delta(X') = \delta - 1$ and $\dim(SX') = 2(n - 1) + 1 - \delta(X') = 2n - \delta = \dim(SX) - 1$. Since $SX' \subseteq SX \cap H$ we also deduce $SX' = SX \cap H$. $\square$

3.4.8. DEFINITION. (**Extremal variety**) A smooth irreducible non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension n is said to be *an extremal variety* if $\delta(X) = \delta > 0$ and if $N = M(n, \delta)$.

In other words an extremal variety is a smooth secant defective variety, which is a maximal element in the partial order defined by isomorphic projection.

We are now in position to refine (3.4.3) in the sharpest form.

3.4.9. THEOREM. (**Maximal embedding of secant defective varieties**, Zak, [**Z2**]) *Suppose $\delta > 0$. Then*

$$M(n, \delta) \leq f([\frac{n}{\delta}], n, \delta).$$

In particular a smooth non-degenerate irreducible projective variety $X \subset \mathbb{P}^N$ with $N = f([\frac{n}{\delta}], n, \delta)$ is linearly normal.

PROOF. By equation (3.4.3) we know that for a given variety $X \subset \mathbb{P}^N$ of dimension n and with $\delta(X) = \delta$, we have $N \leq f(k_0, n, \delta)$. On the other hand by Corollary 3.4.4 we know that $k_0 \leq [\frac{n}{\delta}]$. Fixing n and δ, $y = f(k, n, \delta)$ is a parabola in the plane (k, y), whose vertex has coordinates $(\frac{2n-\delta+2}{2\delta}, \frac{(2n+\delta+2)^2}{8\delta})$; in particular it is an increasing function on the interval $0 \leq k \leq \frac{2n-\delta+2}{2\delta}$. So if $k_0 = [\frac{n}{\delta}]$, there is nothing to prove. If $k_0 < [\frac{n}{\delta}]$, then

$$k_0 \leq [\frac{n}{\delta}] - 1 \leq \frac{n}{\delta} - 1 < \frac{2n-\delta+2}{2\delta},$$

so that

$$N \leq f(k_0, n, \delta) \leq f(\frac{n}{\delta} - 1) = f(\frac{n}{\delta}, n, \delta) - 1 < f([\frac{n}{\delta}], n, \delta),$$

where the last inequality follows from the fact that $f(m, n, \delta) \in \mathbb{N}$ for every $m \in \mathbb{N}$. This finishes the proof. $\square$

Theorem 3.4.9 says that secant defective varieties are allowed to live in a projective space of bounded dimension, the bound being expressed by the value $f([\frac{n}{\delta}], n, \delta)$.

Let us reinterpret some results in the light of the new definitions and of the first properties of the functions $M(n, \delta)$ and $f(k, n, \delta)$, following Zak [**Z2**].

3.4.10. REMARK. (**Case** $\delta > \frac{n}{2}$) If $\delta > \frac{n}{2}$, then $[\frac{n}{\delta}] = 1$, $SX = \mathbb{P}^N$ so that

$$m(n, \delta) = M(n, \delta) = 2n + 1 - \delta = f(1, n, \delta).$$

3.4.11. REMARK. (**Case** $\delta = \frac{n}{2}$) In this case $n \equiv 0$ (mod. 2) and by Theorem 3.4.9

$$N \leq f(2, n, \frac{n}{2}) = \frac{3n}{2} + 2.$$

There are two possibilities:

(1) $SX = \mathbb{P}^N$, $m(n, \frac{n}{2}) = \frac{3n}{2} + 1 = \dim(SX) = N$;
(2) $SX \subsetneq \mathbb{P}^N$, $N = \dim(SX) + 1 = M(n, \frac{n}{2}) = f(2, n, \frac{n}{2}) = \frac{3n}{2} + 2$.

Varieties in case 2) are clearly Severi varieties so the remark furnishes a new proof that Severi varieties are linearly normal.

In the next remark we connect these results with the classical work of Gaetano Scorza.

3.4.12. REMARK. Suppose

$$n \equiv 1 (\text{mod. } 2) \text{ and } SX \subsetneq \mathbb{P}^N.$$

By Remark 3.4.10

$$\delta < \frac{n-1}{2} \text{ and } s = \dim(SX) = 2n + 1 - \delta \geq \frac{3n+3}{2}.$$

We now discuss the extremal case $\delta = \frac{n-1}{2}$ in the above hypothesis.
Suppose $n = 3$ so that $\delta = 1$. By Theorem 3.4.9

$$N \leq f(3, 3, 1) = 9 = s + 2.$$

By the main classification Theorem of [**S1**], see also Theorem 2.3.8, there is only one such 3-fold, $X = \nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$.

If $n > 3$, then

$$N \leq f(\frac{2n}{n-1}, n, \frac{n-1}{2}) = \frac{3n+7}{2} = s + 2.$$

Therefore, for $n \equiv 1$ (mod. 2), $\delta = \frac{n-1}{2}$ there are only the following cases:

(1) $SX = \mathbb{P}^N$, $N = s = m(n, \frac{n-1}{2}) = \frac{3n+3}{2}$;
(2) $N = s + 1 = \frac{3n+5}{2}$;
(3) $N = s + 2 = \frac{3n+7}{2} (= M(n, \frac{n-1}{2})$ if $n > 3)$;
(4) $n = 3$, $N = s + 3 = M(3, 1) = 9$.

All these cases really occur. Examples of case 2) are hyperplane sections of the Severi varieties. For an example as in case 3) one can take $\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$ Segre embedded, while we saw above an example as in case 4).

3.4.1. Scorza varieties. In the previous section we defined extremal varieties and discussed various cases. In particular we saw that if $\delta > \frac{n}{2}$, then $SX = \mathbb{P}^N$ and $X \subset \mathbb{P}^N$ is an extremal variety such that $M(n,\delta) = f([\frac{n}{\delta}], n, \delta) = f(1,n,\delta) = 2n+1-\delta = N = m(n,\delta)$.

By definition of the function $f(k,n,\delta)$ and due to Theorem 3.4.9, an extremal variety $X \subset \mathbb{P}^{M(n,\delta)}$, $\delta > 0$, satisfies $M(n,\delta) = f([\frac{n}{\delta}], n, \delta)$ if and only if $k_0 = [\frac{n}{\delta}]$ and $\delta_k = k\delta$ for $0 \leq k \leq k_0$.

The case of extremal varieties with $k_0 = 1 = [\frac{n}{\delta}]$, i.e. $M(n,\delta) = m(n,\delta)$ does not present particular restrictions and there are infinite examples. On the base of the examples in dimension 3 and 4, classically studied by Scorza in [**S1**] and [**S4**], Zak introduced the following definition.

3.4.13. DEFINITION. (**Scorza variety**, [**Z2**], pg. 121) Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety of dimension n. Then X is said to be *a Scorza variety* if:

(i) $N > m(n,\delta)$, where $\delta = \delta(X) = 2n+1-\dim(SX)$;
(ii) $N = M(n,\delta) < \infty$, i.e. $\delta > 0$ and X is an extremal variety;
(iii) $M(n,\delta) = f([\frac{n}{\delta}], n, \delta)$, where $f(k,n,\delta) = (k+1)(n+1) - \frac{k(k+1)}{2}\delta - 1$.

From now on we will suppose char(K)=0. Under these hypothesis, a smooth non-degenerate irreducible projective variety $X \subset \mathbb{P}^N$ of dimension n is a Scorza variety if and only if $\delta \leq \frac{n}{2}$, $k_0 = [\frac{n}{\delta}]$ and $\delta_k = k\delta$ for $0 \leq k \leq [\frac{n}{\delta}]$.

As we saw in Remark 3.4.11, Severi varieties are instances of Scorza varieties with $\delta = \frac{n}{2}$. So the class of Scorza varieties includes the four Severi varieties. The extraordinary and remarkable classification result due to Zak, which we will try to illustrate in this section, states that there are only few other examples. These examples form infinite series, whose first members are the three classical Severi varieties of dimensions 2,4 and 8. The classification result is the following.

3.4.14. THEOREM. (**Classification of Scorza varieties**, [**Z2**, Chapter V]) *Let $X \subset \mathbb{P}^N$ be a Scorza variety of dimension n. Then X is projectively equivalent to one of the following:*

(1) $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ ($\delta = 1$);
(2) $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$, $a+b=n$, $|a-b| \leq 1$ ($\delta = 2$);
(3) $\mathbb{G}(1, \frac{n}{2}+1) \subset \mathbb{P}^{\frac{n(n+6)}{2}}$, $n \equiv 0$ (mod. 2) ($\delta = 4$);
(4) *the E_6-variety $X \subset \mathbb{P}^{26}$ of dimension 16* ($\delta = 8$).

There is a uniform description of Scorza varieties with $n \equiv 0$ (mod. δ), "*the most interesting case*", according to Zak, [**Z2**, pg. 152]. These varieties have a *determinantal* description as locus of rank 1 matrices in the projective space of suitable Jordan algebras of Hermitian matrices of order $\frac{n}{\delta}+1$ over composition algebras, generalizing the one furnished for Severi varieties, see *loc. cit.* and [**Ch2**]; hence they are realized as suitable quadratic Veronese embedding of *generalized* projective spaces. From this point of view the classification of these Scorza varieties is completely parallel to the classification of the above algebras obtained algebraically by Albert, see for example [**BK**] or [**Ja**].

The first techinical and important towards classification is the following.

3.4.15. THEOREM. (**Entry loci of Scorza varieties**, [**Z2**], pg. 122) *Let $X \subset \mathbb{P}^N$ be a Scorza variety of dimension n, with $\delta = \delta(X)$ and $N = f([\frac{n}{\delta}], n, \delta)$. Let $z \in S^kX$, $2 \leq k \leq k_0 - 1 = [\frac{n}{\delta}] - 1$. Then*

$$\Sigma^k_z(X) \subset C_{T_zS^kX}(S^kX) = \mathbb{P}^{f(k,k\delta,\delta)} = S^k\Sigma^k_z(X)$$

is a Scorza variety such that

$$\dim(\Sigma_z^k(X)) = k\delta,\ \ k_0(\Sigma_z^k(X)) = k,\ \ \delta_i(\Sigma_z^k(X)) = i\delta,\ \ 0 \leq i \leq k.$$

If $\frac{n}{\delta} > k_0 \geq 2$*, then* $\Sigma_z^{k_0}(X)$ *is a Scorza variety of dimension* $k_0\delta < n$ *with* $\delta(\Sigma_z^{k_0}(X)) = \delta$ *and* $< \Sigma_z^{k_0}(X) >= S^{k_0}\Sigma_z^{k_0}(X) = C_{T_z S^{k_0}X}(S^{k_0}X) = \mathbb{P}^{f(k_0, k_0\delta, \delta)}$.

For $z \in SX$ *general point,* $\Sigma_z(X) \subset \mathbb{P}^{\delta+1}$ *is a non-singular quadric hypersurface of dimension* δ.

The following corollary is a fundamental step for the classification of Scorza varieties since it drastically reduces the cases to be considered.

3.4.16. COROLLARY. (**Singular defect of Scorza varieties**, [**Z2**], pg. 125) *Let* $X \subset \mathbb{P}^{f([\frac{n}{\delta}], n, \delta)}$ *be a Scorza variety. Then for* $z \in S^2X$ *general point,* $\Sigma_z^2(X)$ *is a Severi variety so that* $\delta = 1$*, 2, 4, or 8.*

In order to classify Scorza varieties we have to consider only the 4 cases: $\delta = 1$, 2, 4 or 8. The case $\delta = 1$ was proved in Theorem 2.3.8.

3.4.17. THEOREM. (**Classification of Scorza varieties with** $\delta = 1$) *Let* $X \subset \mathbb{P}^N$ *be a smooth non-degenerate irreducible projective variety of dimension* n *such that* $\dim(SX) \leq 2n$*. Then* $N \leq \frac{n(n+3)}{2}$ *and equality holds if and only if* X *is projectively equivalent to* $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$.

In particular $M(n, 1) = \frac{n(n+3)}{2}$ *and* $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^N$ *is the only Scorza variety of dimension* n *with* $\delta = 1$.

The follwoing Corollary of Theorems 3.3.16 and 3.3.4 will imply the calssification of Scorza varieties with $\delta = 2$.

3.4.18. COROLLARY. *Let* $X \subset \mathbb{P}^N$ *be a* QEL*-variety of type* $\delta = 2$*. Then one of the following holds:*

i) n *is even,* $\mathrm{Pic}(X) = \mathbb{Z} < \mathcal{O}(1) >$, $N \leq n + \frac{n(n+2)}{8}$, X *is a Fano variety of index* $\frac{n}{2} + 1$ *and* $Y_x \subset \mathbb{P}^{n-1}$ *is a smooth irreducible variety with one apparent double point of dimension* $\frac{n}{2} - 1$*. Furthermore, if* $N = n + \frac{n(n+2)}{8}$*, then* $Y_x \subset \mathbb{P}^{n-1}$ *is a rational normal scroll of dimension* $\frac{n}{2} - 1$*, which is a variety with one apparent double point.*

ii) $N \leq (n-l)l + n$, $1 \leq l \leq \frac{n}{2}$*, and* X *is projectively equivalent to (an isomorphic projection of) the Segre variety* $\mathbb{P}^l \times \subset \mathbb{P}^{n-l} \subset \mathbb{P}^{(n-l)l+n}$.

PROOF. By Theorem 3.3.16 we have that either we are in case ii) or n is even and $X \subset \mathbb{P}^N$ is a Fano variety of index $\frac{n}{2} + 1$ with $\mathrm{Pic}(X) = \mathbb{Z} < \mathcal{O}(1) >$ such that $Y_x \subset \mathbb{P}^{n-1}$ is a smooth equidimensional scheme such that through a general point of $\mathbb{P}^{n-1}$ tehre passes a unique secant line to Y_x.

If $Y_x \subset \mathbb{P}^{n-1}$ were reducible, then it should consists of at least 3 irreducible components by [**Hw2**, Proposition2] and through a general point of $\mathbb{P}^{n-1}$ there would pass more than one secant line to Y_x. Indeed, by projecting $Y_x \subset \mathbb{P}^{n-1}$ from a general point, we would obtain at least 3 singular points in the projection of Y_x, corresponding to the secant lines passing through the center of projection.

Therefore Y_x is irreducible and it is a smooth irreducible non-degenerate variety with one apparent double point. Thus

$$h^0(Y_x(2)) \leq \frac{\frac{n}{2}(\frac{n}{2} + 1)}{2} = \frac{n(n+2)}{8}$$

by [**Z3**, Corollary 5.4]. Since $N - n = \dim(|II_{x,X}|) + 1 \leq h^0(Y_x(2))$ by Theorem 3.3.4, we get $N \leq n + \frac{n(n+2)}{8}$.

Assume $N = n + \frac{n(n+2)}{8}$. Then $Y_x \subset \mathbb{P}^{n-1}$ is a smooth irreducible non-degenerate variety with one apparent double point defined by $\frac{n(n+2)}{8}$ quadratic equations, so that it is a rational normal scroll by [**Z3**, Corollary 5.8]. □

We now proceed with the classification of Scorza varities with $\delta = 2$. Our proof is different from the one proposed by Zak in [**Z2**, pg. 130] and in our opinion more transparent and direct.

3.4.19. THEOREM. (**Classification of Scorza varieties with** $\delta = 2$) *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $n \geq 4$ and such that* $\dim(SX) < 2n$. *Then*

(1) *if $n = 2m$, then $N \leq m(m+2) = (m+1)^2 - 1$;*
(2) *if $n = 2m+1$, then $N \leq (m+1)(m+2) - 1$.*

Moreover, the inequalities turn into equalities if and only if X is projectively equivalent to $\mathbb{P}^m \times \mathbb{P}^m \subset \mathbb{P}^{m(m+2)}$ or to $\mathbb{P}^m \times \mathbb{P}^{m+1} \subset \mathbb{P}^{m^2+3m+1}$ Segre embedded.

In particular, if m is as above, $M(n,2) = \frac{n(n+4)-n-2m}{4}$ and the above ones are the only Scorza varieties with $\delta = 2$.

PROOF. We have $\delta \geq 2$, so that if $n = 2m$,

$$N \leq f(m, 2m, 2) = m(m+2) = (m+1)^2 - 1,$$

while if $n = 2m+1$, then

$$N \leq f(m, 2m+1, 2) = (m+1)(m+2) - 1$$

and the first part follows.

Suppose equality holds in the above inequalities. Then X is a Scorza variety of dimension $n = 2m$, respectively $n = 2m+1$, with $\delta = 2$. Since for every $n \geq 2$ we have $n + \frac{n(n+2)}{8} \leq \min\{(\frac{n}{2})^2 - 1, \frac{n+1}{2}\frac{n+3}{2}\}$, we can conlude via Corollary 3.4.18. □

Next we analyze the case $\delta = 4$. Firstly we present some preliminary results.

3.4.20. PROPOSITION. *Let $X \subset \mathbb{P}^N$ be an LQEL-manifold of type $\delta = 4$. Then $N \leq \frac{n^2+6n}{8}$ and equality holds if and only if X is projectively equivalent to the Plücker embedding $\mathbb{G}(1, \frac{n}{2}+1) \subset \mathbb{P}^{\frac{n^2+6n}{8}}$.*

PROOF. By Theorem 3.3.9, n is even. If $n = 4$, then the result is trivially true. So we can suppose $n \geq 6$. Let $x \in X$ be a general point. The variety $Y_x \subset \mathbb{P}((\mathbf{T}_x X)^*) = \mathbb{P}^{n-1}$ is smooth, irreducible non-degenerate of dimension $\frac{n+4}{2} - 2 = \frac{n}{2}$, see Theorem 3.3.4. Since $\operatorname{codim}(Y_x) = \frac{n-4}{2} + 1$, we have

$$h^0(\mathcal{I}_{Y_x}(2)) \leq \frac{(\frac{n-4}{2}+1)(\frac{n-4}{2}+2)}{2} = \frac{(n-4)^2 + 6(n-4) + 8}{8}, \tag{3.4.4}$$

by [**Z3**, Corollary 5.4]. By Theorem 3.3.4, we get $N - n = \dim(|II_{x,X}|) + 1 \leq h^0(Y_x(2))$, which combined with (3.4.4) yields $N \leq \frac{n^2+6n}{8}$.

Suppose $N \leq \frac{n^2+6n}{8}$. Thus by [**Z3**, Corollary 5.8], the variety $Y_x \subset \mathbb{P}^{n-1}$ is a variety of minimal degree of dimension $\frac{n}{2}$, so that it is projectively equivalent to the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{\frac{n}{2}-1} \subset \mathbb{P}^{n-1}$.

Therefore $\pi_x(X) = \widetilde{\pi}_x(E) = \mathbb{G}(1, \frac{n}{2} - 1) \subset \mathbb{P}^{\frac{(n-2)(n+4)}{8}}$, Plücker embedded. By the explicit description of $\widetilde{\pi}_x$ and by the fact that $X \subset \mathbb{P}^{\frac{n^2+6n}{8}}$ is an $LQEL$-manifold, it follows that the closure of a general fiber of π_x is a smooth quadric hypersurface of dimension 4 passing through x. Let $\mathcal{Q}_x$ be the irreducible component of dimension $n-4$ of the Hilbert scheme of quadric hypersurfaces of dimension 4 contained in X and passing through x, to which a general fiber of π_x belongs. Let

$$\widetilde{\mathcal{Q}}_x = \{Q \in \mathcal{Q}_x \text{ smooth and such that } Q \not\subseteq \mathrm{T}_x X \cap X\} \subseteq \mathcal{Q}_x.$$

Let $\mathfrak{C}_x \subseteq \mathrm{T}_x X \cap X$ be the cone with vertex x described by the lines parametrized by $Y_x \subset \mathbb{P}^{n-1}$.

We can first blow-up x and then the strict transform of $\mathfrak{C}_x$ on $\mathrm{Bl}_x X$. Let $\alpha_x : Z_x \to X$ be the birational morphism corresponding to these two blow-up's, let E_1 be the strict transform of the first exceptional divisor

E and let E_2 be the exceptional divisor of the second blow-up. The partial resolution of π_x on Z_x is a rational map
$$\psi_x : Z_x \dashrightarrow \mathbb{G}\Big(1, \frac{n}{2} - 1\Big) \subset \mathbb{P}^{\frac{(n-2)(n+4)}{8}},$$
defined along E_1, see Example 3.3.6, which is equivariant for the corresponding $SL(2)$-actions on E_1 and on $\mathbb{G}(1, \frac{n}{2} - 1)$. Every quadric hypersurface in $\widetilde{\mathcal{Q}}_x$ is a closed fiber of ψ_x, so that we can identify $\widetilde{\mathcal{Q}}_x$ with an open subset $U \subseteq \mathbb{G}(1, \frac{n}{2} - 1)$. Moreover
$$W = \psi_x^{-1}(U) \stackrel{\psi_{x|W}}{\longrightarrow} U$$
is a Q^4-bundle over U, birational to an open subset of X containing x.

On $G_n = \mathbb{G}(1, \frac{n}{2}+1) \subset \mathbb{P}^{\frac{n(n+6)}{8}}$, fixing any point w on it, $(T_{G_n} \cap G_n)_{\mathrm{red}}$ is projectively equivalent to the cone over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{\frac{n}{2}-1} \subset \mathbb{P}^{n-1}$. The corresponding map $\psi_w : Z_w \dashrightarrow \mathbb{G}(1, \frac{n}{2} - 1) \subset \mathbb{P}^{\frac{(n-2)(n+4)}{8}}$ is a morphism which is equivariant for the action of $SL(2)$.

Thus we can conclude that there exists a birational map
$$\phi_{x,w} : X \dashrightarrow \mathbb{G}\Big(1, \frac{n}{2} + 1\Big) \subset \mathbb{P}^{\frac{n(n+6)}{8}},$$
sending a general quadric in $\widetilde{\mathcal{Q}}_x$ into a general quadric through w and inducing an isomorphism between these quadric hypersurfaces of dimension 4. Thus $\phi_{x,w}$ is an isomorphism in codimension 1 such that $\phi_{x,w}^*(\mathcal{O}(1)) = \mathcal{O}_X(1)$ (recall that X is a Fano variety with $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$, where $\mathcal{O}_X(1)$ refers to the embedding in $\mathbb{P}^{\frac{n(n+6)}{8}}$). Therefore $\phi_{x,w}$ is an isomorphism which can be realized as the restriction of a projective transformation of $\mathbb{P}^{\frac{n(n+6)}{8}}$ since X (and $\mathbb{G}(1, \frac{n}{2} + 1)$) is linearly normal in $\mathbb{P}^{\frac{n(n+6)}{8}}$ by the first part of the proof of the Proposition. □

3.4.21. THEOREM. (**Classification of Scorza varieties with** $\delta = 4$) *Let $X \subset \mathbb{P}^N$ be a Scorza variety of dimension $n \geq 4$ and with $\delta = 4$. Then $X = \mathbb{G}(1, \frac{n}{2} + 1) \subset \mathbb{P}^{\frac{n^2+6n}{8}}$.*

PROOF. If $X \subset \mathbb{P}^N$ is a Scorza variety with $\delta = 4$, then $n \equiv 0$ mod. 2 by Theorem 3.3.9. Thus by explicit computation we get $N = f([\frac{n}{4}]) = \frac{n^2+6n}{8}$ and the conclusion follows from Proposition 3.4.20. □

The following result is useful for the classification of Severi and Scorza varieties, which will be defined below.

3.4.22. PROPOSITION. *Let $X \subset \mathbb{P}^{26}$ be a $LQEL$-manifold of dimension 16 and type $\delta = 8$. Then $X \subset \mathbb{P}^{26}$ is projectively equivalent to the Cartan variety $E_6 \subset \mathbb{P}^{26}$.*

PROOF. Let $x \in X$ be a general point. The variety $Y_x \subset \mathbb{P}^{15}$ is a QEL-manifold of dimension 10 and type $\delta = 6$, so that it is projectively equivalent to $S^{10} \subset \mathbb{P}^{15}$ by Corollary 3.3.11. Thus $|II_{x,X}| = |H^0(\mathcal{I}_{Y_x}(2)|$. Then $\pi_x(X) = \widetilde{\pi}_x(E) = Q^8 \subset \mathbb{P}^9$, with Q^8 a smooth quadric hypersurface, see [**ESB**, 4.4]. Moreover, the closure of every fiber of $\widetilde{\pi}_x$ is a $\mathbb{P}^7$ and after blowing-up Y_x, the map $\widetilde{\pi}_x$ becomes an equivariant morphism on $\mathrm{Bl}_{Y_x}\mathbb{P}^{15}$. A $\mathbb{P}^7_r \subset \mathbb{P}^{15}$, closure of the fiber $\widetilde{\pi}_x^{-1}(\widetilde{\pi}_x(r))$, $r \in \mathbb{P}^{15} \setminus Y_x$, cuts Y_x in a six dimensional quadric Q^6_r. The variety S^{10} parametrizes one of the family of 4-planes in Q^8. For any point $t = \widetilde{\pi}_x(r) \in Q^8$, put $S_t = \{L \in S^{10} : t \in L\}$. Then S_t is the spinor variety associated to the quadric hypersurface Q^6_r and $S_t \simeq Q^6_r$ by triality; see also [**ESB**, 4.4].

Repeating the construction in Proposition 3.4.20 and arguing exactly in the same way, we can construct a birational map $\phi : X \dashrightarrow E_6 \subset \mathbb{P}^{26}$, which is an isomorphism in codimension 1 and such that $\phi^*(\mathcal{O}_{E_6}(1)) = \mathcal{O}_X(1)$, since once again X and E_6 are Fano varieties with $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathcal{O}_X(1)\rangle$, where $\mathcal{O}_X(1)$ refers to the embedding in $\mathbb{P}^{26}$. Thus ϕ is an isomorphism which can be identified to a projective transformation of $\mathbb{P}^{26}$ since $X \subset \mathbb{P}^{26}$ (and $E_6 \subset \mathbb{P}^{26}$) are linearly normal by Theorem 3.1.5. □

To conclude our tour through the classification of Scorza vareities, we sketch the proof of the following result.

3.4.23. THEOREM. (**Classification of Scorza varieties with** $\delta = 8$) *Let* $X \subset \mathbb{P}^{M(n,8)}$ *be a Scorza variety of dimension* n *and with* $\delta = 8$. *Then* $X = E_6 \subset \mathbb{P}^{16}$.

PROOF. If $X \subset \mathbb{P}^{M(n,8)}$ is a Scorza variety with $\delta = 8$, then $n \equiv 0$ mod. 8 by Theorem 3.3.9. If $k_0 \leq 3$, then $\Sigma_z^3(X)$ is Scorza variety of dimension 24 and secant defect $\delta = 8$ for $z \in S^3X$ general. In the proof of Theorem 3.4.15, Zak shows that in $\Sigma_u^2(X) = E_6 \subset \mathbb{P}^{16}$, $u \in S^2X$ general, there would be a 8 dimensional liner space $L \subset E_6$. This can be easily proved by studying carefully the k-tangential projections of X. Thus for $x \in L$ we would deduce $L \subseteq T_xE_6 \cap E_6$. Since the last intersection is a cone over $S^{10} \subset \mathbb{P}^{15}$ of vertex the point x. We would get a 7 dimensional linear space $M \subset S^{10}$. Reasoning as above we would construct a $\mathbb{P}^6$ contained in $T_yS^{10} \cap S^{10}$, which is a cone over over $\mathbb{G}(1,4) \subset \mathbb{P}^9$, which is impossible. If you do not agree, apply the argument once more and deduce the existence of a p^4 in $T_w\mathbb{G}(1,4) \cap \mathbb{G}(1,4)$, which has dimension 4 being a cone over $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. Thus $k_0 = 2$, $n = 16$ and $N = 26$. The conclusion follows from Proposition 3.4.22. □

3.5. Exercises

3.5.1. EXERCISE. Let $X, Y \subset \mathbb{P}^N$ be closed irreducible subvarieties and assume char(K)=0. Suppose $S(X,Y) \supsetneq X$ and $S(X,Y) \supsetneq Y$ to avoid trivialities. If $z \in S(X,Y)$ is a general point, if $x \in \Sigma_z(X)$ is a general point and if $< z, x > \cap Y = y \in \Sigma_z(Y)$, then y is a smooth point of $\Sigma_z(Y)$,

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(Y) >= T_xX \cap < x, T_yY >,$$
$$T_y\Sigma_z(Y) = T_yY \cap < y, T_x\Sigma_z(X) >= T_yY \cap < y, T_xX >$$

and

$$T_xX \cap T_yY = T_x\Sigma_z(X) \cap T_y\Sigma_z(Y).$$

In particular for $z \in SX$ general point, X not linear, and for $x \in \Sigma_z(X)$ general point, we have that, if $< x, z > \cap X = y \in \Sigma_z(X)$, then y is a smooth point of $\Sigma_z(X)$,

$$T_x\Sigma_z(X) = T_xX \cap < x, T_y\Sigma_z(X) >= T_xX \cap < x, T_yY >$$

and

$$T_xX \cap T_yX = T_x\Sigma_z(X) \cap T_y\Sigma_z(X).$$

(Hint: Let us remark that by assumption and by the generality of z and of x, we can suppose that $y \notin T_xX$ and that $x \notin T_yY$.

Take $S(z, \Sigma_z(X)) = S(z, \Sigma_z(Y))$. Then $\dim(S(z, \Sigma_z(X))) = \dim(\Sigma_z(X)) + 1$. If $u \in < z, x > = < z, y >$ is a general point, then

$$T_uS(z, \Sigma_z(X)) = < z, T_x\Sigma_z(X) >= \mathbb{P}^{\dim(S(z,\Sigma_z(X)))}$$

because $z \notin T_xX$. In particular u is a smooth point of $S(z, \Sigma_z(X))$. By Terracini Lemma, we get $T_uS(z, \Sigma_z(X)) \supseteq < z, T_y\Sigma_z(Y) >$, which together with $z \notin T_yY$ yields $\dim(T_y\Sigma_z(Y)) = \dim(\Sigma_z(Y))$ so that $y \in \Sigma_z(Y)$ is a smooth point. Moreover,

$$T_x\Sigma_z(X) \subseteq T_uS(z, \Sigma_z(Y)) = < z, T_y\Sigma_z(Y) >=< x, T_y\Sigma_z(Y) > \subseteq < x, T_yY > .$$

Since $T_x\Sigma_z(X) \subseteq T_xX$, to conclude it is enough to observe that

$$\begin{aligned}\dim(T_xX \cap < x, T_yY >) &= \dim(X) + \dim(Y) + 1 - \dim(< T_xX, T_yX >)\\ &= \dim(\Sigma_z(X)) = \dim(T_x\Sigma_z(X)).\end{aligned}$$

The other claims follows from symmetry between x and y or are straightforward).

CHAPTER 4

Degenerations of projections and applications

4.1. Degeneration of projections

In this section we present some of the ideas introduced in §3 and §4 of [**CMR**], to which we will constantly refer, and which were later developed in §2 of [**CR**]. This suitable extended Theorem 4.1 of [**CMR**]. A further application of these tecniques appears in [**IR2**], see section 4.2 below.

Let $X \subset \mathbb{P}^N$ be an irreducible, non–degenerate projective variety of dimension n. We fix $k \geq 1$ and we assume that $s^{(k)}(X) = (k+1)n + k$.

Let us fix an integer s such that $N - s^{(k)}(X) \leq s \leq N - s^{(k-1)}(X) - 2$, so that $s^{(k-1)}(X) + 1 \leq N - s - 1 \leq s^{(k)}(X) - 1$. Let $L \subset \mathbb{P}^N$ be a general projective subspace of dimension s and let us consider the projection morphism $\pi_L : S^{k-1}(X) \to \mathbb{P}^{N-s-1}$ of X from L. Notice that, under our assumptions on s, one has:

$$\pi_L(S^k(X)) = \mathbb{P}^{N-s-1}, \quad \pi_L(S^{k-1}(X)) \subset \mathbb{P}^{N-s-1}.$$

Let $p_1, ..., p_k \in X$ be general points and let $x \in < p_1, ..., p_k >$ be a general point, so that $x \in S^{k-1}(X)$ is a general point and $T_{S^{k-1}(X),x} = T_{X,p_1,...,p_k}$. We will now study how the projection $\pi_L : S^{k-1}(X) \to \mathbb{P}^{N-s-1}$ *degenerates* when its centre L tends to a general s-dimensional subspace L_0 containing x, i.e. such that $L_0 \cap S^{k-1}(X) = L_0 \cap T_{X,p_1,...,p_k} = \{x\}$. To be more precise we want to describe the *limit* of a certain *double point scheme* related to π_L in such a degeneration.

Let us describe in detail the set up in which we will work. We let T be a general $\mathbb{P}^{s^{(k-1)}(X)+s+1}$ which is tangent to $S^{k-1}(X)$ at x, i.e. T is a general $\mathbb{P}^{s^{(k-1)}(X)+s+1}$ containing $T_{X,p_1,...,p_k}$. Then we choose a general line ℓ inside T containing x, and we also choose Σ a general $\mathbb{P}^{s-1}$ inside T. For every $t \in \ell$, we let L_t be the span of t and Σ. For $t \in \ell$ a general point, L_t is a general $\mathbb{P}^s$ in $\mathbb{P}^N$. For a general $t \in \ell$, we denote by $\pi_t : S^{k-1}(X) \to \mathbb{P}^{N-s-1}$ the projection morphism of $S^{k-1}(X)$ from L_t. We want to study the limit of π_t when t tends to x. The case $k = 1$ has been considered in [**CMR**].

In order to perform our analysis, consider a neighborhood U of x in ℓ such that π_t is a morphism for all $t \in U \setminus \{x\}$. We will fix a local coordinate on ℓ so that x has the coordinate 0, thus we may identify U with a disk around $x = 0$ in $\mathbb{C}$. Consider the products:

$$\mathcal{X}_1 = X \times U, \quad \mathcal{X}_2 = S^{k-1}(X) \times U, \quad \mathbb{P}_U^{N-s-1} = \mathbb{P}^{N-s-1} \times U$$

The projections π_t, for $t \in U$, fit together to give a morphism $\pi_1 : \mathcal{X}_1 \to \mathbb{P}_U^{N-s-1}$ and a rational map $\pi_2 : \mathcal{X}_2 \dashrightarrow \mathbb{P}_U^{N-s-1}$, which is defined everywhere except at the pair $(x, x) = (x, 0)$. In order to extend it, we have to blow up $\mathcal{X}_2$ at $(x, 0)$. Let $p : \tilde{\mathcal{X}}_2 \to \mathcal{X}$ be this blow–up and let $Z \simeq \mathbb{P}^{s^{(k-1)}(X)}$ be the exceptional divisor. Looking at the obvious morphism $\phi : \tilde{\mathcal{X}}_2 \to U$, we see that this is a flat family of varieties over U. The fibre over a point $t \in U \setminus \{0\}$ is isomorphic to $S^{k-1}(X)$, whereas the fibre over $t = 0$ is of the form $\tilde{S} \cup Z$, where $\tilde{S} \to S^{k-1}(X)$ is the blow up of $S^{k-1}(X)$ at x, and $\tilde{S} \cap Z = E$ is the exceptional divisor of this blow up, the intersection being transverse.

On $\tilde{\mathcal{X}}_2$ the projections π_t, for $t \in U$, fit together now to give a *morphism* $\tilde{\pi} : \tilde{\mathcal{X}}_2 \to \mathbb{P}_U^{N-s-1}$.

By abusing notation, we will denote by π_0 the restriction of $\tilde{\pi}$ to the central fibre $\tilde{S} \cup Z$. The restriction of π_0 to $\tilde{S}$ is determined by the projection of $S^{k-1}(X)$ from the subspace L_0: notice in fact that, since $L_0 \cap S^{k-1}(X) = L_0 \cap T_{X,p_1,\ldots,p_k} = \{x\}$, this projection is not defined on $S^{k-1}(X)$ but it is well defined on $\tilde{S}$.

As for the action of π_0 on the exceptional divisor Z, this is explained by the following lemma, whose proof is analogous to the proof of Lemma 3.1 of [**CMR**], to which we refer for details:

4.1.1. LEMMA. *In the above setting, π_0 maps isomorphically Z to the $s^{(k-1)}(X)$–dimensional linear space Θ which is the projection of T from L_0.*

Now we consider $\mathcal{X}_1 \times_U \tilde{\mathcal{X}}_2$, which has a natural projection map $\psi : \mathcal{X}_1 \times_U \tilde{\mathcal{X}}_2 \to U$. One has a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}_1 \times_U \tilde{\mathcal{X}}_2 & \stackrel{\overline{\pi}}{\to} & \mathbb{P}^{r-s-1}_U \\ \psi \downarrow & & \downarrow \\ U & \stackrel{id_U}{\to} & U \end{array}$$

where $\overline{\pi} = \pi \times \tilde{\pi}$. For the general $t \in U$, the fibre of ψ over t is $X \times S^{k-1}(X)$, and the restriction $\overline{\pi}_t : X \times S^{k-1}(X) \to \mathbb{P}^{r-s-1}$ of $\overline{\pi}$ to it is nothing but $\pi_{t|X} \times \pi_{t|S^{k-1}(X)}$. We denote by $\Delta_t^{(s,k)}$ the double point scheme of $\overline{\pi}_t$. Notice that $\dim(\Delta_t^{(s,k)}) \geq s^{(k)}(X) + s - r$ and, by the generality assumptions, we may assume that equality holds for all $t \neq 0$. Finally consider the flat limit $\tilde{\Delta}_0^{(s,k)}$ of $\Delta_t^{(s,k)}$ inside $\Delta_0^{(s,k)}$. We will call it the *limit double point scheme* of the map $\overline{\pi}_t$, $t \neq 0$. We want to give some information about it. Notice the following lemma, whose proof is similar to the one of Lemma 3.2 of [**CMR**], and therefore we omit it:

4.1.2. LEMMA. *In the above setting, every irreducible component of $\Delta_0^{(s,k)}$ of dimension $s^{(k)}(X) + s - r$ sits in the limit double point scheme $\tilde{\Delta}_0^{(s,k)}$.*

Let us now denote by:

- X_T the scheme cut out by T on X. X_T is cut out on X by $r - s^{(k-1)}(X) - s - 1$ general hyperplanes tangent to X at $p_1, \ldots, p_k$. We call X_T a *general* $(r - s^{(k-1)}(X) - s - 1)$*-tangent section* to X at $p_1, \ldots, p_k$. Remark that each component of X_T has dimension at least $n - (r - s^{(k-1)}(X) - s - 1) = s^{(k)}(X) + s - r$;
- Y_T the image of X_T via the restriction of π_0 to X. By Lemma 4.1.1, Y_T sits in $\Theta = \pi_0(Z)$, which is naturally isomorphic to Z. Hence we may consider Y_T as a subscheme of Z;
- $Z_T \subset X \times Z$ the set of pairs (x, y) with $x \in X_T$ and $y = \pi_0(x) \in Y_T$. Notice that $Z_T \simeq X_T$;
- $\Delta'^{(s,k)}_0$ the double point scheme of the restriction of π_0 to $\tilde{S} \times X$.

With this notation, the following lemma is clear (see Lemma 3.3 of [**CMR**]):

4.1.3. LEMMA. *In the above setting, $\Delta_0^{(s,k)}$ contains as irreducible components $\Delta'^{(s,k)}_0$ on $X \times \tilde{S}$ and Z_T on $X \times Z$.*

As an immediate consequence of Lemma 4.1.2 and Lemma 4.1.3, we have the following proposition (see Proposition 3.4 of [**CMR**]):

4.1.4. PROPOSITION. *In the above setting, every irreducible component of X_T, off $T_{X,p_1,\ldots,p_k}$, of dimension $s^{(k)}(X) + s - r$ gives rise to an irreducible component of Z_T which is contained in the limit double point scheme $\tilde{\Delta}_0^{(s,k)}$.*

So far we have essentially extended word by word the contents of §3 of [**CMR**]. This is not sufficient for our later applications. Indeed we need a deeper understanding of the relation between the double points scheme $\Delta_t^{(s,k)}$ and $(k+1)$–secant $\mathbb{P}^k$'s to X meeting the centre of projection L_t and related degenerations when t goes to 0. We will do this in the following remark.

4.1.5. REMARK. (i) It is interesting to give a different geometric interpretation for the general double point scheme $\Delta_t^{(s,k)}$, for $t \neq 0$. Notice that, by the generality assumption, $L_t \cap S^k(X)$ is a variety of dimension $s^{(k)}(X) + s - r$, which we can assume to be irreducible as soon as $s^{(k)}(X) + s - r > 0$. Take the general point w of it if $s^{(k)}(X) + s - r > 0$, or any point of it if $s^{(k)}(X) + s - r = 0$. Then this is a general point of $S^k(X)$. This means that $w \in < q_0, ..., q_k >$, with $q_0, ..., q_k$ general points on X. Now, for each $i = 0, ..., k$, there is a point $r_i \in < q_0, ..., \hat{q}_i, ...q_k >$ which is collinear with w and q_i. Each pair (q_i, r_i), $i = 0, ..., k$, is a general point of a component of $\Delta_t^{(s,k)}$. Conversely the general point of any component of $\Delta_t^{(s,k)}$ arises in this way.

(ii) Now we specialize to the case $t = 0$. More precisely, consider $Z_T \subset X \times Z$ and a general point (p, q) on an irreducible component of it of dimension $s^{(k)}(X) + s - r$, which therefore sits in the limit double point scheme $\tilde{\Delta}_0^{(s,k)}$. Hence there is a 1–dimensional family $\{(p_t, q_t)\}_{t \in U}$ of pairs of points such that $(p_t, q_t) \in \Delta_t^{(s,k)}$ and $p_0 = p, q_0 = q$.

By part (i) of the present remark, we can look at each pair (p_t, q_t), $t \neq 0$, as belonging to a $(k+1)$–secant $\mathbb{P}^k$ to X, denoted by Π_t, forming a flat family $\{\Pi_t\}_{t \in U \setminus \{0\}}$ and such that $\Pi_t \cap L_t \neq \emptyset$. Consider then the flat limit Π_0, for $t = 0$, of the family $\{\Pi_t\}_{t \in U \setminus \{0\}}$. Since $q \in Z$, clearly Π_0 contains x. Moreover it also contains p. This implies that Π_0 is the span of p with one of the k–secant $\mathbb{P}^{k-1}$'s to X containing $x \in S^{k-1}(X)$.

As an application of the previous remark, we can prove the following crucial theorem, which extends Theorem 4.1 of [**CMR**]:

4.1.6. THEOREM. ([**CR**, Theorem 2.7]) *Let $X \subset \mathbb{P}^N$ be an irreducible, non–degenerate, projective variety such that $s^{(k)}(X) = (k+1)n + k$. Then:*

$$d_{X,k} \cdot \deg(X_k) \leq \nu_k(X).$$

In particular:

(i) *if $N \geq (k+1)(n+1)$ and X is not k–weakly defective, then:*

$$\deg(X_k) \leq \nu_k(X);$$

(ii) *if $N = (k+1)n + k$ then:*

$$d_{X,k} \leq \mu_k(X).$$

PROOF. We let $s = h^{(k)}(X) = r - s^{(k)}(X)$ and we apply Remark 4.1.5 to this situation. Then X_T has $d_{X,k} \cdot \deg(X_k)$ isolated points, which give rise to as many flat limits of $(k+1)$–secant $\mathbb{P}^k$'s to X meeting a general $\mathbb{P}^s$. By the definition of $\nu_k(X)$ the first assertion follows. Then (i) follows from Lemma 1.5.5 and (ii) follows by (1.5.2). ☐

In the last part we shall suppose $k = 1$ and study a slight different degeneration, following §2 of [**IR2**] in order to provide some new applications of this circle of ideas.

Let us recall the setting for the definition of the secant variety. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety and let

$$S_X := \overline{\{(x, y, z) \mid x \neq y, \, z \in \langle x, y \rangle\}} \subset X \times X \times \mathbb{P}^N,$$

be the *abstract secant variety* of $X \subset \mathbb{P}^N$, which is an irreducible projective variety of dimension $2n + 1$. Let us consider the projections of S_X onto the factors $X \times X$ and $\mathbb{P}^N$,

(4.1.1)
$$\begin{tikzcd} & S_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \times X & & \mathbb{P}^N. \end{tikzcd}$$

With this notation we get

$$p_2(S_X) = \overline{\bigcup_{\substack{x \neq y \\ x,y \in X}} \langle x, y\rangle} = SX \subseteq \mathbb{P}^N.$$

Let $L = \langle x, y\rangle$ with $x \in X$ and $y \in X$ general points, i.e. L is a general secant line to X, and let $p \in \langle x, y\rangle \subseteq SX \subseteq \mathbb{P}^N$ be a general point. We fix coordinates on L so that the coordinate of x is 0; let U be an open subset of $\mathbb{A}^1_{\mathbb{C}} \subset L$ containing $0 = x$. Let $p_2 : S_X \to SX \subseteq \mathbb{P}^N$ be as above and let

$$Z_U = p_2^{-1}(U) \subset S_X.$$

By shrinking up U, we can suppose that $p_2 : Z_U \to U$ is flat over $U \setminus \{0\}$ and that $\dim(Z_U)_t = \delta(X)$ for every $t \neq 0$. The projection of $p_1((Z_U)_t)$ onto one of the factors is Σ_t, the entry locus of X with respect to t for every $t \neq 0$.

Moreover, by definition, a point $(r, s) \in X \times X$, $r \neq s$, belongs to $(Z_U)_t$, $t \neq 0$, if and only if $t \in \langle r, s\rangle$, that is if and only if $(r, s) \in p_2^{-1}(t)$. Thus, if $\psi_t : X \dashrightarrow \mathbb{P}^{N-1}$ is the projection from t onto a disjoint $\mathbb{P}^{N-1}$, we can also suppose that ψ_t is a morphism for every $t \neq 0$ and a rational map not defined at $x = 0$ for $t = 0$. The above analysis says that the *abstract entry locus* $(Z_U)_t$, $t \neq 0$, can be considered as the closure in $X \times X$ of the double point locus scheme of ψ_t, minus the diagonal $\Delta_X \subset X \times X$.

Let $T = \langle T_xX, y\rangle$, so that T is a general $\mathbb{P}^{n+1}$ containing T_xX and a general point $y \in X$. By definition $\pi_x^{-1}(\pi_x(y)) = T \cap X \setminus (T_xX \cap X)$. Let

$$F_y = \overline{\pi_x^{-1}(\pi_x(y))},$$

be the closure of the fiber of π_x through y. Every irreducible component of F_y has dimension $\delta(X)$ by Terracini Lemma and by the generality of y. Generic smoothness ensures also that there exists only one irreducible component of F_y through y.

By using the same ideas recalled above, we get the following result.

4.1.7. PROPOSITION. *Let notation be as above. The closure of the fiber of π_x through a general point $y \in X$ is contained in the flat limit of the family $\{(Z_U)_t\}_{t\neq 0}$. In other words, the closure of a general fiber of the tangential projection is a degeneration of the general entry locus of X.*

PROOF. We shall look at ψ_t as a family of morphisms and study the limit of the *double point scheme* $(Z_U)_t$.

Consider the products $\mathcal{X} = X \times U$ and $\mathbb{P}_U = \mathbb{P}^{N-1} \times U$. The projections ψ_t, for $t \in U$, fit together to give a rational map $\psi : \mathcal{X} \dashrightarrow \mathbb{P}_U$, which is defined everywhere except at the pair $(x, 0)$. In order to extend the projection not defined at $x \in X$, we have to blow up $\mathcal{X}$ at $(x, 0)$. Let $\sigma : \widetilde{\mathcal{X}} \to \mathcal{X}$ be this blowing-up and let $Z \simeq \mathbb{P}^n$ be the exceptional divisor. Looking at the obvious morphism $\phi : \widetilde{\mathcal{X}} \to U$, we see that this is a flat family of varieties over U. The fiber $\widetilde{\mathcal{X}}_t$ over a point $t \in U \setminus \{0\}$ is isomorphic to X, whereas the fiber $\widetilde{\mathcal{X}}_0$ over $t = 0$ is of the form $\widetilde{\mathcal{X}}_0 = \widetilde{X} \cup Z$, where $\widetilde{X} \to X$ is the blowing-up of X at x, and $\widetilde{X} \cap Z = E$ is the exceptional divisor of this blowing-up, the intersection being transverse. Reasoning as in [**CMR**, Lemma 3.1], see also Lemma 4.1.1 above, it is easy to see that ψ_0 acts on $\widetilde{X}$ as the projection from the point $0 = x$, while it maps Z isomorphically onto the linear space $\psi_0(T) = \mathbb{P}^n$. This immediately implies that every point of $T \cap X$, different from x, appears in the "double point scheme" of $\psi_0 : \widetilde{X} \cup Z \to \mathbb{P}^{N-1}$. Therefore F_y, being of dimension $\delta(X)$, is contained in the flat limit of $\{(Z_U)_t\}_{t\neq 0}$, proving the assertion. □

For an irreducible variety $X \subset \mathbb{P}^N$ we denoted by $\mu(X)$ the number of secant lines passing through a general point of SX. If $\delta(X) > 0$, then $\mu(X)$ is infinite, while for $\delta(X) = 0$ the above number is finite and in this case

$$\nu(X) = \mu(X) \cdot \deg(SX)$$

is called *the number of apparent double points* of $X \subset \mathbb{P}^N$. With these definitions we obtain the following generalization of [**CMR**, Theorem 4.1] (see also [**CR**, Theorem 2.7]).

4.1.8. THEOREM. ([**IR2**, Theorem 2.3]) *Let $X \subset \mathbb{P}^N$ be as irreducible non-degenerate variety. If $\delta(X) = 0$, then*

$$0 < \deg(\pi_x) \leq \mu(X).$$

In particular for a QEL-manifold of type $\delta = 0$, the general tangential projection is birational.

If $X \subset \mathbb{P}^N$ is a QEL-manifold of type $\delta > 0$, then the general fiber of π_x is irreducible. More precisely the closure of the fiber of π_x passing through a general point $y \in X$ is the entry locus of a general point $p \in \langle x, y \rangle$, i.e. a smooth quadric hypersurface.

PROOF. If $\delta(X) = 0$, then for $t \in U \setminus \{0\}$ the 0-dimensional scheme $(Z_U)_t$ has length equal to $2\mu(X)$. The 0-dimensional scheme F_y contains $\deg(\pi_x)$ isolated points, yielding $2\deg(\pi_x)$ points in the flat limit of $\{(Z_U)_t\}_{t \neq 0}$ by Proposition 4.1.7 and proving the first part.

Suppose X is a QEL-manifold of type $\delta > 0$. Then for every $t \neq 0$ the δ-dimensional scheme $(Z_U)_t$ is a smooth quadric hypersurface by definition of QEL-manifolds. The fiber F_y contains the entry locus Σ_p of a general point $p \in \langle x, y \rangle$, which is a smooth quadric hypersurface of dimension δ passing through x and y. By proposition 4.1.7 the variety F_y is also contained in the flat limit of $\{(Z_U)_t\}_{t \neq 0}$. Therefore F_y coincides with Σ_p. In fact, in this case the family $\{(Z_U)_t\}_{t \neq 0}$ is constant. □

4.2. Rationality of varieties with extremal secant or tangential behaviour

Let us state this highly non-trivial implications of Theorem 4.1.6 and of Theorem 1.5.9.

4.2.1. COROLLARY. *Let k be a positive integer. Let $X \subset \mathbb{P}^N$ be an irreducible, non–degenerate, projective variety of dimension n and let $h := \operatorname{codim}(S^k(X)) \geq 0$,. One has:*

(i) *if X is a $\mathcal{M}^k$–variety then for every m such that $1 \leq m \leq h$, the variety X^m is again a $\mathcal{M}^k$–variety;*
(ii) *if X is a $\mathcal{MA}^{k+1}_{k-1}$-variety then for every m such that $1 \leq m \leq h-1$, the variety X^m is again a $\mathcal{MA}^{k+1}_{k-1}$-variety and X^h is a $\mathcal{OA}^{k+1}_{k-1}$–variety;*
(iii) *if X is either an $\mathcal{MA}^{k+1}_{k-1}$-variety or an $\mathcal{OA}^{k+1}_{k-1}$–variety then $\tau_{X,k} : X \dashrightarrow X_k \subseteq \mathbb{P}^{n+h}$ is birational and X_k is a variety of dimension n of minimal degree $h+1$. In particular X is a rational variety.*

PROOF. Parts (i) follows by Theorem 1.5.9, part (ii). Part (ii) follows by Theorem 1.5.9, parts (ii) and (v). In part (iii), the birationality of $\tau_{X,k}$ follows by Theorem 4.1.6, part (ii). The rest of the assertion follows by Theorem 1.5.9, part (iv). □

4.2.2. REMARK. In the papers [**B1**] and [**B2**], Bronowski considers the case $k = 1$, $h = 0$ and the case $k \geq 2, n = 2, h = 0$. He claims there, without giving a proof, that also the converse of Corollary 4.2.1 holds for $h = 0$. We will call this the *k–th Bronowski's conjecture*, a generalized version of which, for any $h \geq 0$, can be stated as follows: Let $X \subset \mathbb{P}^N$ be an irreducible, non–degenerate, projective variety of dimension n. Set $h := \operatorname{codim}(S^k(X))$. If $\tau_{X,k} : X \dashrightarrow X_k \subseteq \mathbb{P}^{n+h}$ is birational and X_k is a variety of dimension n and of minimal degree $h+1$, then X is either an $\mathcal{MA}^{k+1}_{k-1}$–variety or an $\mathcal{OA}^{k+1}_{k-1}$–variety, according to whether h is positive or zero. We call this the *k–th generalized Bronowski's conjecture*.

Even the curve case $n = 1$ of this conjecture is still open in general. The results in [**CMR**], [**Ru1**], [**Sev**], imply that the above conjecture is true for X smooth if $k = 1, h = 0$ and $1 \leq n \leq 3$. The general smooth surface case $n = 2$, $k \geq 1$, $h \geq 0$ follows from [**CR**, Corollary 8.1]). This interesting conjecture is quite open in general.

Bronowski's conjecture would, for example, imply that the converse of part (ii) of Corollary 4.2.1 holds.

Now we include as another application the following result.

4.2.3. THEOREM. ([**IR2**, Theorem 2.1]) *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate variety and let $x \in X$ be a general point. Then:*

(i) *If X is a QEL-manifold of type $\delta \geq 0$, the projection from a general δ-codimensional subspace of T_xX passing through x is birational onto its image. In particular, if $SX = \mathbb{P}^N$, then X is rational.*
(ii) *Conversely, if the projection from a general δ-codimensional subspace of T_xX passing through x is birational, then X is an LQEL-manifold of type δ.*
(iii) *If $X \subset \mathbb{P}^N$ is an LQEL-manifold of type $\delta > 0$, then X is rational.*

PROOF. Suppose $X \subset \mathbb{P}^N$ is a QEL-manifold of type $\delta \geq 0$. If $\delta = 0$ then the first part of Theorem 4.1.8 yields that π_x is birational onto its image (see also [**CMR**, Corollary 4.2]).

Suppose from now on $\delta > 0$. The projection from a general codimension δ linear subspace $L \subseteq T_xX$ passing through x is a rational map $\pi_L : X \dashrightarrow \mathbb{P}^{N-n+\delta-1}$. The birationality of π_L onto its image follows from the fact that L is obtained by cutting T_xX with δ general hyperplanes through x and by applying the second part of Theorem 4.1.8. If $SX = \mathbb{P}^N$, then $N - n + \delta - 1 = n$. Part (i) is now completely proved.

Suppose we are in the hypothesis of (ii). Let $L = \mathbb{P}^{n-\delta} \subset T_xX$ be a general linear subspace passing through x. L is the tangent space of a general codimension δ linear section of $X \subset \mathbb{P}^N$ passing through x, let us say Z. Thus the restriction of π_L to Z, $\pi_{L|Z} : Z \dashrightarrow \mathbb{P}^{N-\delta-\dim(Z)-1} = \mathbb{P}^{N-n-1}$, is the projection from $L = T_xZ$. Since π_L restricted to X is birational onto its image, also $\pi_{L|Z}$ is easily seen to be birational onto its image. Moreover, looking at $\pi_{L|Z}$ as the projection from T_xZ, we get $\pi_{L|Z}(Z) = \pi_x(X) = W_x \subseteq \mathbb{P}^{N-n-1}$ and hence that $\pi_x : X \dashrightarrow W_x \subseteq \mathbb{P}^{N-n-1}$ has irreducible general fibers of dimension δ.

There is no loss of generality in supposing $\delta = 1$, by passing to a general linear section; see Proposition 2.3.14. Let, for $y \in X$ general,

$$F_y = \overline{\pi_x^{-1}(\pi_x(y))}.$$

We claim that set theoretically $L \cap F_y = \{x\}$. We have $F_y \subset \langle T_xX, y\rangle$ so that $T_xX \cap F_y$ consists of a finite number of points. By the generality of L we get $L \cap F_y \subseteq \{x\}$. Let $t = \pi_x(y)$. Since $\langle L, t\rangle$ is a hyperplane in $\langle T_xX, y\rangle = \langle T_xX, t\rangle$, intersecting $\pi_x^{-1}(\pi_x(y))$ transversally at a unique point, we get that either we are in the case of the claim or F_y is a line. This last case is excluded by Lemma 2.3.15.

Let $q = \pi_{L|Z}^{-1}(t) = \langle L, t\rangle \cap Z \setminus (L \cap Z)$. By definition

$$\langle L, t\rangle = \langle L, q\rangle. \tag{4.2.1}$$

Consider the projection from t onto T_xX, let us say $\psi_t : \langle T_xX, t\rangle \dashrightarrow T_xX$. Let $\widetilde{F}_y = \psi_t(F_y)$. By definition $x \in \widetilde{F}_y$ because $x \in F_y$. Moreover we claim that $L \cap \widetilde{F}_y$ is supported at x, so that $\widetilde{F}_y$ is a line through x. Indeed, if $z \in L \cap \widetilde{F}_y$, then there exists $w \in \langle z, t\rangle \cap F_y \subset \langle L, t\rangle \cap F_y = \langle L, q\rangle \cap F_y$, where the last equality follows from (4.2.1). Thus either $w = x$ or $w = q$ and in any case $x = \psi_t(w) = z$. Therefore $F_y \subset \langle \widetilde{F}_y, t\rangle \simeq \mathbb{P}^2$ and moreover the line $\langle x, y\rangle$ cuts transversally F_y at x and at y. The line $\langle x, y\rangle$ is contained in the plane $\langle \widetilde{F}_y, t\rangle$, so that $\deg(F_y) = 2$ and F_y is a smooth conic passing through x and y, concluding the proof.

Let us prove part (iii). Fix a general point $x \in X$ and denote by $\mathcal{Q}_x$ the family of δ-dimensional quadric hypersurfaces contained in X and passing through x. Let $\mathcal{F}_x \to \mathcal{Q}_x$ be the universal family, which has a section corresponding to the point x.

Assume first $\delta = 1$. We see that smooth conics through x are parameterized by an open subset of $\mathbb{P}(\mathbf{T}_x^*X)$. Moreover, Lemma 2.3.13 shows that the tautological morphism from $\mathcal{F}_x$ to X is birational. So, $\mathcal{F}_x$, and also X, is rational. This classical reasoning, certainly familiar to Scorza, appears in a rather general form in [**IN**, Proposition 3.1].

Suppose now that $\delta \geq 2$ and fix H a general hyperplane section of X through x. Using Lemma 2.3.13, we see that sending a quadric hypersurface through x to its trace on H yields a birational map between the families $\mathcal{Q}_x(X)$ and $\mathcal{Q}_x(H)$. So, we see inductively that $\mathcal{Q}_x(X)$ is a rational variety of dimension $n - \delta$. Therefore, $\mathcal{F}_x$ is rational, as the family $\mathcal{F}_x \to \mathcal{Q}_x$ has a section. Being birational to $\mathcal{F}_x$ by Lemma 2.3.13, X is rational too. $\square$

4.2.4. REMARK. We can generalize *Bronowski Conjecture*, recalled above, to the following: a smooth irreducible n-dimensional variety $X \subset \mathbb{P}^{2n+1-\delta(X)}$ is a QEL-manifold if and only if the projection from a general codimension $\delta(X)$ linear subspace of T_xX passing through x is birational. Theorem 4.2.3 proves one implication, yielding the rationality of QEL-manifolds and extending [**CMR**, Corollary 4.2]. One may consult [**CR**] for other generalizations of the above conjecture to higher secant varieties.

It is worth mentioning that the above results reveal the following interesting picture for the tangential projections of QEL-manifolds of type $\delta \geq 0$ with $SX = \mathbb{P}^N$: for $\delta = 0$ we project from the whole space and we have varieties with one apparent double point; at the other extreme we found the stereographic projection of quadric hypersurfaces, the only QEL-manifolds of type equal to their dimension.

Bibliography

[Åd] B. Ådlandsvik, *Joins and higher secant varieties*, Math. Scand. 61 (1987), 213–222.

[AR] A. Alzati, F. Russo, *Special subhomaloidal systems of quadrics and varieties with one apparent double point*, Math. Proc. Camb. Phil. Soc. 134 (2003), 65-82.

[ACGH] A. Arbarello, M. Cornalba, Ph. Griffiths, J. Harris *Geometry of algebraic curves*, Sprinder-Verlag, Grundlheren der math. Wissenschaften, 267 (1984).

[At] M. F. Atiyah, *Complex fibre bundles and ruled surfaces*, Proc. London Math. Soc. 5 (1955), 407–434.

[B1] L. Bădescu, *Special chapters of projective geometry*, Rend. Sem. Matem. Fisico Milano, 69 (1999-2000), 239–326.

[B2] L. Bădescu, *Projective Geometry and Formal Geometry*, Birkhäuser, 2004.

[BS] L. Bădescu, M. Schenider, *A criterion for extending meromorphic functions*, Math. Ann. 305 (1996), 393–402.

[BL] W. Barth, M. E. Larsen, *On the homotopy groups of complex projective algebraic manifolds*, Math. Scand. 30 (1972), 88–94.

[Be] E. Bertini, *Introduzione alla Geometria Proiettiva degli Iperspazi*, seconda edizione, Principato, 1923.

[BK] H. Braun, M. Koecher, *Jordan-Algebren*, Springer-Verlag, Berlin and New York, 1966.

[B1] J. Bronowski, *The sum of powers as canonical expressions*, Proc. Cam. Phil. Soc. 29 (1933), 69–82.

[B2] J. Bronowski, *Surfaces whose prime sections are hyperelliptic*, J. London Mat. Soc. 8 (1933), 308–312.

[Ca] G. Castelnuovo, *Massima dimensione dei sistemi lineari di curve piane di dato genere*, Annali di Mat. 18 (1890).

[CJ1] M. L. Catalano-Johnson, *The possible dimensions of the higher secant varieties*, Amer. J. Math. 118 (1996), 355–361.

[CJ2] M. L. Catalano-Johnson, *The homogeneous ideal of higher secant varieties*, J. Pure Appl. Algebra 158 (2001), 123–129.

[Ch1] P. E. Chaput, *Severi Varieties*, Math. Z. 240 (2002), 451–459, or math.AG/0102042.

[Ch2] P. E. Chaput, *Scorza varieties and Jordan algebras*, math.AG/0208207

[CC1] L. Chiantini, C. Ciliberto, *Weakly defective varieties*, Trans. A.M.S. 354 (2001), 151–178.

[CC2] L. Chiantini, C. Ciliberto, *On the concept of k–secant order of a variety*, J. London Math. Soc. 73 (2006), 436–454.

[CMR] C. Ciliberto, M. Mella, F. Russo, *Varieties with one apparent double point*, J. Alg. Geometry, Journ. of Alg. Geom. 13 (2004), 475–512.

[CR] C. Ciliberto, F. Russo, *Varieties with minimal secant degree and linear systems of maximal dimension on surfaces*, Adv. in Math. 200 (2006), 1–50.

[Cu] C. W. Curtis, *Linear Algebra–An Introductory Approach*, Undergraduate Texts in Mathematics, corrected sixth edition, Springer Verlag, 1997.

[De1] O. Debarre, *Higher-Dimensional Algebraic Geometry*, Universitext, Springer Verlag, U.S.A., 2001.

[De2] O. Debarre, *Bend and Break*, 2007, available at www-irma.u-strasbg.fr/ debarre/Grenoble.pdf

[D1] P. Deligne, *Letters to W. Fulton*, july/november 1979.

[D2] P. Deligne, Le groupe fondamental du complément d' une courbe plane n' ayant que des points doubles ordinaries est abélien, Sém. Bourbaki, n. 543, 1979/80, L.N.M. 842, Springer -Verlag, 1981, 1–10.

[dP] P. del Pezzo *Sulle superficie dell' n-esimo ordine immerse nello spazio a n dimensioni*, Rend. Circ. Mat. Palermo 12 (1887), 241–271.

[Ed] W. L. Edge, *The number of apparent double points of certain loci*, Proc. Cambridge Philos. Soc. 28 (1932),285–299.

[Ew] B. A. Edwards, *Algebraic loci of dimension $r-3$ in space of r dimensions, of which a chord cannot be drawn through an arbitrary point of space*, J. London Math. Soc. 2 (1927), 155–158.

[E1] L. Ein, *Varieties with small dual variety I*, Invent. Math. 86 (1986), 63–74.

[E2] L. Ein, *Vanishing theorems for varieties of low codimension*, Springer L.N.M. 1311 (1988), 71–75.

[ESB] L. Ein, N. Shepherd-Barron, *Some special Cremona transformations*, Amer. J. Math. 111 (1989), 783–800.

[EH] D. Eisenbud, J. Harris, *On varieties of minimal degree*, Alg. Geometry Bowdin 1985, Proc. Symp. in Pure Math. 46 (1987), 3–13.

[En] F. Enriques, *Sulla massima dimensione dei sistemi lineari di dato genere appartenenti a una superficie algebrica*, Atti Reale Acc. Scienze Torino 29 (1894), 275–296.

[EC] F. Enriques, O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, vol. 1, Zanichelli, Bologna, 1918.

[Fa] G. Faltings, *Formale Geometrie und homogene Räume*, Inv. Math. 64 (1981), 123–165.

[FP] G. Fischer, J. Piontkowski, *Ruled Varieties–An Introduction to Algebraic Differential Geometry*, Advanced Lectures in Mathematics, Vieweg, 2001.

[Fj] T. Fujita, *Classification theory of polarized varieties*, London Math. Soc. Lec. Notes Ser. 155 (1990).

[FR] T. Fujita, J. Roberts, *Varieties with small secant varieties: the extremal case*, Amer. Jour. Math. 103 (1981), 953–976.

[Fu1] W. Fulton, *Intersection theory*, Ergebnisse der Math., Springer Verlag, 1984.

[Fu2] W. Fulton, *On the Topology of Algebraic Varieties*, Proc. Symp. Pure Math. 46 (1987), 15–46.

[FH] W. Fulton, J. Hansen, *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. of Math. 110 (1979), 159–166.

[FL] W. Fulton, R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, L.N.M. 862, Springer-Verlag, 1981, 26–92.

[Gi] G. Z. Giambelli, *Ordine di una varietá piú ampia di quella rappresentata coll' annullare tutti i minori di dato ordine estratti da una matrice generica di forme*, Mem. R. Ist. Lomb. 11 (1904), 101–135.

[Go] N. Goldstein, *Ampleness and connectedness in complex G/P*, Trans. Amer. Math. Soc. 274 (1982), 361–373.

[GH] P. Griffiths, J. Harris, *Algebraic geometry and local differential geometry*, Ann. Sci. Ecole Norm. Sup. 12 (1979), 355-432.

[Gr1] A. Grothendieck, *Techniques de construction e théorems d' existence en geometrie algebrique IV: les sch' emas de Hilbert*, Seminaire Bourbaki, 1960/61, Exp. 221, Asterisque hors serie 6, Soc. Math. Fr., 1997.

[Gr2] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théormes de Lefschetz locaux et globaux*, Séminaire de Géomé trie Algébrique (1962), North Holland, 1968.

[Ha] J. Harris, *Algebraic Geometry, A First Course*, G. T. M. 133, Springer Verlag, 1992.

[H1] R. Hartshorne, *Ample subvarieties of algebraic varieties*, L.N.M. 156, Springer-Verlag, 1970.

[H2] R. Hartshorne, *Varieties of small codimension in projective space*, Bull. A. M. S. 80 (1974), 1017-1032.

[HCV] D. Hilbert, S. Cohn-Vossen, *Geometry and the Imagination*, Chelsea Pub. Co., New York, 1952 (translated from German *Anschauliche Geometrie*, 1932).

[HR] A. Holme, J. Roberts, *Zak's theorem on superadditivity*, Ark. Mat. 32 (1994), 99–120.

[Hw1] J. M. Hwang, *Geometry of Minimal Rational Curves on Fano Manifolds*, in "Vanishing Theorems and Effective Results in Algebraic Geometry", I.C.T.P. Lecture Notes, vol. 6, 335–393, 2001.

[Hw2] J. M. Hwang, *On the degrees of Fano four-folds of Picard number 1*, J. reine angew. Math. 556 (2003), 225-235.

[HK] J.M. Hwang, S. Kebekus, *Geometry of chains of minimal rational curves*, J. Reine Angew. Math. 584 (2005), 173–194.

[Io] P. Ionescu, *Embedded projective varieties of small invariants*, Springer L.N.M. 1056 (1984), 142–186.

[IN] P. Ionescu, D. Naie, *Rationality properties of manifolds containing quasi-lines*, Internat. J. Math. 14 (2003), 1053–1080.

[IR1] P. Ionescu, F. Russo, Conic-connected manifolds, preprint 2006, also math.AG/0701885.

[IR2] P. Ionescu, F. Russo, *Varieties with quadratic entry locus, II*, preprint 2007, also math.AG/0703531.

[IL] T. A. Ivey, J. M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, Graduate Studies in Mathematics, 61, Amer. Math. Soc., Providence, RI, 2003.

[Ja] N. Jacobson, *Structure and representations of Jordan algebras*, Amer. Math. Soc. Colloq. Publ., 39, Amer. Math. Soc., Providence, RI, 1968.

[Jo] K. Johnson, *Immersion and embedding of projective varieties*, Acta Math. 140 (1978), 49–74.

[Ju] J. P. Jouanolou, *Théorémes de Bertini et applications*, Prog. Math. 42, Birkauser, 1983.

[KS] Y. Kachi, E. Sato, *Segre's Reflexivity and an Inductive Characterization of Hyperquadrics*, Mem. Amer. Math. Soc., vol. 160, no. 763, 2002.

[Ke1] G. Kempf, *On the geometry of a theorem of Riemann*, Ann. of Math. 98 (1973), 178–185.

[Ke2] G. Kempf, *Algebraic Varieties*, London Math. Soc. Lectures Notes 172, Cambridge University Press, England, 1993.

[Ko] J. Kollár, *Rational Curves on Algebraic Varieties*, Erg. der Math. 32, Springer Verlag, Germany, 1996.

[KMM] J. Kollár, Y. Miyaoka, S. Mori, *Rationally connected varieties*, J. Algebraic Geom. 1 (1992), 429–448.

[KM] J. Kollár, S. Mori, *Birational Geometry of Algebraic Varieties*, Camb. Tracts in Math. 134, Camb. University Press, Cambridge, U. K., 1998.

[La] T. Y. Lam, *Introduction to Quadratic Forms over Fields*, Graduate Studies in Mathematics, 67, Amer. Math. Soc., U.S.A., 2005.

[L1] J. M. Landsberg, *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. Jour. 85 (1996), 605–634.

[L2] J. M. Landsberg, *On the infinitesimal rigidity of homogeneous varieties*, Compositio Math. 118 (1999), 189–201.

[L3] J. M. Landsberg, *Griffiths-Harris rigidity of compact hermitian symmetric spaces*, J. Diff. Geom. 74 (2006), 395–405.

[Ln] H. Lange, *Higher secant varieties of curves and the theorem of Nagata on ruled surfaces*, Manuscripta Math. 47 (1984), 263–269.

[LV] R. Lazarsfeld, A. van de Ven, *Topics in the geometry of projective space, Recent work by F. L. Zak*, DMV Seminar 4, Birkhäuser, (1984).

[Mo1] S. Mori, *Projective manifolds with ample tangent bundle*, Ann. Math. 110 (1979), 593–606.

[Mo2] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math. 116 (1982), 133–176.

[Mk] S. Mukai, *Biregular classification of Fano threefolds and Fano manifolds of coindex* 3, Proc. Nat. Acad. Sci. USA 86 (1989), 3000–3002.

[Mu] D. Mumford, *The Red Book of Varieties and Schemes*, L. N. M. 1358, Srpinger-Verlag, 1988.

[Oh] M. Ohno, *On degenerate secant varieties whose Gauss map have the largest image*, Pacific J. Math. 187 (1999), 151–175.

[P1] F. Palatini, *Sulle superficie algebriche i cui S_h $(h+1)$-seganti non riempiono lo spazio ambiente*, Atti Accad. Torino 41 (1906), 634–640.

[P2] F. Palatini, *Sulle varietá algebriche per le quali sono minori dell' ordinario, senza riempire lo spazio ambiente, una o alcune delle varietá formate da spazi seganti*, Atti Accad. Torino 44 (1909), 362–375.

[Ro] T. G. Room, *The Geometry of Determinantal Loci*, Cambridge Univerity Press, Cambridge, U. K., 1938.

[Ru1] F. Russo, *On a theorem of Severi*, Math. Ann. 316 (2000), 1–17.

[Ru2] F. Russo, *Varieties with quadratic entry locus, I*, preprint 2006, also math.AG/0701889.

[RS] F. Russo, A. Simis, *On birational maps and Jacobian matrices*, Comp. Math. 126 (2001), 335–358.

[S1] G. Scorza, *Sulla determinazione delle varietá a tre dimensioni di S_r $(r \geq 7)$ i cui S_3 tangenti si tagliano a due a due*, Rend. Circ. Mat. Palermo 25 (1908), 193–204.

[S2] G. Scorza, *Un problema sui sistemi lineari di curve appartenenti a una superficie algebrica*, Rend. Reale Ist. Lombardo Scienze e Lettere 41 (1908), 913–920.

[S3] G. Scorza, *Le varietá a curve sezioni ellittiche*, Ann. Mat. Pura Appl. 15 (1908), 217–273.

[S4] G. Scorza, *Sulle varietá a quattro dimensioni di S_r $(r \geq 9)$ i cui S_4 tangenti si tagliano a due a due*, Rend. Circ. Mat. Palermo 27 (1909), 148–178.

[S5] G. Scorza, Opere Scelte, I–IV, Ed. Cremonese, 1960.

[Sg] C. Segre, *Sulle varietá che rappresentano le coppie di punti di due piani o spazi*, Rend. Circ. Mat. Palermo 5 (1891), 192–204.

[Sm] J. G. Semple, *On representations of the S_k's of S_n and of the Grassmann manifold $\mathbb{G}(k,n)$*, Proc. London. Math. Soc. 32 (1931), 200–221.

[SR] J. G. Semple, L. Roth, *Introduction to Algebraic Geometry*, Oxford University Press, 1949 and 1986.

[Sr] E. Sernesi, *Deformations of algebraic schemes*, Grund. der Math. Wiss., vol. 334, Springer Verlag, 2006.

[Sev] F. Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni e ai suoi punti tripli apparenti*, Rend. Circ. Mat. Palermo 15 (1901), 33–51.

[Sh] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer-Verlag, 1974.

[SUV] A. Simis, B. Ulrich, W. V. Vasconcelos, *Tangent star cones*, J. Reine angew. Math. 483 (1997), 23–59.

[T1] A. Terracini, *Sulle V_k per cui la varieta' degli S_h $(h+1)$-secanti ha dimensione minore dell' ordinario*, Rend. Circ. Mat. Palermo 31 (1911), 392–396.

[T2] A. Terracini, *Alcune questioni sugli spazi tangenti e osculatori ad una varietá*, I, II, III, in *Selecta Alessandro Terracini*, vol. I, 24–90, Edizioni Cremonese, Italy, 1968.

[T3] A. Terracini, *Su due problemi concernenti la determinazione di alcune classi di superficie, considerate da G. Scorza e F. Palatini.*, Atti Soc. Natur. e Matem. Modena 6 (1921-22), 3–16.

[T4] A. Terracini, *Selecta*, I–II, Ed. Cremonese, Roma, 1968.

[Ve] P. Vermeire, *Some results on secant varieties leading to a geometric flip construction*, Comp. Math. 125 (2001), 263–282.

[We] H. Weyl, *Emmy Noether*, Scripta Mathematica 3 (1935), 201–220.

[Z1] F. L. Zak, *Severi varieties*, Math. USSR Sbornik 54 (1986), 113–127.

[Z2] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monografhs, vol. 127, Amer. Math. Soc. 1993.

[Z3] F. L. Zak, *Projective invariants of quadratic embeddings*, Math. Ann. 313 (1999), 507–545.