AFFINE VARIETIES WITH TORUS ACTION

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Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let N be a lattice of rank n and let T be the algebraic torus whose 1-parameter subgroup lattice is N, i.e.,

$$\mathrm{T} = N \otimes_{\mathbb{Z}} \mathbf{k}^* \simeq (\mathbf{k}^*)^n .$$

Definition. A T-variety X is a normal algebraic variety endowed with a effective regular action of T. We define the complexity of X as $\dim_{\mathbf{k}} X - \dim_{\mathbf{k}} \mathrm{T}$.

The most accessible class of T-varieties corresponds to the case of complexity 0, i.e. T-varieties with a dense open orbit. These are called toric varieties. There is a well established theory of toric varieties. In particular, there is a description of all affine toric varieties by means of polyhedral cones in the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ [Oda88].

As in the case of toric varieties, there is a combinatorial descriptions of T-varieties of greater complexity. In [KKMS73] a description of T-varieties of complexity 1 is given under some mild assumptions, and in [Tim08] it is done in full generality. In 2006, Altmann and Hausen [AH06] gave a combinatorial description of affine T-varieties of arbitrary complexity.

Most of my research consists in generalizing the theory of toric varieties to higher complexity in terms of the description given by Altmann and Hausen.

Algebraic group actions on T-varieties. Now let G be a connected linear algebraic group. We say that a regular G-action on an affine T-variety X is *compatible* if the image of G in $\mathrm{Aut}(X)$ is normalized but not centralized by T. Furthermore, we say that a compatible G-action is of *fiber type* if the general orbits of the G-action are contained in the orbit closures of the T-action, and of *horizontal type* otherwise.

Let $\mathbb{G}_{\mathrm{a}} = (\mathbf{k}, +)$ be the additive group. In [Lie10a] we give a classification of compatible $\mathbb{G}_{\mathrm{a}}$-actions on affine T-varieties of complexity 1 in terms of the AH-description. In [Lie10b] we generalize further this classification to include all $\mathbb{G}_{\mathrm{a}}$-actions of fiber type on affine T-varieties of arbitrary complexity. Moreover, in [Lie11] we give a nice application of [Lie10a], to compute the roots of the affine Cremona group.

In a joint work with Arzhantsev [AL12] we give a classification of compatible SL_2-actions on an affine T-variety X in the case where the T-action is of complexity 1, or the SL_2-action is of fiber type. As an application we reprove the well known classification of affine quasi-homogeneous SL_2-threefolds due to Popov [Pop73]. We also provide a description in terms of the AH-description of all special SL_2-actions on affine varieties.

In a joint work with Arzhantsev, Hausen y Herppich [AHHL12] we apply some ideas from [Lie10a] to give a description of the connected component of the authomorphism group of a complete rational T-variety of complexity 1.

Singularities of T-varieties. For an affine T-variety X of arbitrary complexity, in [LS10] we compute a partial desingularization having only toric singularities in terms of the AH-description. We use this desingularization to give criterion for X to have only rational singularities and a partial criterion for X to be Cohen-Macaulay. We also provide criteria X to be $\mathbb{Q}$-Gorenstein, factorial, $\mathbb{Q}$-factorial, or log-terminal.

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