Cremona transformations, diffeomorphisms of surfaces and approximation by (-1)-curves

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joint work with
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Every C^∞-map $f\colon S^1 \to S^1$ is approximated by rational maps

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X real algebraic variety
Is a given C^∞-map $f\colon S^1 \to X$ approximated by rational curves?
[Recall: $\mathbb{P}^1(\mathbb{R}) \sim S^1$.]

Rational curves

Example

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \big(t^2+1, t(t^2+1)\big) \end{aligned}$$

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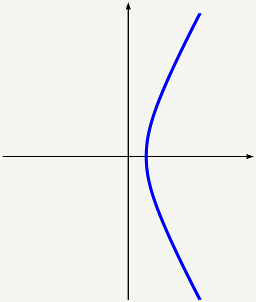

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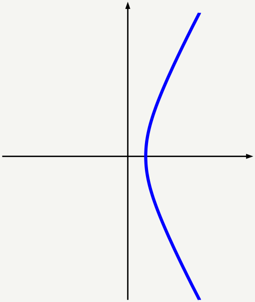
Compactification

$\mathbb{R} \hookrightarrow \mathbb{P}^1(\mathbb{R}) \xrightarrow{\hat{f}} X \xleftarrow{bir} \mathbb{R}^2$

X rational surface



Approximating by rational curves

X nonsingular real algebraic variety
$\mathcal{C}^\infty(S^1, X)$:= space of maps endowed with the $\mathcal{C}^\infty$-topology
$\mathcal{A}_X \subset \mathcal{C}^\infty(S^1, X)$:= subset of rational curves $\mathbb{P}^1(\mathbb{R}) \to X$

Definition

Let $f \in \mathcal{C}^\infty(S^1, X)$ be a $\mathcal{C}^\infty$-map
f is approximated by rational curves
$\Leftrightarrow$
$f \in \overline{\mathcal{A}_X}$.

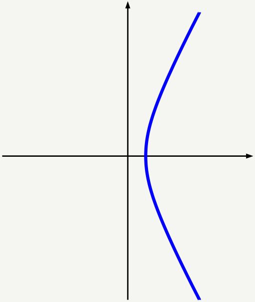
Theorem (Bochnak, Kucharz, 1999)

Let X be a nonsingular real rational variety, then any $\mathcal{C}^\infty$-map $\mathbb{P}^1(\mathbb{R}) \to X$ is approximated by rational curves.

Smoothness?

Remark

$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & \big(t^2+1, t(t^2+1)\big) \end{array}$$



Smoothness?

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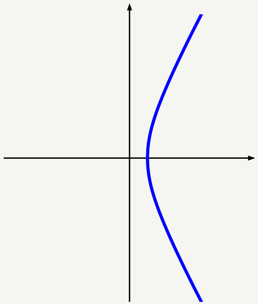
$$f : \mathbb{C} \longrightarrow \mathbb{C}^2$$
$$t \longmapsto \big(t^2+1, t(t^2+1)\big)$$

$f(\mathbb{R}) \subsetneq f(\mathbb{C}) \cap \mathbb{R}^2$



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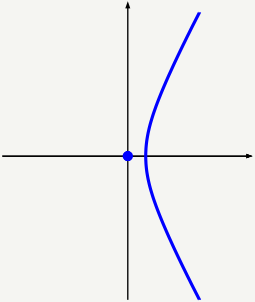
Indeed: $f(i) = f(-i) = (0,0)$



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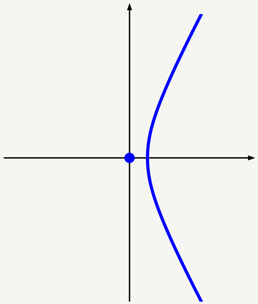
$$y^2 = x^2(x-1)$$

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C}^2 \end{array}$$



Approximating by smooth rational curves

$\mathcal{B}_X \subset \mathcal{A}_X \subset \mathcal{C}^\infty(S^1, X)$:= subset of smooth rational curves $\mathbb{P}^1(\mathbb{R}) \to X$

Definition

$f \in \mathcal{C}^\infty(S^1, X)$ is approximated by smooth rational curves
$\Leftrightarrow$
$f \in \overline{\mathcal{B}_X}$.

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Corollary

Let X be a nonsingular real rational variety, then any embedded circle is approximated by smooth rational curves.

Real rational surfaces

Theorem (Comessatti, 1914)

X orientable nonsingular real rational surface
$\Rightarrow X$ diffeomorphic to the sphere S^2 or to the torus $S^1 \times S^1$

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Conversely:
$S^2 \sim$ rational model $\{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$
$S^1 \times S^1 \sim$ rational model $\{x^2 + y^2 = z^2 + t^2 = 1\} \subset \mathbb{R}^4$
$\mathbb{RP}^2 \sim$ rational model $\mathbb{P}^2(\mathbb{R})$
$\#^h \mathbb{RP}^2 \sim$ rational model $B_{p_1, p_2, \ldots, p_{h-1}} \mathbb{P}^2(\mathbb{R})$ (blow-up at $h-1$ points)

Classification of rational models

$S^1 := \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 = 1\}$
Real algebraic manifold := compact connected submanifold of $\mathbb{R}^n$ defined by real polynomial equations, for some n.
X, Y real algebraic manifolds, $f\colon X \to Y$ map
f algebraic := (i) real rational (ii) defined $\forall x \in X$
f isomorphism := (i) algebraic, (ii) f^{-1} exists (iii) f^{-1} algebraic

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(Implicit function Theorem does not hold in the algebraic setting.)

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Theorem (Biswas, Huisman, 2007)

Two nonsingular real rational surfaces are isomorphic if and only if they are diffeomorphic.

Real (-1)-curves

Let $L \subset X$ be a real algebraic curve on a real algebraic surface

Definition

L is a (-1)-curve iff
$\exists$ birational morphism $\pi \colon X \to Y$ such that $\pi(L)$ is a smooth point on Y and π restricted to $X \setminus L \to Y \setminus \pi(L)$ is an isomorphism.

By Casteluovo's criterium, $\exists$ such a birational morphism $\pi \colon X \to Y$ iff there exists a real algebraic surface X' and a real algebraic isomorphism $\Phi \colon X \to X'$ such that $L' := \Phi(L)$ is rational, irreducible and nonsingular and $L' \cdot L' = -1$ (self-intersection over complex points).

Approximating by (-1)-curves

Theorem

X nonsingular real rational surface and $L \subset X$ a nonsingular curve, the following assertions are equivalent:

1. *X is nonorientable near L and one of the following is satisfied:*
   $X \setminus L$ is a punctured sphere, or
   $X \setminus L$ is a punctured torus, or
   $X \setminus L$ is nonorientable.
2. *L is homotopic to a (-1)-curve*
3. *L admits $\mathcal{C}^\infty$-approximation by (-1)-curves*

Proof of the approximation by smooth rational curves

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4. The rest of the talk is devoted to deduce the approximation result!

Density of Aut(X)

Recall: $f : X \to X$ automorphism $\Leftrightarrow$
(i) f birational map, (ii) f is a self-diffeomorphism on the real locus
$\mathrm{Aut}(X)$:= group of real algebraic automorphisms $X \to X$
Remark: let $V|_{\mathbb{R}}$ such that $V(\mathbb{R}) = X$, then
$\mathrm{Aut}_{\mathbb{R}}(V) \subset \mathrm{Aut}(X) \subset \mathrm{Bir}_{\mathbb{R}}(V)$

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Theorem (Kollár, M. 2009)

- *$S = S^2$, $S^1 \times S^1$, or any non-orientable surface,*
  $\Rightarrow \exists$ real algebraic model $X \sim S$ such that $\overline{\mathrm{Aut}(X)} = \mathrm{Diff}(X)$ for the C^∞-topology.

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- *S any orientable surface of genus ≥ 2,*
  $\Rightarrow \forall$ model $X \sim S$, $\mathrm{Aut}(X)$ is not dense in $\mathrm{Diff}(X)$, even for the C^0-topology.

Cremona transformation (around 1860)

On $\mathbb{P}^3$ take $(x : y : z : t) \mapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{t}\right) = (yzt : ztx : txy : xyz)$

Base locus = 6 edges of a tetraedron T.
Move vertices to $(1, \pm i, 0, 0), (0, 0, 1, \pm i)$, get:
$\sigma\colon (x : y : z : t) \mapsto \big((x^2 + y^2)z : (x^2 + y^2)t : (z^2 + t^2)x : (z^2 + t^2)y\big)$

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Each quadric
$Q_{abcdef} := a(x^2 + y^2) + b(z^2 + t^2) + cxz + dyt + ext + fyz$
(i) passes through the vertices of T,
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$\sigma\colon Q_{abcdef}(\mathbb{R}) \xrightarrow{\cong} Q_{abcdfe}(\mathbb{R})$

Action on spheres

$S^2 := \{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+z^2=1\}$
$Q_0 := \{(x,y,z,t) \in \mathbb{P}^3,\ x^2+y^2+z^2-t^2=0\}$

Take Q_{abcdef} with $Q_{abcdef}(\mathbb{R}) \sim S^2$, $\Rightarrow Q_{abcdfe}(\mathbb{R}) \sim S^2$,
then both are equivalent to Q_0 up to linear change of coordinates.
Get: $\sigma_{abcdef} : S^2 \xrightarrow{\cong} S^2$, well defined up to $O(3,1)$.

Theorem

The Cremona transformations with imaginary base points and $O(3,1)$ *generate* $\mathrm{Aut}(S^2)$ *which is dense in* $\mathrm{Diff}(S^2)$.

Theorem (Lukackiĭ 1977)

$SO(m+1,1)$ *is a maximal closed subgroup of* $\mathrm{Diff}_0(S^m)$.

Rational models of non-orientable surfaces: ($\chi(R_g) = 2 - g$)
$R_g \sim B_{p_1,\dots,p_g} S^2$, the sphere blown-up at g points
Let $q_1, \dots, q_n \in R_g$ n distinct points (n can be zero.)

Theorem

$\mathrm{Aut}(R_g, q_1, \dots, q_n)$ *is dense in* $\mathrm{Diff}(R_g, q_1, \dots, q_n)$ *in the* C^∞*-topology on* R_g.

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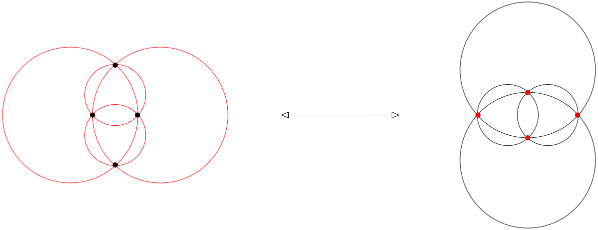
Steps of the proof:

1. Marked points
   [Huisman, M. 2007: $\mathrm{Aut}(S^m)$ acts ∞-transitively on S^m, $\forall m > 1$]
   $\Rightarrow \mathrm{Aut}(S^2, p_1, \dots, p_{g+n})$ is dense in $\mathrm{Diff}(S^2, p_1, \dots, p_{g+n})$ for any finite set of distinct points $p_1, \dots, p_{g+n} \in S^2$.
2. Identity components
   [Fragmentation Lemma]
   $\Rightarrow \mathrm{Aut}_0(R_g, q_1, \dots, q_n)$ is dense in $\mathrm{Diff}_0(R_g, q_1, \dots, q_n)$.
3. Mapping class group
   $\mathrm{Aut}(R_g, q_1, \dots, q_n)$ surjects to $\mathcal{M}(R_g, q_1, \dots, q_n)$.

Cremona transformation with real base points



Factored as:

$S^2 \longleftarrow B_{p_1,\ldots,p_4} S^2 \cong B_{q_1,\ldots,q_4} S^2 \longrightarrow S^2$

Proposition

Cremona transformations act transitively on isotopy classes of g disjoint Möbius bands in R_g.

Cremona σ: $B_{p_1,\ldots,p_4} S^2 \cong B_{q_1,\ldots,q_4} S^2$, $\exists \Phi \in \mathrm{Aut}(S^2)$ such that $\Phi(p_i) = q_i$, get $\Phi \circ \sigma$:

$B_{p_1,\ldots,p_4} S^2 \xrightarrow{\sigma} B_{q_1,\ldots,q_4} S^2 \xrightarrow{\Phi} B_{p_1,\ldots,p_4} S^2$

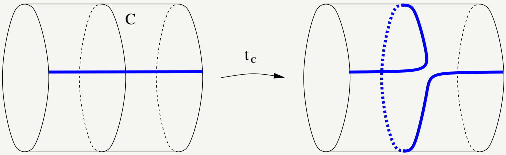
The mapping class group

R smooth compact surface

$\mathcal{M}(R, q_1, \ldots, q_n) := \pi_0(\mathrm{Diff}(R, q_1, \ldots, q_n))$

Theorem (Dehn 1938)

When R orientable, $\mathcal{M}$ is generated by Dehn twists around simple closed curves:



Theorem

When R non-orientable, Dehn twists generate an index 2 subgroup of $\mathcal{M}$, need to add cross-cap slides.

Reduction of the set of generators

Chillingworth (1969), and Korkmaz (2002) with base points
Recall $R_g = B_{p_1,\dots,p_g} S^2$

Theorem

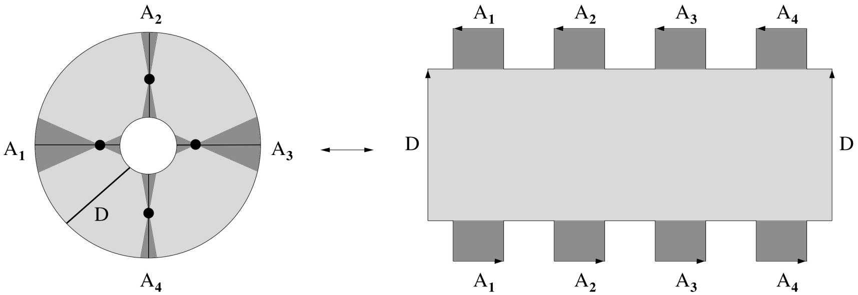
Dehn twists around lifts of simple closed curves of S^2 passing through an even number of the p_i (no self-intersection at the p_i) suffice.

With lantern relation $\Rightarrow$

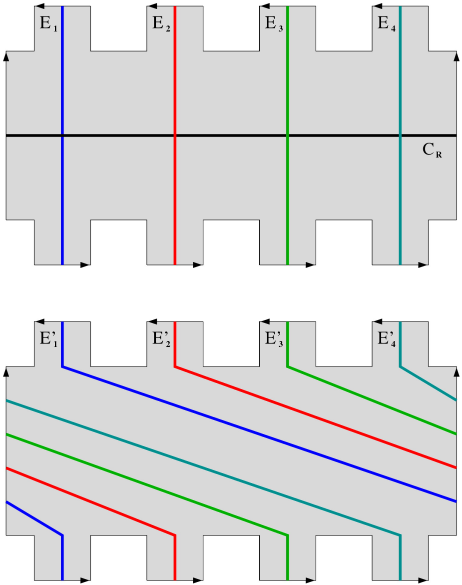
Corollary

Dehn twists around lifts of simple closed curves of S^2 passing through 0, 2 or 4 of the p_i suffice.

Two models of the annulus blown up in 4 points



The 4 exceptional curves and Dehn twist around C_R



Deformation

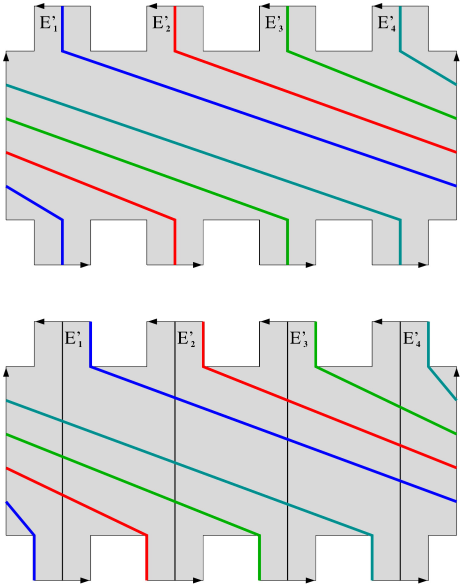
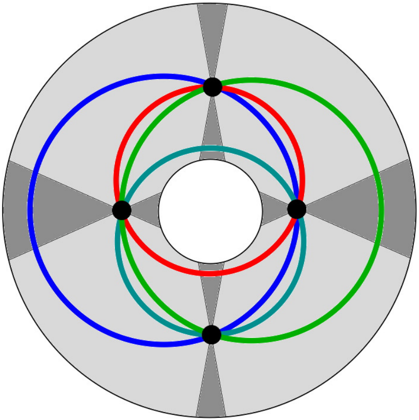


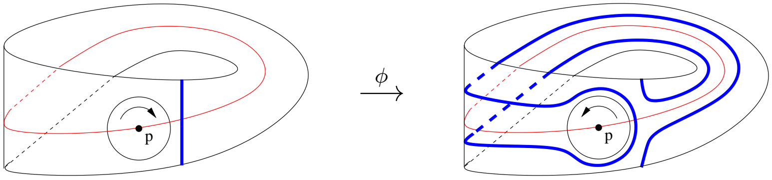
Image of the four exceptional curves



Cremona with 4 real base points represents the Dehn twist around C_R passing through the 4 base points.

Cross-cap slide ($g \geq 2$)

Let $R_g = B_{q,p,p_3,\dots p_g} S^2$, and $D \subset S^2$ a disc containing q, p and none of the other points.
Consider the Möbius band $B_q D$ and slide a small disc around red curve:



Then glue to $B_q S^2 \setminus B_q D$.
Realized by Cremona with 2 real base points.

Theorem

For any g, the Cremona transformations with $4, 2$ or 0 real base points generate the (non-orientable) mapping class group $\mathcal{M}_g$.

Orientable surfaces

Let X be an orientable real algebraic surface.

1. $\mathbb{C}X$ rational or ruled $\Rightarrow X \sim S^2$, or $X \sim S^1 \times S^1$ (Comessatti, 1914);
2. $\mathbb{C}X$ K3 or abelian $\Rightarrow \mathrm{Aut}(X)$ preserves a volume form;
3. $\mathbb{C}X$ elliptic $\Rightarrow \mathrm{Aut}(X)$ preserves fibration;
4. $\mathbb{C}X$ general type $\Rightarrow \mathrm{Aut}(X)$ is finite.

Generalization of the ∞-transitivity on spheres

Theorem (Blanc, Mangolte 2009)

Let X be a nonsingular real projective surface. Then $\mathrm{Aut}(X)$ has a very transitive action on X if and only if the following holds:

1. *X is geometrically rational, and*
2. $\#X \leq 2$, *or*
   $\#X = 3$, *and there is no pair of homeomorphic connected components, or*
   $\#X = 3$ *for a few other particular X.*

Furthermore, when $\mathrm{Aut}(X)$ is not very transitive, it is not even 2-transitive.

Generalization of the density of Aut $\subset$ Diff

Up to this point, the question of density is left open only for some geometrically rational surfaces with 2, 3, 4 or 5 connected components. The following result deals with the non-density for most of the surfaces with at least 3 connected components

Proposition (Blanc, Mangolte 2009)

Let X be a geometrically rational surface.

- *If $\#X \geq 5$, then $\mathrm{Aut}(X)$ is not dense in $\mathrm{Diff}(X)$;*
- *if $\#X = 3$ or $\#X = 4$, then $\mathrm{Aut}(X)$ is not dense in $\mathrm{Diff}(X)$ for a general X, but could be dense in some special cases;*
- *if $\#X = 2$, the density of $\mathrm{Aut} \subset \mathrm{Diff}$ remains an open question*