Log Canonical Thresholds cats and tigers (arXiv:1203.0995)

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The Calabi Problem

- Any complex projective manifold X has a Kähler metric g (for instance, restrict the Fubini-Study metric of $\mathbb{P}^n$).
- Can this metric be chosen to have constant Ricci tensor? This means:

$$\mathrm{Ric}(g) = k \cdot g$$

- **Eugenio Calabi** asked a similar question in 1954. Consider three cases
  - $c_1(X) < 0$ (General type): proved by Aubin and Yau (1976)
  - $c_1(X) = 0$ (Calabi Yau manifolds): proved by Yau (1978).
  - $c_1(X) > 0$ (Fano manifolds): not always. Classification open.

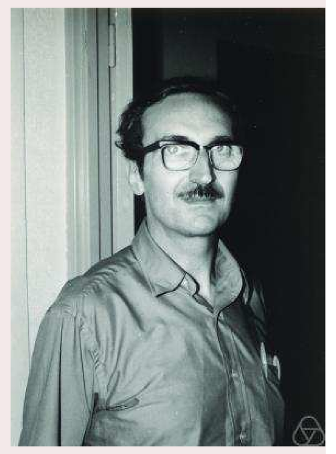

Figure: Eugenio Calabi (source: Wikipedia).

I'm a birationalist . Take me out of here!

- This is *really* an algebraic problem. Enter the hero: the **log canonical threshold** of X. Let X be a normal variety and D effective$\mathbb{Q}$-divisor.
- Given a pair (X, D) , define

$$\mathrm{lct}(X, D) = \sup\{\lambda \mid (X, \lambda D) \text{ is log canonical}\}.$$

- Example: $\mathrm{lct}(= \mathbb{A}^2, \{x^2 - y^3 = 0\}) = \frac{5}{6}$.

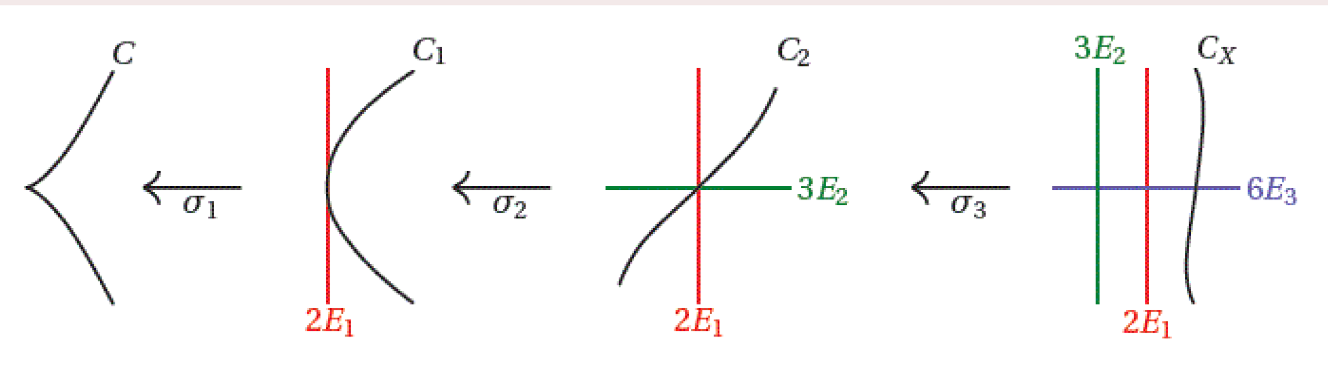

Figure: Resolution of a cusp. Picture by Andrew Wilson.

- The existence of metrics is a global problem. We need a **global** version:

$$\mathrm{lct}(X) = \inf\{\mathrm{lct}(X, D) \mid \forall D \sim_{\mathbb{Q}} -K_X\}.$$

- If X is Fano, Demailly and Kollar showed that $\mathrm{lct}(X) = \alpha(X)$, **Tian's α invariant**.

Theorem (Tian; Demailly, Kollár; Nadel)

Let X be a Fano variety with quotient singularities such that the inequality

$$\mathrm{lct}(X) > \frac{\dim(X)}{\dim(X) + 1}$$

holds. Then X has an orbifold Kähler–Einstein metric.

Running away from differential geometry

- The theorem is not sharp: $\mathrm{lct}(\mathbb{P}^2) = \frac{1}{2}$ but $\mathbb{P}^2$ is Kähler-Einstein.
- Possible **algebraic** characterisation of Kähler-Einstein: K-stability.
- It is known: $\mathrm{lct}(X) > \frac{\dim X}{\dim X + 1} \Rightarrow X$ is KE $\Rightarrow X$ is K-stable.
- It is conjectured (Donaldson's program): K-stability $\Rightarrow$ $\exists$ KE.
- Problem: K-stability is long to define, hard to prove and difficult to use.
- If $\mathrm{lct}(X)$ is algebraic and K-stability is algebraic... do we need KE?

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Theorem (Odaka, Sano (2010))

Assume resolution of singularities and let X be a $\mathbb{Q}$-Fano variety of dimension n over $\bar{k} = k$ and suppose that $\mathrm{lct}(X) > \frac{n}{n+1}$ (resp. $\mathrm{lct}(X) \geq \frac{n}{n+1}$). Then, $(X, \mathcal{O}_X(-K_X))$ is K-stable (resp. K-semistable).

Are del Pezzo surfaces K-stable?

- Cheltsov (2007) computed $\mathrm{lct}(X)$ for complex smooth del Pezzos.
- His proof is mostly algebraic, but uses $k = \mathbb{C}$ when $K_X^2 = 2, 3, 4$.
- What about $\mathrm{char}(k) > 0$?

Theorem (fragment)

Let X be a nonsingular del Pezzo surface over an algebraically closed field k. Then:

$$\mathrm{lct}(X) = \begin{cases} 1 \text{ when } K_X^2 = 1 \text{ and } |-K_X| \text{ has no cuspidal curves} \\ 5/6 \text{ when } K_X^2 = 1 \text{ and } |-K_X| \text{ has a cuspidal curve} \\ 5/6 \text{ when } K_X^2 = 2 \text{ and } |-K_X| \text{ has no tacnodal curves} \\ 3/4 \text{ when } K_X^2 = 2 \text{ and } |-K_X| \text{ has a tacnodal curve} \\ 2/3 \text{ when } K_X^2 = 4 \end{cases}$$

In particular all these are K-semistable.

Proof 1/2 (idea $K_X^2 = 4$)

- Find *worst* effective divisor $D_0 = L_1 + L_2 + C \sim -K_X$ as in last blow-up of the cusp resolution: $\mathrm{lct}(X, D_0) = 2/3$.
- Suppose $(X, \frac{2}{3}D \sim_{\mathbb{Q}} \frac{2}{3}(-K_X))$ not log canonical at some $p \in X$. Then $\mathrm{mult}_p(D) > 3/2$.
- Let $\pi : \widetilde{X} \to X$ be the blow-up of p with exceptional curve E. It can be shown that at some $q \in E$:

$$\frac{2}{3}(\mathrm{mult}_p(D) + \mathrm{mult}_q(\widetilde{D}) > 2.$$

Lemma (Convexity)

Given X non-singular and $D \sim_{\mathbb{Q}} B = \sum b_i B_i$ be effective such that (X, B) is log canonical but (X, D) is not, we can choose $\alpha \in [0, 1) \bigcap \mathbb{Q}$ such that $\exists B_i \subset \mathrm{Supp}(B)$ irreducible with $B_i \not\subset \mathrm{Supp}(D')$ for

$$D' = \frac{1}{1 - \alpha}(D - \alpha B) \sim_{\mathbb{Q}} D$$

such that $(X, D')$is not log canonical and D' is effective.

Lemma (Auxiliary divisors)

Let $p \in X$, $q \in \widetilde{X}$. There is $H = \sum j_i J_i \sim_{\mathbb{Q}} -K_X$, an effective $\mathbb{Q}$-divisor such that:

(i) $\frac{2}{3}H$ *is log canonical.*
(ii) $p \in J_i \; \forall J_i$ *and* $q \in \widetilde{J}_i$, $\forall J_i$ *such that* $-K_X \cdot J_i > 1$.
(iii) $\deg J_i \leq 3 \; \forall J_i$.
(iv) *All J_i are irreducible.*

Proof 2/2 (idea $K_X^2 = 4$)

These divisors are constructed from curves in $\mathbb{P}^2$ case by case ($\deg C = -K_X \cdot C$):

- $p \notin$ a line, $q \notin$ a conic.
- $p \notin$ a line, $q \in$ a conic...

Using convexity we know $\exists J_j \not\subset \mathrm{Supp}(D)$, and get a contradiction:

$$3 - \mathrm{mult}_p D \geq J_j \cdot D - \mathrm{mult}_p D = \widetilde{J}_j \cdot \widetilde{D} \geq \mathrm{mult}_p D.$$

Conjecture (Stabilisation of lct)

For X log terminal Fano, the equality

$$\mathrm{lct}(X) = \mathrm{lct}(X, D)$$

holds for some effective $\mathbb{Q}$-divisor $D \equiv -K_X$ on X.

Cats and Tigers

Keel and McKernan defined a **tiger** of index m to be a divisor $D \sim -mK_X$ such that $(X, \frac{1}{m}D)$ is not Kawamata log terminal. Suppose the previous conjecture is true. Consider m_0 minimal such that $\mathrm{lct}(X) = \mathrm{lct}(X, \frac{1}{m_0}D_0)$. Any tiger D of index $m_0 \leq m$ is a **cat**.

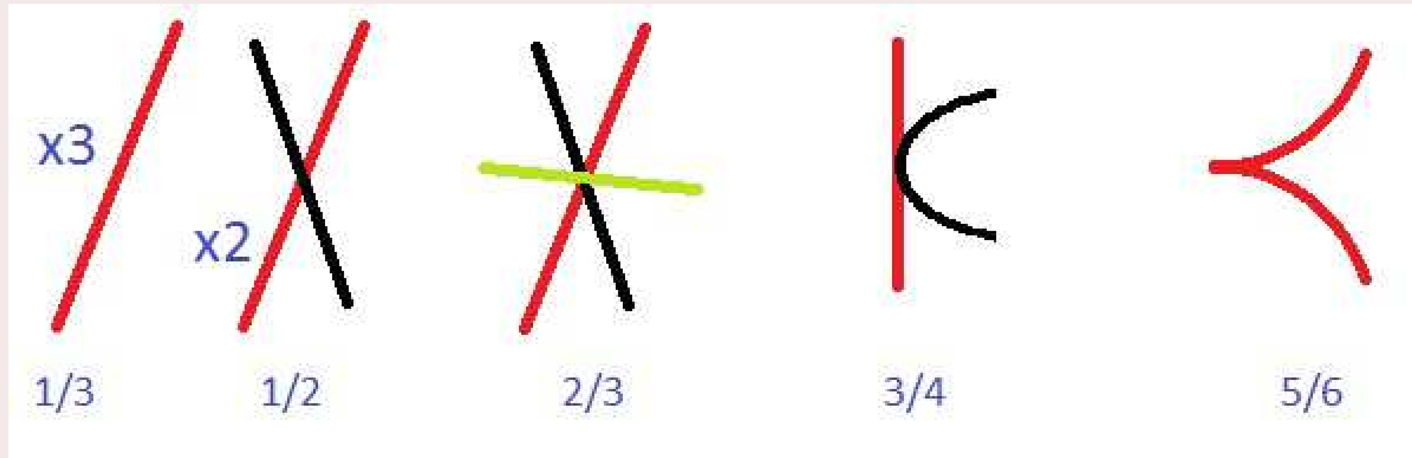

Figure: Cats of $\mathbb{P}^2$ and their lct.

Conjecture (All tigers have a cat)

Every tiger D on a log terminal surface X contains a cat in its support.

Comments

- In all known cases $m_0 \leq 2$. First conjecture verified in many cases.
- Tigers are obstructions to Kähler-Einstein metrics. If the conjecture is true, the study of obstructions becomes simpler.
- If the conjectures are true it makes sense to classify cats to find all tigers.
- Cat Conjecture true for $K_X^2 = 1, 2$ (smooth). In higher degree it is harder since the number of cats grow.
- To create very singular curves $D \sim -mK_X$ we need big m, but then discrepancy of $(X, \frac{1}{m}D)$ drops.

Acknowledgments

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