On complex-analytic and homological rigidity of smooth Schubert cycles on rational homogeneous spaces of Picard number 1

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Abstract: In a series of works with J.-M. Hwang we have developed a geometric theory of uniruled projective manifolds basing on the study of varieties of minimal rational tangents (VMRTs), which are roughly speaking tangents to minimal rational curves. When X is of Picard number 1, the idea that geometric properties at a general base point can be propagated by means of the adjunction of minimal rational curves permeates throughout the geometric theory. For instance, in the proof of Cartan-Fubini extension (in the sense that a germ of VMRT-preserving local biholomorphisms between Fano manifolds of Picard number 1 satisfying some mild conditions necessarily extend to a biholomorphism), analytic continuation is first of all performed over families of minimal rational curves emanating from base points which lie in a small open set and completed by the successive adjunction of families of minimal rational curves. Methods of the geometric theory basing on VMRTs have especially been applied to rational homogeneous manifolds $X = G/P$ of Picard number 1 to yield various rigidity results such as a solution to the Lazarsfeld problem and results on rigidity under algebraic deformation.

A projective subvariety S of X is said to be saturated if minimal rational curves tangent to Z must necessarily lie on S and if S is uniruled by such minimal rational curves. This is the case whenever X is a rational homogeneous manifold of Picard 1 and $Z \subset X$ is a homogeneous submanifold defined by a subdiagram of the marked Dynkin diagram associated to X. In recent works of the speaker with J. Hong we have embarked on a study of saturated subvarieties of uniruled projective manifolds (of Picard number 1). A method of analytic continuation has been devised to yield a non-equidimensional analogue of Cartan-Fubini extension for germs of holomorphic immersions between uniruled projective manifolds of Picard number 1 under the assumption that VMRTs are sent to linear sections of VMRTs and under a mild geometric condition expressible in terms of second fundamental forms on linear sections of VMRTs. Under these conditions, a VMRT-respecting germ of holomorphic immersion between germs of uniruled projective manifolds of Picard number 1 is proven to extend rationally to a map whose total transform is a saturated subvariety S of the target manifold X. For the case of homogeneous submanifolds $S \subset X$ of rational homogeneous spaces $X = G/P$ of Picard number 1, we derive a theorem of characterization of $S \subset X$ in terms of VMRTs whenever the maximal parabolic P corresponds to a long simple root and $S \subset X$ is non-linear. In the proof we have developed a notion of parallel transport of VMRTs along minimal rational curves, a geometric notion which has turned out to be very useful in problems of characterization of certain projective submanifolds, e.g., in a recent work of Hong-Mok yielding homological characterization of certain smooth Schubert varieties of $X = G/P$.