ISOMETRIES BETWEEN OPEN SETS OF CARNOT GROUPS AND GLOBAL ISOMETRIES OF SUBFINSLER HOMOGENEOUS MANIFOLDS

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Abstract. We show that isometries between open sets of Carnot groups are affine. This result generalizes a result of Hamenstädt. Our proof does not rely on her proof. In addition, we study global isometries of general homogeneous manifolds equipped with left-invariant subFinsler distances. We show that each isometry is determined by the blow up at one point. For proving the results, we consider the action of isometries on the space of Killing vector fields. We make use of results by Capogna-Cowling and by Gleason-Montgomery-Zippin for obtaining smoothness of the isometric action.

Contents

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1. Introduction

A fundamental problem in geometric analysis is the study of spaces that are isometrically homogeneous, i.e., metric spaces on which the group of isometries acts transitively. Such spaces have particular differentiable structures under the additional assumptions of being of finite dimension, locally compact, and the distance being intrinsic. Indeed, one can characterize such spaces as particular subFinsler manifolds, by using the theory of locally compact groups and methods from Lipschitz analysis on metric spaces, [Gle52, BM46, MZ74, Ber89a, Ber89b]. Despite the fact that the group of global isometries of such manifolds is a Lie group and acts smoothly and by smooth maps, the local isometries are still not completely understood. Here, with the term 'local isometry' we mean isometry between open subsets.

In this paper we give a complete description of the space of local isometries for those homogeneous spaces that also admit dilations. These spaces, called Carnot groups, are particular nilpotent groups equipped with general left-invariant geodesic distances.

Our method of proof also shows that, as in Riemannian geometry, global isometries of homogeneous spaces are uniquely determined by their 'first-order' expansion at a point. Such a characterization for isometries was known already in some specific cases, e.g., for Riemannian manifolds, Banach spaces, and subRiemannian Carnot groups.

The study of isometries of distinguished Riemannian manifolds, such as homogeneous spaces, symmetric spaces, and Lie groups, has been a flourishing subject. References for the regularity of isometries are the classical papers [MS39, Pal57b], see also [CH70, Tay06]. For the general theory of transformations groups, we refer to [Pal57a, Kob95, CE80]. Regarding the Finsler category, we mention the work [DH02]. Banach spaces are classical and the space of isometries is well studied, see [FJ93, BL00]. There has been some effort in understanding isometries of subRiemannian manifolds, see [Str86, Str89, Ham90, Kis03, Hla12]. For sub-Finsler homogeneous spaces, we refer to [Ber88, Ber89a], see also [CM06]. Regarding Carnot groups, in the deep paper [Ham90], U. Hamenstädt showed that isometries are affine, in the case that the isometry is globally defined and that the distance is subRiemannian (and not just subFinsler). We say that an isometry of a group (equipped with a left-invariant distance) is affine if it is the composition of a left translation with a group isomorphism.

We generalize Hamenstädt's result to the setting of a subFinsler distance and isometries defined only on some open set. We need to point out that, first, to obtain such a local result, one cannot use the same argument as in [Ham90] to deduce smoothness of the map. Actually, the issue of smoothness was a subtle point, which was clarified only later by I. Kishimoto in [Kis03], for global isometries. Moreover, in Hamenstädt's strategy, one needs to consider a blow down of the isometry, which requires the map to be globally defined. Hence, we shall provide a new method of proof.

1.1. **Statements of main results.**

Let G be a Lie group and H be a closed subgroup. Let $M = G/H$ be the homogeneous manifold of left cosets. Hence, the group G acts transitively on M on the left. Let Δ be a G-invariant subbundle of the tangent bundle TM. We assume that Δ is bracket-generating and call it *horizontal bundle*. Fix a norm on Δ_p, for $p \in M$, and assume that it is G-invariant. Then the Carnot-Carathéodory distance between two points of the manifold is the infimum of the lengths of curves tangent to Δ and connecting the two points. Since the length is measured using the norm, such a distance is also called *subFinsler*. If $G = M$ and, setting $V_1 := \Delta_e$ and $V_{j+1} := [V_1, V_j]$, we have the property that

$$\mathrm{Lie}(G) = V_1 \oplus \cdots \oplus V_s,$$

then the space is called *subFinsler Carnot group*. See Section 2 for more detailed definitions, notation, and properties of such spaces.

Our first theorem characterizes local isometries of a subFinsler Carnot group as affine maps.

Theorem 1.1. *Let $(\mathbb{G}, d)$ be a subFinsler Carnot group. Let $\Omega_1, \Omega_2 \subset \mathbb{G}$ be two open sets. Let $f : \Omega_1 \to \Omega_2$ be an isometry. Then there exists a left translation τ and a group isomorphism ϕ of $\mathbb{G}$ such that f is the restriction to Ω_1 of $\tau \circ \phi$, which is an isometry.*

Note that in the statement above we require the domain Ω_1 to be open. In Section 3.4, we shall see that such an assumption is necessary, unlike in the Euclidean case. However, connectivity is not required.

With a similar method as for the proof of the first theorem, we show that global isometries of a subFinsler homogeneous space are characterized by their value at one point and the differential at this point. In [Str86, Str89], the same conclusions were obtained for smooth isometries of particular subRiemannian manifolds.

Theorem 1.2. *Let $M = (G/H, d)$ be a connected homogeneous manifold equipped with a G-invariant subFinsler distance with horizontal subbundle Δ. Let $f : M \to M$ be an isometry. Then f is an analytic map. Moreover, if $h : M \to M$ is another isometry with the properties that $f(p) = h(p)$ and $(df)_p|_{\Delta_p} = (dh)_p|_{\Delta_p}$, for some $p \in M$, then $f = h$.*

Even under the additional assumption that the manifold in Theorem 1.2 is actually a Lie group, one cannot have the same conclusions as in Theorem 1.1. Indeed, for general Lie groups, it is not true that isometries are necessarily affine maps. See the discussion on the inversion map on $\mathbb{S}^3$ in Section 3.4. We should also point out that, in general, local isometries of homogeneous spaces do not extend to global isometries, e.g., in the case of the flat cylinder (see Section 3.4).

By [MM95], every quasiconformal mapping between two Carnot-Carathéodory spaces admits a blow-up map at almost every point, that is an isomorphism between two Carnot groups. If the two Carnot-Carathéodory spaces are Carnot groups, such a result was proven in [Pan89]. For this reason, the blow-up map is also called Pansu differential. Whenever f is an isometry of M, as in Theorem 1.2, then f is smooth and $df(p)|_{\Delta_p}$ coincides with the differential of the blow up at p, restricted to Δ_p. In other words, Theorem 1.2 states that every isometry f of M is determined by $f(p)$ and by its blow-up at p.

We complete the introduction with a problem.

Question 1.3. Given a connected and simply connected Lie group (G, d) equipped with a left-invariant subFinsler distance, let $\mathbb{G}$ be the tangent cone of G at e. For which group automorphism ϕ of $\mathbb{G}$ there exists an isometry of G that has ϕ as blow up at the identity?

There are obvious necessary conditions. Namely, the differential of ϕ should preserves the strata and be an isometry when restricted to the first stratum. However, these conditions are not sufficient, see the discussion on the Riemannian Heisenberg group in Section 3.4.

1.2. **Structure of the paper.** The proof of our results is divided into several steps. The overall strategy of the two theorems is similar. However, in few instances we need different approaches. Regarding global isometries of subFinsler homogeneous manifolds, we show their smoothness in Section 2.2, relying on the classical solutions of the Hilbert 5^{th} problem. This method works only for mappings that are globally defined. For the local isometries of Theorem 1.1 we shall rely on a regularity result of 1-quasiconformal mappings in [CC06], that we reformulate in Theorem 2.8.

Once we know that isometries are smooth, in Section 2.3 we consider the action of their differentials on vector fields that generate flows of isometries (Killing vector fields). For local isometries, we need an extension result for Killing vector fields that is proven in Section 2.4 and that relies on a method developed in [Tan70]. In Section 3.1 we prove Theorem 1.2. In Section 3.2 we provide a weak version of Theorem 1.1. Namely, in Theorem 3.3, we assume that Ω_1 is connected and that the distance is subRiemannian. In Section 3.3 we complete the proof of Theorem 1.1. We then devote Section 3.4 to a number of final remarks.

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2. Preliminaries

2.1. **General notation.** Let G be a Lie group. Denote by $\mathfrak{g}$ or by $\mathrm{Lie}(G)$ the Lie algebra of G whose elements are tangent vectors at the identity e of G. For $Y \in \mathfrak{g}$, we denote by $\tilde{Y}$ the left-invariant vector field that coincides with Y in e. So $[X,Y] := [\tilde{X},\tilde{Y}]_e$.

Let H be a closed subgroup of G. Hence, the space G/H of left cosets gH, with $g \in G$, has a natural structure of analytic manifold, see [Hel01, page 123]. The group G is a Lie transformation group of $M = G/H$. Namely, every element $g \in G$ acts by left translations on M, i.e., induces the diffeomorphism

$$g'H \mapsto g \cdot (g'H) := gg'H. \tag{2.1}$$

For $Y \in \mathfrak{g}$, we denote by $Y^\dagger$ the vector field of M whose flow $\Phi^t_{Y^\dagger}$ at time t is

$$\Phi^t_{Y^\dagger}(p) = \exp(tY) \cdot p, \qquad \forall p \in M. \tag{2.2}$$

It is well known (see [Hel01, Theorem 3.4]) that, for $X, Y \in \mathfrak{g}$, we have

$$[X,Y]^\dagger = -[X^\dagger, Y^\dagger]. \tag{2.3}$$

We shall fix a G-invariant subbundle Δ of the tangent bundle TM of M. The choice of such a subbundle can be seen in the following way. In the homogeneous manifold M we denote by o the coset H and call it the *origin* of M. Notice that the action of H on M fixes the origin. There is a one-to-one correspondence between H-invariant subspaces Δ_o in $T_o(M)$ and Ad_H-invariant subspaces V in $\mathrm{Lie}(G)$ that contains $\mathrm{Lie}(H)$. We choose such a subspace Δ_o in $T_o(G/H)$, and therefore, such a $V \subseteq \mathfrak{g}$. Then, for all $gH \in G/H$, the subbundle Δ is defined as

$$\Delta_{gH} := g_* \Delta_o,$$

where g_* is the differential of the map in (2.1). The subbundle is well defined, i.e., the definition does not depend on the representative in gH, exactly because Δ_o is H-invariant.

If the subspace $V \subset \mathfrak{g}$ associated to Δ_0 has the property that $\mathfrak{g}$ is the smallest Lie subalgebra of $\mathfrak{g}$ containing V, then V (or, equivalently, Δ) is said to be *bracket-generating*.

We shall fix a G-invariant norm on Δ. Notice that there are cases where norms with the aforementioned property do not exist. The choice of such a norm can be seen in the following way. Fix a seminorm on V that is Ad_H-invariant and for which the kernel is $\mathrm{Lie}(H)$. The projection from G to M gives an H-invariant norm $\|\cdot\|$ on Δ_o. Hence, we have an induced G-invariant norm on Δ by

$$\|v\| = \left\|(g^{-1})_* v\right\|, \qquad \forall v \in \Delta_{gH}.$$

Since the initial norm is Ad_H-invariant, it follows that the above equation is independent on the choice of representative in gH.

An absolutely continuous curve $\gamma : [0,1] \to M$ is said to be *horizontal* (with respect to Δ) if the derivative $\dot{\gamma}(t)$ belongs to Δ, for almost every $t \in [0,1]$. Each horizontal curve γ has an associated length defined as

$$L(\gamma) := \int_0^1 \|\dot{\gamma}(t)\| \,\mathrm{d}t.$$

Definition 2.4 (SubFinsler homogeneous manifolds)**.** Let $M = G/H$ be a homogeneous space formed by a Lie group G modulo a closed subgroup H. We are given a bracket-generating G-invariant subbundle $\Delta \subseteq TM$ and a G-invariant norm $\|\cdot\|$ on Δ. Equivalently, we are given an Ad_H-invariant and bracket-generating subspace $V \subseteq \mathrm{Lie}(G)$, with $V \supseteq \mathrm{Lie}(H)$, and an Ad_H-invariant seminorm $\|\cdot\|$ on V whose kernel is $\mathrm{Lie}(H)$. The *subFinsler distance* (also known as Finsler Carnot-Carathéodory distance) between two points $p, q \in M$ is defined as

$$d(p,q) := \inf\{L(\gamma) \mid \gamma \text{ horizontal and } \gamma(0) = p, \gamma(1) = q\}. \tag{2.5}$$

We call the pair (M, d) a subFinsler homogeneous manifold.

By Chow's Theorem [Mon02, Chapter 2], the topology of (M, d) is the topology of M as manifold. Notice that, by construction, the above subFinsler distance is left-invariant, i.e., every left translation (2.1) is an isometry of (M, d). Since subFinsler homogeneous manifolds are locally compact and isometrically homogeneous, they are complete metric spaces. Hence, by Hopf-Rinow Theorem, they are proper spaces, i.e., boundedly compact.

By the work of V. N. Berestovskiĭ, we know that the above-defined subFinsler homogeneous manifolds are the only metric spaces that are isometrically homogeneous, are locally compact, have finite topological dimension, and whose distance is a geodesic distance. Such a result is based on Montgomery-Zippin's characterization of Lie groups, see Theorem 2.9.

Theorem 2.6 (Consequence of [MZ74], [Ber89b], and [Mit85])**.** *Let X be a locally compact and finite-dimensional topological space. Assume that X is equipped with an intrinsic distance d such that its isometry group $\mathrm{Iso}(X, d)$ acts transitively on X. Then (X, d) is isometric to a subFinsler homogeneous manifold.*

If, moreover, the space (X, d) admits a non-trivial dilation, i.e., there exists $\lambda > 1$ such that $(X, \lambda d)$ is isometric to (X, d), then (X, d) is a subFinsler Carnot group (see definition below).

By the above result, subFinsler Carnot groups are special cases of subFinsler homogeneous manifold. We refer to [Mon02, page 38] for the easy proof that such spaces admit dilations, for all $\lambda > 1$.

Definition 2.7 (SubFinsler Carnot groups)**.** Given a subspace V_1 of the Lie algebra of a Lie group G, define by recurrence the subspaces V_j as

$$V_j := [V_1, V_{j-1}], \qquad \forall j > 1.$$

If one has that

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_k \oplus \cdots ,$$

then G is said to be a (nilpotent) *stratified group* and V_1 is called the *first stratum* (of the stratification $\{V_j\}$). If d is the subFinsler distance associated to G, V_1, and some norm $\|\cdot\|$ on V_1, then the pair (G, d) is called *subFinsler Carnot group*, or simply Carnot group.

If the norm in Definition 2.4 comes from a scalar product, then the associated distance is called *Carnot-Carathéodory* or *subRiemannian*. If this is the case for a subFinsler Carnot group, then we call it *subRiemannian Carnot group*. We shall use the notation $\mathbb{G}$, rather than G, to emphasize that we are dealing with a Carnot group, rather than a general Lie group.

One can show that a curve in a subFinsler manifold has finite length if and only if it is a horizontal curve, up to reparametrization. Consequently in the case that an isometry f of a subFinsler homogeneous manifold is C^1, then it is a *contact map*, i.e., its differential preserves the subbundle. Namely,

$$df_p(\Delta_p) \subseteq \Delta_{f(p)}, \qquad \forall p \in G.$$

Here and hereafter, we use both notations df or f_* to denote the differential of a differentiable map f, viewed as push-forward operator on vectors or on vector fields.

We shall need to show that isometries between open sets of a subRiemannian Carnot group are analytic maps. Such a regularity result follows from the work of Capogna and Cowling on 1-quasiconformal mappings. We state here a weaker form of their result ([CC06, Theorem 1.1]).

Theorem 2.8 (Consequence of [CC06])**.** *Let $\Omega \subseteq \mathbb{G}$ be an open set of a subRiemannian Carnot group $\mathbb{G}$. Let $f : \Omega \to \mathbb{G}$ be a biLipschitz embedding.*

(i) *If f is an isometry, then f is analytic.*
(ii) *If for a.e. $p \in \Omega$, the blow-up of f at p is an isometry of $\mathbb{G}$, then f is analytic.*

2.2. **Smoothness of the isometric action.** By definition, an isometry is a map that preserves the distance. Hence, there is no a priori smoothness assumption. In this section, we shall explain why, in the case of subFinsler homogeneous manifolds, one has in fact that (global) isometries are smooth maps forming a Lie group and the action is smooth. Smoothness of the action of local isometries between open sets of Carnot groups will follow from a different reasoning, see Corollary 2.18.

The smoothness of global maps is a consequence of the work of Gleason [Gle52] and Montgomery and Zippin [MZ52, MZ74]. In particular, one has the following general result.

Theorem 2.9 (Gleason-Montgomery-Zippin)**.** *If a second countable and locally compact group H acts by isometries, continuously, effectively, and transitively on a locally compact, locally connected, and finite-dimensional metric space X then H is a Lie group and X is a differentiable manifold.*

To obtain the regularity of the action in the group parameters we use the following theorem, which is a generalization of Bochner-Montgomery's result [BM45].

Theorem 2.10 ([MZ74, page 213])**.** *Let $H \times M \to M$, $(h, x) \mapsto h(x)$, be a (continuous) action of a Lie group H on an analytic manifold M. Assume that, for all $h \in H$, the map $x \mapsto h(x)$ is analytic. Then $h(x)$ is analytic in h and x simultaneously.*

For studying local isometries, we will make use of another result of Montgomery, see [Mon45a, Mon45b]. We obtain a Lie group structure for *compact* groups acting by analytic maps. The result holds more generally, see [MZ74, page 208, Theorem 2], but we only need the following weaker result.

Theorem 2.11 ([Mon45a, Theorem 3])**.** *If H is a compact effective group acting on a connected analytic manifold M and if each transformation of H is analytic then H does not contain arbitrarily small subgroups other than the identity; or, in other words, H considered in itself is a Lie group.*

From Theorem 2.9, we deduce the following consequence, which was partially observed in [Kis03] as well.

Corollary 2.12 (Consequence of Hilbert 5^{th} theory)**.** *Let G be a Lie group acting on an analytic manifold M. Assume that the action is transitive and analytic. Let d be a G-invariant and boundedly compact distance on M, inducing the manifold topology. Then the isometry group* $\mathrm{Iso}(M)$ *is a Lie group, the action*

$$\begin{array}{rcl} \mathrm{Iso}(M) \times M & \to & M \\ (f, p) & \mapsto & f(p) \end{array} \tag{2.13}$$

is analytic, and the space

$$\mathrm{Iso}_o(M) = \{f \in \mathrm{Iso}(M) \mid f(o) = o\}$$

is a compact Lie group.

Proof. By Ascoli-Arzelà Theorem we have that $\mathrm{Iso}(M)$ is locally compact and $\mathrm{Iso}_o(M)$ is compact (both equipped with the compact open topology). Obviously they both are groups with the composition as multiplication. Furthermore, since G acts transitively on M, so does $\mathrm{Iso}(M)$. Therefore, by Theorem 2.9 it follows that $\mathrm{Iso}(M)$ is a Lie group. Being a compact subgroup, $\mathrm{Iso}_o(M)$ is a Lie group as well.

For the proof that the action of $\mathrm{Iso}(M)$ on M is analytic, let us explicit the analytic structures considered. The group G and the manifold M are given with their analytic structures, which we denote ω_G and ω_M, respectively. Hence, by assumption, the action

$$(G, \omega_G) \times (M, \omega_M) \longrightarrow (M, \omega_M), \tag{2.14}$$

given by (2.1), is analytic. The group $\mathrm{Iso}(M)$ has an analytic structure ω_I of Lie group and, since it is acting transitively (and continuously) on M, there exists an analytic structure $\tilde{\omega}_M$ on M such that the map

$$(\mathrm{Iso}(M), \omega_I) \times (M, \tilde{\omega}_M) \longrightarrow (M, \tilde{\omega}_M), \tag{2.15}$$

given by (2.13), is analytic, see [Hel01, page 123]. Every element of G induces an isometry. Hence, we have a map

$$\iota : (G, \omega_G) \longrightarrow (\mathrm{Iso}(M), \omega_I), \tag{2.16}$$

induced by (2.1). The map ι is a continuous homomorphism. By [Hel01, Theorem 2.6] we have that ι is in fact analytic. By composition of (2.15) and (2.16), we have that

$$(G, \omega_G) \times (M, \tilde{\omega}_M) \longrightarrow (M, \tilde{\omega}_M), \tag{2.17}$$

again given by (2.1), is analytic. By [Hel01, Theorem 4.2] there is a unique analytic structure on M for which the action given by (2.1) is analytic. Therefore, we conclude that $\omega_M = \tilde{\omega}_M$. Hence, the map (2.13) is analytic when M is given the initial analytic structure ω_M. □

From Theorem 2.11 we deduce the following consequence, which will be important in the study of local isometries of Carnot groups.

Corollary 2.18 (Consequence of Hilbert 5^{th} theory). *Let B be a closed ball in a Carnot group centered at the identity e. Let $\mathcal{I} := \{f : B \to B \text{ isometry} \mid f(e) = e\}$. Then*

(i) *$\mathcal{I}$ is compact;*
(ii) *$\mathcal{I}$ is a Lie group;*
(iii) *for all $x \in B$ the map*

$$\begin{array}{ccc} \mathcal{I} & \to & B \\ f & \mapsto & f(x) \end{array}$$

is analytic.

Proof. Regarding part (i), since $\mathcal{I}$ is a family of equicontinuous and equibounded maps, by Ascoli-Arzelà Theorem, the family is precompact. Since $\mathcal{I}$ is obviously closed, then it is compact.

Part (ii) will follow from Theorem 2.11. Indeed, the set $\mathcal{I}$ is closed under composition. Hence it is a compact group acting on B. By Theorem 2.8, each element of $\mathcal{I}$ is an analytic map. Then Theorem 2.11 implies that $\mathcal{I}$ is a Lie group.

Part (iii) follows immediately from Theorem 2.10. □

2.3. **The action of an isometry on Killing vector fields.** Let now $M = (G/H, d)$ be a subFinsler homogeneous space with horizontal bundle Δ. In this section we define a filtration of the space of (global) vector fields that are infinitesimal generators of isometries of M. The properties of this space that are shown here are crucial to prove both our results.

Definition 2.19 (Killing vector fields $\mathcal{K}$). A vector field Z on M is said to be a *Killing vector field* if, for all $t \in \mathbb{R}$, the flow Φ_Z^t at time t is an isometry. We denote by $\mathcal{K}$ the collection of all Killing vector fields.

One can show that $\mathcal{K}$ is closed under sum and Lie bracket. Notice that if $Y \in V$, then the vector field $Y^\dagger$ is a Killing vector field. Indeed, the flows of these vector fields are one parameter groups of the left action of G (see (2.2)). The space $\mathcal{K}$ is the Lie algebra of the group of (global) isometries of M. Such a group, by Corollary 2.12, is a Lie group. Hence, $\mathcal{K}$ is a finite dimensional Lie algebra. We recall that isometries that are smooth are in particular contact maps. Therefore, given a Killing vector field Z with flow Φ_Z^t, the fact that Φ_Z^t is contact implies that

$$[Z, \Gamma(\Delta)] \subset \Gamma(\Delta), \tag{2.20}$$

where $\Gamma(\Delta)$ denotes the space of smooth sections of the subbundle Δ. Indeed, if $W \in \Gamma(\Delta)$, we have

$$[Z, W] = \mathcal{L}_Z(W) = -\frac{d}{dt}(\Phi_Z^t)_*(W)|_{t=0} \in \Gamma(\Delta),$$

where $\mathcal{L}_Z(W)$ denotes the Lie derivative along Z of W.

As in Section 2.1, we denote by o the origin in M. We write $\mathcal{K}_{-1} := \{Z \in \mathcal{K} \mid Z_o \in \Delta_o\}$. Moreover, we denote $\mathcal{K}_0 := \{Z \in \mathcal{K} \mid Z_o = 0\}$ and inductively, for every $j \geq 1$, we define

$$\mathcal{K}_j := \{Z \in \mathcal{K}_0 \mid [Z, Y^\dagger] \in \mathcal{K}_{j-1} \, \forall Y \in V\}.$$

Notice that $\mathcal{K}_{i+1} \subseteq \mathcal{K}_i$ for every $i \geq -1$. Moreover, each $\mathcal{K}_j$ is a vector space.

If Z is a Killing vector field, we choose $Y \in \mathfrak{g}$ such that $(Y^\dagger)_o = Z_o$ and therefore we decompose Z as

$$Z = Y^\dagger + Z', \tag{2.21}$$

with $Z' \in \mathcal{K}_0$. We obtain the decomposition $\mathcal{K} = \mathfrak{g}^\dagger + \mathcal{K}_0$, which is not necessarily a direct sum. We show the following property.

Lemma 2.22. $[\mathcal{K}_0, \mathcal{K}_j] \subseteq \mathcal{K}_j, \ \forall j \geq 0.$

Proof. We prove the lemma using induction. If $j = 0$, the statement is true, because the bracket of two vector fields that vanish at 0 vanishes at 0 as well.

Suppose now that the claim is verified for all indexes from 0 to $j-1$ and pick $Z \in \mathcal{K}_0$ and $Z' \in \mathcal{K}_j$. Clearly $[Z, Z']_o = 0$, i.e., $[Z, Z'] \in \mathcal{K}_0$. It remains to prove that $[[Z, Z'], Y^\dagger] \in \mathcal{K}_{j-1}$, for every $Y \in V$. First notice that the Jacobi identity yields

$$[[Z, Z'], Y^\dagger] = [[Z, Y^\dagger], Z'] - [[Z', Y^\dagger], Z]. \tag{2.23}$$

Recall the identification between V and Δ_o given in Section 2 and let $Y' \in \Gamma(\Delta)$ be such that $Y'_o = Y$. Then $[Z, Y^\dagger] = [Z, Y'] + [Z, Y^\dagger - Y']$. Therefore, it follows that $[Z, Y^\dagger]_o = [Z, Y']_o$. We recall that Z satisfies (2.20), so that in particular $[Z, Y^\dagger]_o \in \Delta_o$. Thus (2.21) gives $[Z, Y^\dagger] = W^\dagger + W'$, for some $W \in V$ and $W' \in \mathcal{K}_0$. Since $\mathcal{K}_j \subseteq \mathcal{K}_{j-1}$, by induction we conclude that

$$[[Z, Y^\dagger], Z'] = [W^\dagger, Z'] + [W', Z'] \in \mathcal{K}_{j-1}. \tag{2.24}$$

Again by the induction hypothesis we have that $[[Z', Y^\dagger], Z] \in \mathcal{K}_{j-1}$, which, together with (2.24) and (2.23), finishes the proof. □

Lemma 2.25. *Let f be an isometry between open and connected subsets of M. We assume that f is smooth and that f_*Z uniquely extends to a Killing vector field for every $Z \in \mathcal{K}$, for which we shall abuse the notation f_*Z. Moreover, we suppose that $df_o|_{\Delta_o}$ is the identity, and that $f(o) = o$. Then*

(i) *if $Z \in \mathcal{K}_{-1}$, then $f_*Z \in Z + \mathcal{K}_0$;*

(ii) *if $Z \in \mathcal{K}_j$ with $j \geq -1$, then $f_*Z \in Z + \mathcal{K}_{j+1}$.*

Proof. The idea of the proof is the following. The first part is a consequence of the fact that df_o is the identity on Δ_o. The second part will follow by induction and Lemma 2.22.

Regarding the proof of (i), we first note that by hypothesis f_* induces an isomorphism on $\mathcal{K}$. Thus for $Z \in \mathcal{K}$ with $Z_o \in \Delta_o$, by (2.21) there exist $Y \in \mathfrak{g}$ and $Z' \in \mathcal{K}_0$ such that $f_*Z = Y^\dagger + Z'$. Since $f(o) = o$, we have that

$$Y^\dagger_o = (f_*Z)_o = df_o Z_o = Z_o.$$

So

$$f_*Z = Z + (Y^\dagger - Z) + Z' \in Z + \mathcal{K}_0.$$

Hence (i) is proven.

Regarding the proof of (ii), we proceed by induction on j. The case $j = -1$ is given by (i). Now suppose that (ii) holds for every index from -1 to $j-1$ and choose $Z \in \mathcal{K}_j$. Clearly $(f_*Z - Z)_o = 0$. We are left to prove that $[f_*Z - Z, Y^\dagger] \in \mathcal{K}_j$, for every $Y \in V$. Using (i) and the induction hypothesis, we have

$$\begin{aligned}[f_*Z - Z, Y^\dagger] &= [f_*Z, Y^\dagger] - [Z, Y^\dagger] \\ &\in [f_*Z, f_*Y^\dagger + \mathcal{K}_0] - [Z, Y^\dagger] \\ &= f_*[Z, Y^\dagger] + [f_*Z, \mathcal{K}_0] - [Z, Y^\dagger] \subseteq \mathcal{K}_j + [f_*Z, \mathcal{K}_0]\end{aligned} \tag{2.26}$$

Notice that if $f_*Z \in \mathcal{K}_j$, then Lemma 2.22 together with (2.26) imply (ii). Since $\mathcal{K}_j \subseteq \mathcal{K}_{j-1}$, the induction hypothesis implies that $f_*Z \in Z + \mathcal{K}_j$. Since $Z \in \mathcal{K}_j$, we conclude that $f_*Z \in \mathcal{K}_j$. □

2.4. **Unique extension for Killing vector fields.** In this section we shall prove that an isometry defined among some open subsets of a subRiemannian Carnot group induces an isomorphism of the space $\mathcal{K}$ of Killing vector fields. This fact will be important in order to apply the results of the previous section to the proof of Theorem 1.1. A key tool throughout this section is Tanaka's prolongation theory. On the one hand, we shall use a finiteness criterium [Tan70, Corollary 2, page 76] to show finite dimensionality of Tanaka's prolongation, which will be in turn isomorphic to $\mathcal{K}$. On the other hand, the discussion in [Tan70, Section 6] will imply that a locally defined Killing vector field is uniquely extended to an element of $\mathcal{K}$.

We only recall what of Tanaka's theory is essential for our purposes. For an insight, we refer the reader to [Tan70, Yam93]. A recent survey can also be found in [OW11].

A linear self map $u : \mathfrak{g} \to \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a derivation if

$$u[X, Y] = [u(X), Y] + [X, u(Y)], \qquad \forall X, Y \in \mathfrak{g}.$$

Given a nilpotent Lie algebra $\mathfrak{g}$ with stratification $V_1 \oplus \ldots \oplus V_s$, we set

$$\mathrm{Der}_0(\mathfrak{g}) := \{u : \mathfrak{g} \to \mathfrak{g} \text{ derivation } \mid u(V_j) \subseteq V_j, \forall j = 1, \ldots, s\}.$$

Once we fix a scalar product on V_1, we denote by $O(V_1)$ the Lie group of linear isometries of V_1, and by $\mathfrak{o}(V_1)$ its Lie algebra. In particular, $\mathfrak{o}(V_1) \subseteq \mathfrak{gl}(V_1)$.

We recall now the definition of the Tanaka prolongation of the Lie algebra $\mathfrak{g}$ with respect to the subspace

$$\mathfrak{g}_0 := \{u \in \mathrm{Der}_0(\mathfrak{g}) \mid u|_{V_1} \in \mathfrak{o}(V_1)\}.$$

Set $\mathfrak{g}_j := V_{-j}$, for $j \in \{-s, \ldots, -1\}$. By induction, if $k \geq 1$, define

$$\mathfrak{g}_k := \{u \in \bigoplus_{j=-s}^{-1} \mathfrak{g}_{j+k} \otimes \mathfrak{g}_j^* \mid u[X,Y] = [u(X),Y] + [X,u(Y)], \forall X,Y \in \mathfrak{g}\},$$

where, if $X \in \mathfrak{g}$ and $u \in \mathfrak{g}_k$ with $k \geq 0$, we set

$$[u, X] := u(X). \tag{2.27}$$

The *Tanaka prolongation* of the Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ is the sum

$$\mathrm{Prol}(\mathfrak{g}) := \bigoplus_{j \geq -s} \mathfrak{g}_j.$$

The Lie algebra structure of $\mathfrak{g}$ is extended to $\mathrm{Prol}(\mathfrak{g})$ by (2.27) and by the inductively defined formula

$$[u,v](X) := [u(X), v] + [u, v(X)], \quad \forall u \in \mathfrak{g}_i, v \in \mathfrak{g}_j, X \in \mathfrak{g}. \tag{2.28}$$

Lemma 2.29. $\mathrm{Prol}(\mathfrak{g})$ *is a graded finite dimensional Lie algebra. Namely* $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$, $\forall i, j \geq -s$.

Proof. The grading property and the Jacobi identity easily follow from the definition of the Lie bracket. By [Tan70, Corollary 2, pag 76], for showing finite dimensionality it is enough to prove that the space

$$\mathfrak{h}_0 := \{u \in \mathfrak{g}_0 \mid u[X,Y] = 0, \forall X, Y \in \mathfrak{g}\}$$

has finite prolongation in the sense of Singer and Sternberg, see [SS65] or [Kob95]. The set $\mathfrak{h}_0$ can be identified with a subalgebra of $\mathfrak{o}(V_1)$. By an easy argument the first prolongation of $\mathfrak{o}(V_1)$ is trivial, see [Kob95, page 8]. □

We observe that the definition of prolongation that we provided above adapts to every choice of a subalgebra of $\mathrm{Der}_0(\mathfrak{g})$. Tanaka constructed these prolongation algebras to describe the infinitesimal generators of mappings preserving some nonintegrable geometric structures. If, e.g., we prolong with respect to all of $\mathrm{Der}_0(\mathfrak{g})$, then we would obtain a characterization of those vector fields generating flows of contact mappings. More precisely, one has that the prolongation algebra and the corresponding space of vector fields identify as Lie algebras whenever the prolongation if finite.

In our case, since $\mathrm{Prol}(\mathfrak{g})$ is finite by Lemma 2.29, Tanaka's method leads to the identification of Lie algebras

$$\mathrm{Prol}(\mathfrak{g}) \cong \mathcal{K}. \tag{2.30}$$

Now we observe that this same identification also applies to vector fields that are defined only locally.

Definition 2.31. (Local Killing vector fields $\mathcal{K}_\Omega$) Let $(\mathbb{G}, d)$ be a Carnot group. Let Z be a vector field on $\Omega \subseteq \mathbb{G}$. We say that Z is a *Killing vector field on* Ω if, for every $p \in \Omega$, there exist $t_0 > 0$ and $\Omega_0 \subset \Omega$ neighborhood of p such that, for all $t \in [0, t_0]$, the flow $\Phi^t_Z|_{\Omega_0}$ at time t is an isometric embedding of Ω_0 inside $\mathbb{G}$, with respect to the Carnot-Carathéodory distance on $\mathbb{G}$. We denote by $\mathcal{K}_\Omega$ the collection of all Killing vector fields on an open set Ω of $\mathbb{G}$.

The idea behind the isomorphism of $\mathcal{K}_\Omega$ and $\mathcal{K}$ is the following. To any vector field in $\mathcal{K}_\Omega$ we associate an element of $\mathrm{Prol}(\mathfrak{g})$, the Tanaka prolongation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. By Lemma 2.29, $\mathrm{Prol}(\mathfrak{g})$ is a finite dimensional Lie algebra. One should think that the elements in $\mathrm{Prol}(\mathfrak{g})$ are the coefficients of the Taylor expansion at a fixed point of the coordinates of the vector field. A vector field Z is in $\mathcal{K}_\Omega$ if and only if the coefficients of Z with respect to a fixed basis of left-invariant vector fields satisfy a particular system of PDE's, whose solutions are polynomials. Hence the coefficients of Z are polynomials. The same polynomials define an extension of Z for which we abuse the notation Z. Since the system of PDE's is in fact polynomial and linear, then the value of this operator on the coefficients of Z is a polynomial that is null on an open set. Therefore it is 0 everywhere. Hence, the vector field Z is still a solution. Thus, $Z \in \mathcal{K}$.

Lemma 2.32. *Let $\mathbb{G}$ be a subRiemannian Carnot group. If Ω is a connected open subset of $\mathbb{G}$, then the restriction function from $\mathcal{K} = \mathcal{K}_{\mathbb{G}}$ to $\mathcal{K}_\Omega$ is a bijection.*

Proof. One of the fundamental point in Tanaka's work is that a locally defined infinitesimal generator of mappings preserving some geometric structure can be extended to the whole group, whenever the prolongation describing that type of mappings is finite. From Section 6 in [Tan70], it follows in particular that to any vector field in $\mathcal{K}_\Omega$ we can associate an element of $\mathrm{Prol}(\mathfrak{g})$, the Tanaka prolongation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. So (2.30) concludes the proof. □

Remark 2.33. Let f be an isometry between two connected open sets of a Carnot group. By Theorem 2.8, the map f is smooth. By Lemma 2.32, the vector field $f_* Z$, which is in $\mathcal{K}_\Omega$, uniquely extends to a vector field in $\mathcal{K}$. Therefore, the isometry f induces an isomorphism of $\mathcal{K}$.

3. Isometries of subFinsler homogeneous spaces

This section is devoted to the proof of Theorem 1.2 and Theorem 1.1. We prove the theorem on global isometries immediately in Section 3.1. For the proof of Theorem 1.1, we first need to consider the situation of a distance defined by a scalar product on V_1 and isometries defined on connected subsets (see Theorem 3.3). Then we consider the general case in Section 3.3.

3.1. **Global isometries of subFinsler homogeneous spaces.** We prove now Theorem 1.2. We shall only make use of Corollary 2.12 and Lemma 2.25 of the previous discussion.

Proof of Theorem 1.2. From Corollary 2.12, it follows that f and h are smooth. We suppose that we have $f(o) = h(o)$ and $(dh)_o|_{\Delta_o} = df_o|_{\Delta_o}$. We plan to show that $f = h$.

Up to replacing f with $h^{-1} \circ f$, we may assume that $f(o) = o$ and that df_o is the identity on Δ_o. Let f_* be the push forward operator on vector fields. Since f is smooth and globally defined, the map f_* is a Lie algebra isomorphism on $\mathcal{K}$, the space of Killing vector fields. We shall prove that $f_* Y^\dagger = Y^\dagger$, for all $Y \in \mathfrak{g}$. First, pick $Y \in V$ and assume by contradiction that $f_* Y^\dagger \neq Y^\dagger$. Then, using Lemma 2.25, there exists $Z \in \mathcal{K}_j$, with $j \geq 0$ and $Z \neq 0$, such that

$$f_* Y^\dagger \in Y^\dagger + Z + \mathcal{K}_{j+1}.$$

Using again Lemma 2.25, we have

$$\begin{aligned} f_*^2 Y^\dagger &= f_* f_* Y^\dagger \\ &\in f_* Y^\dagger + f_* Z + f_* \mathcal{K}_{j+1} \\ &\subseteq Y^\dagger + Z + \mathcal{K}_{j+1} + Z + \mathcal{K}_{j+1} + \mathcal{K}_{j+1} = Y^\dagger + 2Z + \mathcal{K}_{j+1}. \end{aligned}$$

By iteration, we get

$$f_*^n Y^\dagger \in Y^\dagger + nZ + \mathcal{K}_{j+1}, \qquad \forall n \in \mathbb{N}. \tag{3.1}$$

Notice that, for all $K \in \mathcal{K}$,

$$f_*^n K \in \{g_* K \mid g \in \mathrm{Iso}_o(M)\}.$$

Since $\mathrm{Iso}_o(M)$ is compact (see Corollary 2.12), the family $\{g_*\}_{g \in \mathrm{Iso}_o(M)}$ is a collection of bounded operators. So, on the one hand, $f_*^n Y^\dagger$ belongs to a bounded set. On the other hand, any bounded set has empty intersection with the affine space $Y^\dagger + nZ + \mathcal{K}_{j+1}$, for n big enough. Hence, (3.1) is contradicted. Thus $f_* Y^\dagger = Y^\dagger$ for every $Y \in V$. Notice that formula (2.3) implies that vector fields of the form $Y^\dagger$ as Y varies in V bracket-generate all vector fields $Y^\dagger$ with $Y \in \mathfrak{g}$. Since f_* commutes with the bracket of vector fields, we conclude that

$$f_* Y^\dagger = Y^\dagger,$$

for all $Y \in \mathfrak{g}$. This implies

$$f(\exp(tY) \cdot f^{-1}(gH)) = \exp(tY) gH$$

for every $Y \in \mathfrak{g}$. In particular, choosing $g = o$ and since $f(o) = o$, we obtain

$$f(\exp(tY)H) = \exp(tY)H, \tag{3.2}$$

for every $Y \in \mathfrak{g}$. Let now $U \subseteq G$ be a neighborhood of e with the property that for every $g \in U$ there exists $Y \in \mathfrak{g}$ such that $\exp Y = g$. Then (3.2) implies that f is the identity when restricted to the left cosets of U. Since M is supposed to be connected, we conclude that f is the identity everywhere. □

3.2. **Isometries of open sets in Carnot groups.** In this section we prove Theorem 1.1 in a particular case. Namely, we show that any isometry defined between two connected and open subsets of a Carnot group $\mathbb{G}$ endowed with a subRiemannian metric is affine.

The structure of the proof of the following theorem is similar to that one of Theorem 1.2. Since now we deal with isometries defined on subsets, we shall rely on [CC06] for the smoothness of such maps. Moreover, Lemma 2.32 and the observation thereafter allow us to extend any local Killing vector field to a global one. Consequently an isometry defined on an open subset of a Carnot group $\mathbb{G}$ will induce an isomorphism of $\mathcal{K}$.

Theorem 3.3. *Let $(\mathbb{G}, d)$ be a subRiemannian Carnot group. Let $\Omega_1, \Omega_2 \subset \mathbb{G}$ be two* connected *open sets. Let $f : \Omega_1 \to \Omega_2$ be an isometry. If $f(e) = e$, then f is the restriction to Ω_1 of a group isomorphism of $\mathbb{G}$.*

Proof. We have that f is analytic from Theorem 2.8. Let V_1 be the first stratum of $\mathbb{G}$. Let ϕ be the blow up of f at e, i.e., the Pansu differential at the identity, see [War08]. Notice that $d\phi_e|_{V_1} = df_e|_{V_1}$. By [Pan89], the map ϕ is a group isomorphism and, moreover, it is an isometry, being the limit of isometries. We plan to show that $f = \phi|_{\Omega_1}$. Up to replacing f with $\phi^{-1} \circ f$, we may assume that

$$df_e Y = Y, \qquad \forall Y \in V_1.$$

We now prove that $f_* Y^\dagger = Y^\dagger$, for all $Y \in \mathfrak{g}$. Let $Y \in V_1$. Assume by contradiction that $f_* Y^\dagger \neq Y^\dagger$. By means of Lemma 2.32 and the remark thereafter, the map f_* induces a Lie algebra isomorphism of the Killing vector fields $\mathcal{K}$. Therefore, proceeding as in the first part of the proof of Theorem 1.2, we show that

$$f_*^n Y^\dagger \in Y^\dagger + nZ + \mathcal{K}_{j+1}, \qquad \forall n \in \mathbb{N}, \tag{3.4}$$

for some nonzero element $Z \in \mathcal{K}_j$. Let $\mathcal{I}$ be the group of isometries preserving a ball on which f is defined. Notice that, for all $K \in \mathcal{K}$,

$$f_*^n K \in \{g_* K \mid g \in \mathcal{I}\}.$$

By (iii) of Corollary 2.18 the differentials g_* depend smoothly on $g \in \mathcal{I}$. Since $\mathcal{I}$ is compact, then the family $\{g_*\}_{g \in \mathcal{I}}$ is a collection of bounded operators. In particular, the set $\{g_* Y^\dagger \mid g \in \mathcal{I}\}$ is bounded. So, on the one hand, $f_*^n Y^\dagger$ belongs to a bounded set. On the other hand, any bounded set has empty intersection with the affine space $Y^\dagger + nZ + \mathcal{K}_{j+1}$, for n big enough. Hence, (3.4) is contradicted. Thus $f_* Y^\dagger = Y^\dagger$, for every $Y \in V_1$. Since f_* is a Lie algebra automorphism, we conclude that f_* is the identity on right-invariant vector fields. But $f(e) = e$, so f is the identity. □

3.3. **Generalization to nonconnected set and to subFinsler Carnot groups.** We shall extend Theorem 3.3 to the case of subsets of $\mathbb{G}$ that are not necessarily connected. In order to do that, we use the following theorem.

Theorem 3.5 ([Agr09, Theorem 1])**.** *Let M be a (analytic) subRiemannian manifold and set $q_0 \in M$. Then there exists an open and dense subset $\Sigma_{q_0} \subseteq M$ such that for any $q \in \Sigma_{q_0}$ there exists a unique length minimizing curve γ connecting q_0 to q. Moreover, such γ is analytic.*

The following result holds for general subRiemannian manifolds. We show that an isometry is completely determined on its behavior on an open set.

Proposition 3.6. *Let M be a (analytic) subRiemannian manifold. Let $f : \Omega_1 \to \Omega_2$ be an isometry among two open sets in M. Assume that f is the identity on an open subset of Ω_1. Then f is the identity.*

Proof. Let $\Omega \subseteq \Omega_1$ be the open subset such that $f|_\Omega$ is the identity. Pick $q \in \Omega_1$. According to the notation in Theorem 3.5, consider $\Sigma = \Sigma_q \cap \Sigma_{f(q)}$. Fix $p \in \Sigma \cap \Omega$. Since $p \in \Sigma_q$, Theorem 3.5 implies that there exists a unique and analytic length minimizing curve γ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since Ω is open, one can choose $s_0 \in (0, 1)$ such that $p' := \gamma(s_0) \neq p$ and $\gamma(s_0) \in \Omega$. Denote by ρ a length minimizing curve such that $\rho(0) = p'$ and $\rho(1) = f(q)$. Let $\tilde{\gamma}$ be the curve formed by joining $\gamma|_{[0,s_0]}$ with ρ. We claim that $\tilde{\gamma}$ minimizes the length between p and $f(q)$. Indeed, since $f(p) = p$, $f(p') = p'$, and f is an isometry, we have

$$\begin{aligned} d(p, f(q)) &= d(p, q) = d(p, p') + d(p', q) \\ &= d(p, p') + d(p', f(q)). \end{aligned}$$

So $\tilde{\gamma}$ realizes the distance from p to $f(q)$. Since $p \in \Sigma_{f(q)}$, it follows that $\tilde{\gamma}$ is analytic. Since γ and $\tilde{\gamma}$ coincide on an interval, they are both analytic, and have the same length, they coincide. In particular, $q = f(q)$. □

Using the proposition above, together with Theorem 3.3, we obtain Theorem 1.1 in the case of subRiemannian Carnot groups. At this point, Theorem 1.1 will be proved once we extend the results to the case of a subFinsler metric. This is the content of the following lemma, which can be easily extended to subFinsler homogeneous manifolds.

Lemma 3.7. *Let G be a Lie group equipped with a left-invariant subFinsler distance. Then there exists a left-invariant subRiemannian distance d_{SR} with same horizontal bundle as d_{SF} such that any C^1 isometry among open subsets of G is an isometry with respect to d_{SR}.*

Proof. The proof is an application of John's Ellipsoid Theorem, see [Joh48]. Let Δ_p be the horizontal bundle at p. Denote $K_p = \{v \in \Delta_p \mid \|v\| \leq 1\}$, where $\|\cdot\|$ is the norm defining d_{SF}. John's Ellipsoid Theorem states that there exists a unique ellipsoid E_p contained in K_p with maximal volume.

Let f be any C^1 d_{SF}-isometry. We claim that for any p in the domain of f, we have

$$df_p(E_p) = E_{f(p)}. \tag{3.8}$$

Indeed, df_p restricts to a linear isometry between $(\Delta_p, \|\cdot\|)$ and $(\Delta_{f(p)}, \|\cdot\|)$. In particular, $df_p(K_p) = K_{f(p)}$ and $df_p(E_p)$ is an ellipsoid contained in $K_{f(p)}$. Since $E_{f(p)}$ is the maximal-volume ellipsoid, it follows that $\mathrm{vol}(df_p(E_p)) \leq \mathrm{vol}(E_{f(p)})$. Since df_p^{-1} also restricts to a linear isometry, we obtain the reverse inequality and, by uniqueness, we have (3.8). In particular, choosing f to be a left translation L_p, with $p \in G$, we have that $E_p = (dL_p)_e E_e$. Therefore, $\{E_p\}_{p \in G}$ define a left-invariant scalar product on Δ_p which in turn gives a left-invariant subRiemannian distance on G. Moreover, the equation (3.8) implies that any C^1 isometry with respect to d_{SF} is also an isometry with respect to d_{SR}. □

Proof of Theorem 1.1. Let $(\mathbb{G}, d_{SF})$ be a subFinsler Carnot group. Let $\Omega_1, \Omega_2 \subseteq \mathbb{G}$ be two open sets and $f : \Omega_1 \to \Omega_2$ an isometry, which a priori is not smooth. By Lemma 3.7, we have a subRiemannian distance d_{SR} on $\mathbb{G}$ satisfying the following properties: $(\mathbb{G}, d_{SR})$ is a subRiemannian Carnot group and any C^1 isometry of $\mathbb{G}$ with respect to d_{SF} is an isometry with respect to d_{SR}.

Since d_{SR} is biLipschitz equivalent to d_{SF}, it follows that f is biLipshitz with respect to d_{SR}. By Pansu Theorem [Pan89], the blow up exists a.e. and it is a group isomorphism, hence C^1. Since any blow up of f is an isometry of $(\mathbb{G}, d_{SF})$, Lemma 3.7 implies that the blow up of f at a.e. point is an isometry of $(\mathbb{G}, d_{SR})$. By Theorem 2.8.(ii), the map f is analytic. Again using Lemma 3.7, we have that the map f is an isometry with respect to d_{SR}. Up to composing with a translation, we may assume $f(e) = e$. By Theorem 3.3, we obtain that on the connected component Ω of Ω_1 containing e, the map f is a group isomorphism ϕ. Then the map $\phi^{-1} \circ f$ is an isometry that is the identity on Ω. By Proposition 3.6, we get that $\phi^{-1} \circ f$ is the identity on Ω_1, which finishes the proof. □

3.4. **Afterthoughts.** Not even in Euclidean space it is true that all group isomorphisms are isometries. However, an automorphism of a subFinsler Carnot group is an isometry if and only if its differential preserves the first stratum (ad hence all strata) and restricted to the first stratum preserves the norm defining the subFinsler distance. Hence we have a complete description of local isometries of subFinsler Carnot groups.

Unlikely in the Euclidean space, Theorem 1.1 cannot be generalized to arbitrary subsets. Here we present a counterexample. We take the subRiemannian Heisenberg group $(\mathbb{H}, d_{SR})$ and we define exponential coordinates (x, y, z) with respect to the basis of its Lie algebra given by vectors X, Y and Z. The only nonzero bracket relation is $[X, Y] = Z$. Consider the three coordinate axes, namely,

$$E := \exp(\mathbb{R}X) \cup \exp(\mathbb{R}Y) \cup \exp(\mathbb{R}Z).$$

Then the map $(x, y, z) \mapsto (x, y, -z)$ is an isometry of E into itself. However, this map is not the restriction of a group isomorphism.

Given an isometry f of a subFinsler homogeneous space G/H, the differential of the blow-up at a point p equals the differential at p, when they are both restricted to Δ_p. Therefore Theorem 1.2 claims that $\mathrm{Iso}_o(G/H)$ injects into $\mathrm{Iso}_o((G/H)_o)$, where $(G/H)_o$ denotes the Gromov tangent cone of G/H at o, which is a Carnot group. However, it is not true that isometries of $(G/H)_o$ are always blow-ups of isometries of G/H. In fact, we can find counterexamples even in the domain of Riemannian Lie groups. Take for instance the three dimensional Heisenberg group, endowed with a Riemannian distance. We denote it by $(\mathbb{H}, d_R)$. Then its tangent cone at every point is the Euclidean 3-space, which contains all the rotations among its isometries. However, rotations with respect to horizontal lines are not isometries for $(\mathbb{H}, d_R)$. This follows from the observation that $\mathrm{Iso}(\mathbb{H}, d_R) = \mathrm{Iso}(\mathbb{H}, d_{SR})$. The identification of the two isometry groups rests upon the study of length minimizing curves: for both metric models of $\mathbb{H}$ the only infinite geodesics are the 1-parameter groups corresponding to horizontal vectors. Since isometries must preserve infinite geodesics, it follows that the horizontal space is preserved by the differential of any isometry.

We notice that the statements of Theorem 1.1 and Theorem 1.2 become equivalent if $G/H = \mathbb{G}$ and if $\Omega_1 = \mathbb{G}$. If this is not the case, we cannot conclude that a global isometry of a subFinsler Lie group G is affine. Indeed, take the three dimensional sphere $\mathbb{S}^3$ viewed as the space of quaternions with euclidean norm equal to one. The manifold $\mathbb{S}^3$ is then a Riemannian Lie group. It is easy to check that the inversion map $p \mapsto p^{-1}$ on $\mathbb{S}^3$ is an isometry that is not a group isomorphism.

For general Lie groups, isometries between open sets might not be restrictions of global isometries. For example, every point in the flat cylinder $\mathbb{R}\times\mathbb{S}^1$ has a neighborhood isometric to a disk in $\mathbb{R}^2$. Hence, all rotations are isometries of such a neighborhood. Of course, not all of them extend to global isometries.

Last, the fact that isometries of a Carnot group $\mathbb{G}$ are affine maps implies that $\mathcal{K}_j = \{0\}$ for every $j \geq 1$. As a by-product, the Tanaka prolongation $\mathrm{Prol}(\mathfrak{g})$ defined in Section 2.4 is $\mathfrak{g} \oplus \mathfrak{g}_0$. Although Tanaka's method proves the finiteness of $\mathrm{Prol}(\mathfrak{g})$ (see Lemma 2.29) , we are not aware of a direct method to show that in general $\mathrm{Prol}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}_0$.

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