Ancient Solutions to the Navier-Stokes Equations

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Definition

A divergence free field $u \in L_\infty(Q_-;\mathbb{R}^n)$ is called a weak bounded ancient solution (or simply bounded ancient solution) to the Navier-Stokes equations if

$$\int\limits_{Q_-} (u \cdot \partial_t w + u \otimes u : \nabla w + u \cdot \Delta w) dz = 0$$

for any $w \in C^\infty_{0,0}(Q_-)$.

Let u be an arbitrary bounded ancient solution. For any number $m>1$,

$$|\nabla u|+|\nabla^2 u|+|\nabla p_{u\otimes u}|\in \mathcal{L}_m(Q_-).$$

Moreover, for each $t_0\leq 0$, there exists a function $b_{t_0}\in L_\infty(t_0-1,t_0)$ with the following property

$$\sup_{t_0\leq 0}\|b_{t_0}\|_{L_\infty(t_0-1,t_0)}\leq c(n)<+\infty.$$

If we let $u^{t_0}(x,t)=u(x,t)+b_{t_0}(t)$ in $Q^{t_0}=\mathbb{R}^n\times]t_0-1,t_0[$, then, for any number $m>1$ and for any point $x_0\in\mathbb{R}^n$, the uniform estimate

$$\|u^{t_0}\|_{W^{2,1}_m(Q(z_0,1))}\leq c(m,n)<+\infty,\qquad z_0=(x_0,t_0),$$

is valid and, for a.a. $z=(z,t)\in Q^{t_0}$, functions u and u^{t_0} obey the system of equations

$$\partial_t u^{t_0}+\mathrm{div} u\otimes u-\Delta u=-\nabla p_{u\otimes u},\qquad \mathrm{div} u=0.$$

The first equation of the latter system can be rewritten in the following way

$$\partial_t u + \mathrm{div} u \otimes u - \Delta u = -\nabla p_{u\otimes u} - b'_{t_0}, \qquad b'_{t_0}(t) = db_{t_0}(t)/dt,$$

in Q^{t_0} in the sense of distributions. So, the real pressure field in Q^{t_0} is the following distribution $p_{u\otimes u} + b'_{t_0} \cdot x$

We can find a measurable vector-valued function b defined on $]-\infty, 0[$ and having the following property. For any $t_0 \leq 0$, there exists a constant vector c_{t_0} such that

$$\sup_{t_0 \leq 0} \|b - c_{t_0}\|_{L_\infty(t_0-1,t_0)} < +\infty.$$

Moreover, the Navier-Stokes system takes the form

$$\partial_t u + \mathrm{div} u \otimes u - \Delta u = -\nabla(p_{u\otimes u} + b' \cdot x), \qquad \mathrm{div} u = 0$$

in Q_- in the sense of distributions.

A vector field u is called a mild bounded ancient solution if u is a bounded ancient solution and there exists a pressure field $p \in L_\infty(-\infty, 0; BMO(\mathbb{R}^3))$ such that

$$\int\limits_{Q_-} \Big(u \cdot (\partial_t \varphi + \Delta \varphi) + u \otimes u : \nabla \varphi \Big) dz = - \int\limits_{Q_-} p \, \mathrm{div} \, \varphi \qquad (1)$$

for any $\varphi \in C_0^\infty(Q_-)$. Without loss of generality, we may assume that $p = p_{u \otimes u}$.

Lemma

Let u a mild bounded ancient solution. Then u is of class C^∞ and moreover

$$\sup_{(x,t)\in Q_-}\left(|\partial_t^k\nabla^l u(x,t)|+|\partial_t^k\nabla^{l+1}p(x,t)|\right)+$$

$$+\|\partial_t^k p\|_{L_\infty(BMO)}\leq C(k,l,\|p\|_{L_\infty(BMO)})<\infty$$

for any $k,l=0,1,....$

Any mild bounded ancient solution has the following property: for any $A<0$, they can be presented in the form

$$u_i(x,t)=\int\limits_{\mathbb{R}^3}\Gamma(x-y,t-A)u_i(y,A)dy+$$

$$+\int\limits_A^t\int\limits_{\mathbb{R}^3}K_{ijm}(x,y,t-\tau)u_j(y,\tau)u_m(y,\tau)dyd\tau, \qquad (2)$$

where Γ is a standard heat kernel and K is obtained from the Oseen tensor

$$T(x,t)=\Gamma(x,t)\mathbb{I}-\nabla^2\Phi(x,t),$$

where

$$\Delta\Phi(x,t)=\Gamma(x,t),$$

in the following way

$$K_{ijm}(x,t)=T_{ij,m}(x,t).$$

A vector-valued function u is a mild bounded ancient solution if and only if u is a bounded and, for any $A < 0$, (2) holds.

Any mild bounded ancient solution is a constant.

Theorem

Let u be a mild bounded ancient solution to the Navier-Stokes equations.Assume that

$$\sup_{0<r<\infty} M_{s,l}(u;r) = \int_{-\infty}^{0} \Big(\int_{\mathbb{R}^3} |u(x,t)|^s dx \Big)^{\frac{l}{s}} < \infty$$

with $3/s + 2/l = 1$ *and* $l < \infty$*. Then* $u \equiv 0$ *in* Q_-*.*

Theorem

Assume that $n = 2$ and u is an arbitrary bounded ancient solution. Then $u(x, t) = b(t)$ for any $x \in \mathbb{R}^2$.

Lemma

Let functions

$$\omega \in \mathcal{W}^{2,1}_m(Q_-) = \{u \in W^{2,1}_{m,\mathrm{loc}}(Q_-) : \sup_{z_0 \in Q_-} \|u\|_{W^{2,1}_m(Q(z_0,1))} < \infty\},$$

with $m > 3$, *and* $u \in L_\infty(Q_-)$ *satisfy the equation*

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = 0 \qquad in \quad Q_-$$

and the inequality

$$|u| \leq 1 \qquad in \quad Q_-.$$

Then, for any positive numbers ε *and* R, *there exists a point* $z_0 = (x_0, t_0)$, $x_0 \in \mathbb{R}^2$ *and* $t_0 \leq 0$, *such that*

$$\omega(z) \geq M - \varepsilon, \qquad z \in Q(z_0, R),$$

where $M = \sup_{z \in Q_-} \omega(z)$.

Theorem

STRONG MAXIMUM PRINCIPLE *Let functions* $w \in W^{2,1}_m(Q(z_0,R))$ *with* $m > n+1$ *and* $a \in L_\infty(Q(z_0,R);\mathbb{R}^n)$ *satisfy the equation*

$$\partial_t w + a \cdot \nabla w - \Delta w = 0 \qquad in \quad Q(z_0,R).$$

Let, in addition,

$$w(z_0) = \sup_{z \in Q(z_0,R)} w(z).$$

Then

$$w(z) = w(z_0) \qquad in \quad Q(z_0,R).$$

Theorem

Let u be an arbitrary axially symmetric bounded ancient solution with zero swirl. Then $u(x,t)=b(t)$ for any $x\in\mathbb{R}^3$ and for any $t\leq 0$. Moreover, $u_1(x,t)=0$ and $u_2(x,t)=0$ for the same x and t or, equivalently, $u_\varrho(\varrho,x_3,t)=0$ for any $\varrho>0$, for any $x_3\in\mathbb{R}$, and for any $t\leq 0$.

Theorem

Let u be an arbitrary axially symmetric bounded ancient solution satisfying assumption

$$|u(x,t)| \leq \frac{A}{|x'|}, \qquad x = (x', x_3) \in \mathbb{R}^3,\ -\infty < t \leq 0, \qquad (1)$$

where A is a positive constant independent of x and t. Then $u \equiv 0$ in Q_-.

$$\partial_t v + v \cdot \nabla v - \triangle v = -\nabla q, \quad \mathrm{div}\, v = 0 \tag{2}$$

in $\mathbb{R}^3\times]0,\infty[$,

$$v(x,0) = v_0(x) \in C^\infty_{0,0}(\mathbb{R}^3) \tag{3}$$

for any $x \in \mathbb{R}^3$. Here, $C^\infty_{0,0}(\mathbb{R}^3) = \{v \in C^\infty_0(\mathbb{R}^3) : \ \mathrm{div}\, v = 0\}$.

$$g(t) = \sup_{0<\tau\leq t} M(\tau) \to \infty \qquad \text{as} \quad t \to T - 0, \tag{4}$$

where

$$M(t) = \sup_{x\in\mathbb{R}^3} |v(x,t)|.$$

Let ω be a open set in $\mathbb{R}^3$. We say that a pair u and p is a *suitable weak* solution to the Navier-Stokes equations in $\omega\times]T_1,T[$ if u and p satisfy the conditions:

$$u \in L_{2,\infty}(\omega\times]T_1,T[) \cap L_2(T_1,T;W^1_2(\omega)); \tag{5}$$

$$p \in L_{\frac{3}{2}}(\omega\times]T_1,T[); \tag{6}$$

$$\partial_t u + u\cdot\nabla u - \Delta u = -\nabla p, \qquad \mathrm{div}\, u = 0 \tag{7}$$

in the sense of distributions;
the local energy inequality

$$\left.\begin{array}{l}\displaystyle\int\limits_{\omega} \varphi(x,t)|u(x,t)|^2\,dx + 2\int\limits_{\omega\times]T_1,t[} \varphi|\nabla u|^2\,dxdt' \\ \displaystyle\leq \int\limits_{\omega\times]T_1,t[} (|u|^2(\Delta\varphi+\partial_t\varphi) + u\cdot\nabla\varphi(|u|^2+2q))\,dxdt'\end{array}\right\} \tag{8}$$

holds for a.a. $t\in]T_1,T[$ and all nonnegative functions $\varphi\in C^\infty_0(\omega\times]T_1,\infty[)$.

Now, we are in a position to explain ε-regularity theory. Quantities that are invariant with respect to the Navier-Stokes scaling

$$\begin{aligned} v^{\lambda}(y,s) &= \lambda v(x_0+\lambda y, t_0+\lambda^2 s), \\ q^{\lambda}(y,s) &= \lambda^2 q(x_0+\lambda y, t_0+\lambda^2 s). \end{aligned} \tag{9}$$

play the crucial role in this theory. By the definition, such quantities are defined on parabolic balls $Q(r)$ and have the property

$$F(v,q;r) = F(v^{\lambda},q^{\lambda};r/\lambda).$$

Suppose that v and q are a suitable weak solution to the Navier-Stokes equations in Q. There exist universal positive constants ε and $\{c_k|\}_{k=1}^{\infty}$ such that if $F(v, q; 1) < \varepsilon$ then $|\nabla^k v(0)| < c_k$, $k = 0, 1, 2, ...$ Moreover, the function $z \mapsto \nabla^k v(z)$ is Hölder continuous (relative to the usual parabolic metric) with any exponent less $1/3$ in the closure of $Q(1/2)$.

$$F(v, q; r) = \frac{1}{r^2} \int\limits_{Q(r)} \left(|v|^3 + |q|^{\frac{3}{2}} \right) dz.$$

Let v and q be a suitable weak solution in Q. There exists a universal positive constant ε with the property: if $\sup_{0<r<1} F(v;r) < \varepsilon$ *then $z = 0$ is a regular point. Moreover, for any $k = 0, 1, 2, ...$, the function $z \mapsto \nabla^k v(z)$ is Hölder continuous with any exponent less $1/3$ in the closure of $Q(r)$ for some positive r.*

$$F(v;r) = M_{s,l}(v;r) = \|v\|^{l}_{s,l,Q(r)} = \int\limits_{-r^2}^{0} \Big(\int\limits_{B(r)} |v|^s dx \Big)^{\frac{l}{s}} dt$$

provided

$$\frac{3}{s} + \frac{2}{l} = 1$$

$$A(v; z_0, r) \equiv \sup_{t_0 - r^2 \leq t \leq t_0} \frac{1}{r} \int\limits_{B(x_0, r)} |v(x,t)|^2 \, dx,$$

$$E(v; z_0, r) \equiv \frac{1}{r} \int\limits_{Q(z_0, r)} |\nabla v|^2 \, dz,$$

$$C(v; z_0, r) \equiv \frac{1}{r^2} \int\limits_{Q(z_0, r)} |v|^3 \, dz,$$

$$D_0(q; z_0, r) \equiv \frac{1}{r^2} \int\limits_{Q(z_0, r)} |q - [q]_{x_0, r}|^{\frac{3}{2}} \, dz.$$

Proposition

Let v *and* q *be a suitable weak solution to the Navier-Stokes equations in* Q*. Given* $M > 0$*, there exists a positive number* $\varepsilon(M)$ *having the property: if two inequalities* $\limsup_{r\to 0} E(r) < M$ *and*

$$\liminf_{r\to 0} E(r) < \varepsilon(M)$$

hold, then $z = 0$ *is a regular point of* v*.*

$$G_1(v;r) = \sup_{z=(x,t)\in Q(r)} |x||v(z)|,$$
$$G_2(v;r) = \sup_{z=(x,t)\in Q(r)} \sqrt{-t}|v(z)|.$$

Proposition

Let v and q be a suitable weak solution to the Navier-Stokes equations in Q and $z = 0$ be a singular point of v. There exist two functions $\widetilde{v}$ and $\widetilde{q}$ having the following properties:
(i) $\widetilde{v} \in L_3(Q)$ and $\widetilde{q} \in L_{\frac{3}{2}}(Q)$ obey the Navier-Stokes equations in Q in the sense of distributions;
(ii) $\widetilde{v} \in L_\infty(B\times]-1,-a^2[)$ for all $a \in]0,1[$;
(iii) there exists a number $0 < r_1 < 1$ such that $\widetilde{v} \in L_\infty(\{(x,t) : r_1 < |x| < 1, \ -1 < t < 0\})$.
Moreover, functions $\widetilde{v}$ and $\widetilde{q}$ are obtained from v and q with the help of the space-times shift and the Navier-Stokes scaling and the origin remains to be a singular point of $\widetilde{v}$.

$$u^{(k)}(y,s) = \lambda_k v(x,t), \qquad p^{(k)}(y,s) = \lambda_k^2 q(x,t)$$

with

$$x = x^{(k)} + \lambda_k y, \qquad x = t_k + \lambda_k^2 s,$$

where $x^{(k)} \in \mathbb{R}^3$, $-1 < t_k \leq 0$, and $\lambda_k > 0$ are parameters of the scaling and $\lambda_k \to 0$ as $k \to +\infty$.

$\lambda_k = 1/M_k$, where a sequence M_k is defined as

$$M_k = \|v(,t_k)\|_{\infty,\overline{B}(r)} = |v(x^{(k)},t_k)|$$

with $x^{(k)} \in \overline{B}(r_1)$ for sufficiently large k.

Proposition

There exist a subsequence of $u^{(k)}$ (still denoted by $u^{(k)}$) and a mild bounded ancient solution u such that, for any $a>0$, the sequence $u^{(k)}$ converges uniformly to u on the closure of the set $Q(a) = B(a)\times]-a^2,0[$. The function u has the additional properties: $|u| \leq 1$ in $\mathbb{R}^3\times]-\infty,0[$ and $|u(0)| = 1$.

what happens if we drop the condition on smallness of scale-invariant quantities, assuming their uniform boundedness only, i.e, $\sup_{0<r<1} F(v, r) < +\infty$.

Boundedness of

$$\sup_{0<r<1} G_2(v;r) = G_2(v,1) = G_{20} < +\infty$$

can be rewritten in the form

$$|v(z)| \leq \frac{G_{20}}{\sqrt{-t}}$$

for all $z = (x,t) \in Q$. If v satisfies the above inequality and $z = 0$ is still a singular point of v, we say that a *singularity of Type I* or *Type I blowup* takes place at $t = 0$.

Proposition

Let functions v *and* q *be a suitable weak solution to the Navier-Stokes equations in* Q*.*
(i) If $\min\{G_1(v;1), G_2(v;1)\} < +\infty$*, then*

$$g = \sup_{0<r<1} \{A(v;r) + C(v;r) + D(q;r) + E(v;r)\} < +\infty.$$

(ii) If

$$g' = \min\{\sup_{0<r<1} A(v;r), \sup_{0<r<1} C(v;r), \sup_{0<r<1} E(v;r)\} < +\infty,$$

then $g < +\infty$*.*

validity of the conjecture would rule out Type I blowups

$$\int_{Q_-^+} \Big(u\cdot(\partial_t\varphi+\Delta\varphi)+u\otimes u:\nabla\varphi\Big)dz=0 \tag{10}$$

for any $\varphi\in C^\infty_{0,0}(Q_-)$ with $\varphi(x',0,t)=0$ for any $x'\in\mathbb{R}^2$ and for any $-\infty<t<0$;

$$\int_{Q_-^+} u\cdot\nabla q dz=0 \tag{11}$$

for any $q\in C^\infty_0(Q_-)$.

$$u(x,t) = (u_1(x_3,t), u_2(x_3,t), 0). \tag{12}$$

A bounded function u is a mild bounded ancient solution if and only if there exists a pressure p such that $p = p^1 + p^2$, where

$$\triangle p^1 = -\mathrm{divdiv}\, u \otimes u \qquad (13)$$

in Q_-^+ with $p^1_{,3}(x', 0, t) = 0$ and $p^2(\cdot, t)$ is a harmonic function in $\mathbb{R}^3_+$ whose gradient satisfies the estimate

$$|\nabla p^2(x,t)| \leq c \ln(2 + 1/x_3) \qquad (14)$$

for all $(x,t) \in Q_-^+$ and has the property

$$\sup_{x' \in \mathbb{R}^2} |\nabla p^2(x,t)| \to 0 \qquad (15)$$

as $x_3 \to \infty$; u and p satisfy (11) and

$$\int\limits_{Q_-^+} \Big(u \cdot (\partial_t \varphi + \Delta \varphi) + u \otimes u : \nabla \varphi + p \mathrm{div}\, \varphi \Big) dxdt = 0 \qquad (16)$$

for any $\varphi \in C_0^\infty(Q_-)$ with $\varphi(x', 0, t) = 0$ for any $x' \in \mathbb{R}^2$ and for any $t < 0$.

If u is a mild bounded ancient solution, then $\nabla u \in L_\infty(Q^+_-)$. The function u is infinitely smooth in spatial variables in upper half space $x_3 > 0$.

Conjecture There is no non-trivial mild bounded ancient solution to the Navier-Stokes equations in the half space.

$$\partial_t v + v \cdot \nabla v - \triangle v = -\nabla q, \quad \text{div}\, v = 0 \tag{17}$$

in $\mathbb{R}^3_+ \times]0, \infty[$,

$$v(x', 0, t) = 0 \tag{18}$$

for any $x' \in \mathbb{R}^2$ and $t > 0$,

$$v(x, 0) = v_0(x) \in C^\infty_{0,0}(\mathbb{R}^3_+) \tag{19}$$

for any $x \in \mathbb{R}^3_+$. Here, $C^\infty_{0,0}(\mathbb{R}^3_+) = \{v \in C^\infty_0(\mathbb{R}^3_+) : \ \mathrm{div}\, v = 0\}$.

Assume that the initial boundary value problem has a solution that blows up at time T. There exists at least one non-trivial (non-zero) mild bounded ancient solution either in the whole space or in the half space.

Assume that for some positive constant A

$$|v(x,t)| \leq \frac{A}{x_3} \tag{20}$$

for all $x \in \mathbb{R}^3_+$ and $t \in]0, T[$. There exists a positive constant ε such that if the initial boundary value problem has a solution that blows up at time T. Then there must be

$$A \geq \varepsilon. \tag{21}$$