Unstability and interpolation

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MODERN ALGEBRA AND CLASSICAL GEOMETRY

Together with Edoardo Sernesi

June 27, 2017

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LC, Edoardo Sernesi

Nodal curves on surfaces of general type.

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The name comes from Severi's result on the varieties which parameterize singular nodal curves in $\mathbb{P}^2$.

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Definition

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When S has general type, the previous tool fails and one needs new ideas of investigation.

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New idea (Edoardo)

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I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces.* Ann. Math. 127 (1988), 309-316.

M. Beltrametti, P. Francia, A. J. Sommese, *On Reiders method and higher order embeddings.* Duke Math. J. 58 (1989), 425-439.

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Then, work with the arithmetic of S, M, K_S, L_D, in order to exclude the existence of M.

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Theorem (LC, E. Sernesi)

Let S be a smooth surface such that K_S is ample and let C be an irreducible curve on S such that $C =_{num} pK_S$; $p \geq 2$; $p \in \mathbb{Q}$ and the linear system L_D of C has smooth general member D. Assume that C has $\delta \geq 1$ nodes and no other singularities.
If $\delta < p(p-2)K_S^2/4$, then $V_{\delta,D}$ is smooth of codimension δ at C.

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Corollary

Let S be a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3$ with plane section H. Assume that $C \in |nH|$ has δ nodes and no other singularities and

$$\delta < \frac{nd(n-2d+8)}{4}.$$

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The result of the corollary is sharp, at least for $d = 5$.

Results with make use of the idea.

Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *New asymptotics in the geometry of equisingular families of curves*, Internat. Math. Res. Notices 13 (1997), 595–611.

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LC and Thomas Markwig, *Triple-Point Defective Ruled Surfaces*, J. Pure Appl. Alg. 212, n.6 (2008), 1337–1346.

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The basic interpolation problem

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B. Segre's conjecture in $\mathbb{P}^2$

If $\dim |L - \sum m_i P_i| >$ expected dimension, then $|L - \sum m_i P_i|$ has a fixed component of multiplicity ≥ 2.

More general surfaces S

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Some results

J. Huizenga, *Interpolation on surfaces in* $\mathbb{P}^3$. Trans. Amer. Math. Soc. 365 (2013), 623-644.

C. De Volder, A. Laface, *Recent results on linear systems on generic K3 surfaces.* Rend. Sem. Mat. Univ. Pol. Torino 63 (2005) 91-94.

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A. Landesman, A. Patel, *Interpolation Problems: Del Pezzo Surfaces.* (2016) arXiv:1601.05840

E. Mezzetti, R.M. Miró Roig, *Togliatti systems and Galois coverings.* (2016) arXiv:1611.05620

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Then, imposing **one** triple point gives only 5 conditions (expected 6).

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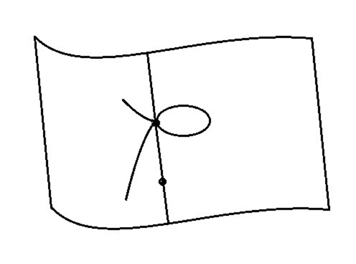

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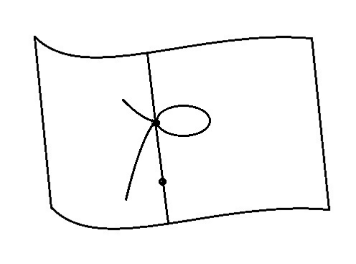

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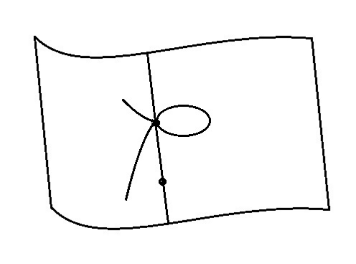

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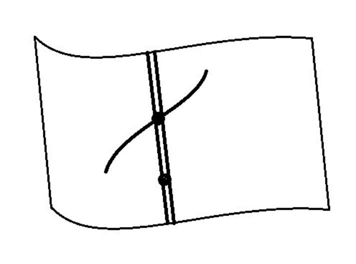

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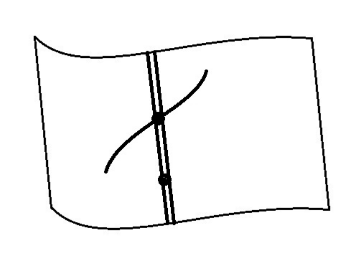

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The **Togliatti system**. A rather special projection of the 3-Veronese surface.

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CLASSICALLY

C. Segre and T.G. Room apparently stated that

"every triple-point defective surface is ruled".

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In Dye's example the degree is $8 = L^2 = (L - K_S)^2$

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(in the sense that the embedding sends the rulings to lines).

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The equisingular deformations of a triple point

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T. Keilen, *Families of Curves with Prescribed Singularities*. PhD thesis, Universit´at Kaiserslautern (2001).,
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Fix x, y local coordinates of a general point $P \in S$. Fix a curve $C \in |L_D|$ with a triple point in P.

Then the tangent space to the variety $V^3_{L_D}$ of curves in $|L_D|$ with a triple point has tangent space at C defined by

$$T(V^3_{L_D}) = H^0(\mathcal{I}_Z(L_D))/ < C >$$

where Z is the scheme, supported at P, defined by an ideal J isomorphic to either:

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J defines a (locally) complete intersection scheme Z (of length $3, 4$).

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Thus, by Serre's construction, we find a rank 2 vector bundle $\mathcal{E}$ with a section supported at the equisingular scheme Z.

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Let L be a very ample line bundle on S, such that $L - K_S$ is ample and base-point-free.
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Examples of ruled surfaces satisfying the assumption (and their description) can be found in:
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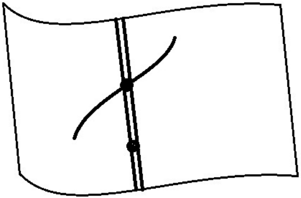

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Straightforward application to blow ups of $\mathbb{P}^2$

Fix multiplicities $m_1 \leq m_2 \leq \cdots \leq m_k$. Let H denote the class of a line in $\mathbb{P}^2$ and assume that, for $P_1, \ldots, P_k$ general in $\mathbb{P}^2$, the linear system $rH - m_1P_1 - \cdots - m_kP_k$ is defective. Let $f : S \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the points $P_2, \ldots, P_k$ and set $L := rf^*H - m_2E_2 - \cdots - m_kE_k$, where each E_i is the exceptional divisor at P_i.

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Assume that L is very ample on S, of the expected dimension, and that $L - K_S$ is ample and base-point-free, with $(L - K_S)^2 > 16$.

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Fix multiplicities $m_1 \leq m_2 \leq \cdots \leq m_k$. Let H denote the class of a line in $\mathbb{P}^2$ and assume that, for $P_1, \ldots, P_k$ general in $\mathbb{P}^2$, the linear system $rH - m_1 P_1 - \cdots - m_k P_k$ is defective. Let $f : S \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at the points $P_2, \ldots, P_k$ and set $L := rf^*H - m_2 E_2 - \cdots - m_k E_k$, where each E_i is the exceptional divisor at P_i.

Assume that L is very ample on S, of the expected dimension, and that $L - K_S$ is ample and base-point-free, with $(L - K_S)^2 > 16$.

Assume, finally, that $m_1 \leq 3$.

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Then $m_1 = 3$ and the general element of the linear system is non-reduced. Moreover L embeds S as a ruled surface. (Segre's conjecture holds).

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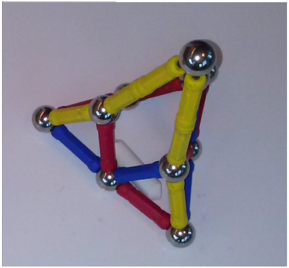

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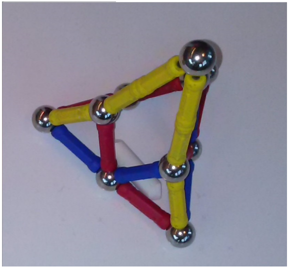

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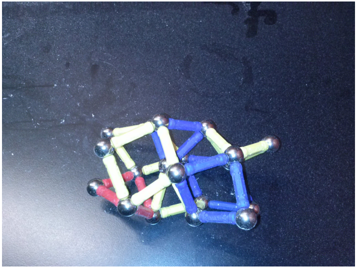

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