Well-posedness in smooth function spaces for the moving-boundary 3-D compressible Euler equations in physical vacuum

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References for this talk

[2] D. Coutand and S. Shkoller, Well-posedness in smooth function spaces for the moving-boundary 1-D compressible Euler equations in physical vacuum, Comm. Pure Appl. Math. (2011), 328–366.

[3] D. Coutand and S. Shkoller, Well-posedness in smooth function spaces for the moving-boundary 3-D compressible Euler equations in physical vacuum, Arch. Rational Mech. Anal., (2012), 515–616.

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John von Neumann on Vacuum

At the conference Problems of Cosmical Aerodynamics, held in Paris, 1949 where one session chaired by J. von Neumann was devoted to the existence and uniqueness or multiplicity of solutions of the Euler equations. Questions raised by von Neumann concerning the analysis of the Euler system in the presence of a vacuum led to many interesting discussions, which are still enlightening at present, among von Neumann and several participants including Heisenberg, who asked von Neumann to explain the role of boundary conditions near the vacuum front, and the relation of the vanishing density at the vacuum boundary with the nonlinear theory of turbulent boundary layers wherein the viscosity of the gas vanishes. J. von Neumann answered with the following response:

"The boundary layer theory for a fluid of low viscosity certainly furnishes a monumental warning. The naive and yet prima facie seemingly reasonable procedure would be to apply the ordinary equations of the ideal fluid and then to expect that viscosity will somehow take care of itself in a narrow region along the wall. We have learned that this procedure may lead to great errors; a complete theory of the boundary layer may give you completely different conditions also for the flow in the bulk of the field. It is possible that the same discipline will be necessary for the boundary with a vacuum."

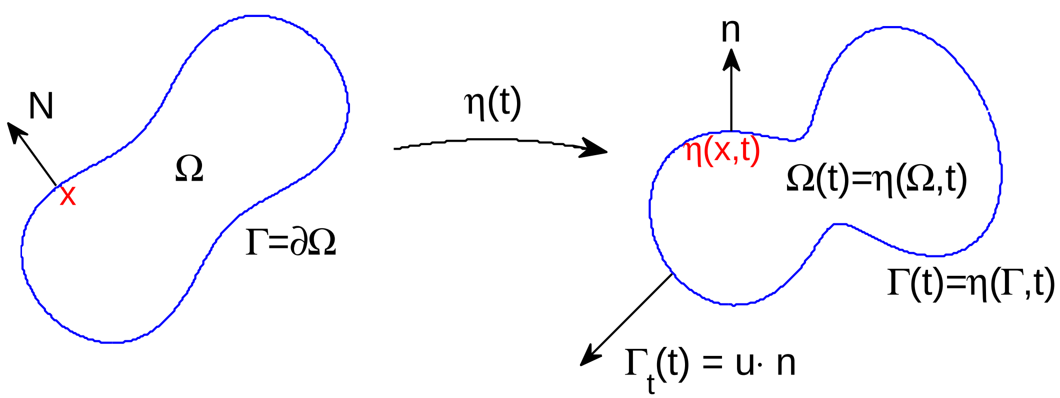
The “pair-instability” Supernova

- **The Biggest Big-Bang**, Scientific American, Sep. 30, 2010.
- 100 times more energy than an ordinary supernova

FIGURE 1: STARBURST: The remains of a supernova, as captured in a composite image by three NASA telescopes. Image: COURTESY OF NASA/CXC/SAO/ESA/ASU/JPL/CALTECH/UNIVERSITY OF MINNESOTA

Compressible Euler equations with free-boundary

$$
\begin{aligned}
\rho\left[\partial_t \mathbf{u} + (\mathbf{u}\cdot D)\mathbf{u}\right] + Dp(\rho) &= 0 && \text{in } \ \Omega(t)\,, \\
\rho_t + \operatorname{div}(\rho\mathbf{u}) &= 0 && \text{in } \ \Omega(t)\,, \\
\text{(vacuum BC)} \quad p(\rho) &= 0 && \text{on } \ \Gamma(t) := \partial\Omega(t)\,, \\
\text{(boundary motion)} \quad \partial_t \Gamma(t) &= \mathbf{u}\cdot n \\
\text{(initial data)} \quad (\mathbf{u}(0), \rho(0), \Omega(0)) &= (\mathbf{u}_0, \rho_0, \Omega)
\end{aligned}
$$



Equation of state

- **Equation of state**:

$$p(\rho) = \rho^\gamma, \quad \gamma > 1\,.$$
$$\rho > 0 \ \text{ in } \ \Omega(t)\,, \quad \rho = 0 \ \text{ on } \ \Gamma(t)\,.$$

- **Set** $\gamma = 2$:

$$\rho\,[\mathbf{u}_t + (\mathbf{u}\cdot D)\mathbf{u}] + D\rho^2 = 0\,.$$

or equivalently

$$\mathbf{u}_t + (\mathbf{u}\cdot D)\mathbf{u} + 2D\rho = 0\,.$$

Physical Vacuum Boundary

- $\underbrace{\mathbf{u}_t + (\mathbf{u} \cdot D)\mathbf{u}}_{\text{acceleration}} = -\frac{\gamma}{\gamma-1} D\rho^{\gamma-1}$ so gas *accelerates* if $\boxed{\frac{\partial}{\partial n}\rho^{\gamma-1} \neq 0}$.

- $\Gamma(t)$ is a *physical vacuum* boundary if $\rho_0^{\gamma-1} \sim \text{dist}(\Gamma)$ or

$$\rho_0^{\gamma-1}(x) \geq C \text{dist}(x, \Gamma) \quad x \text{ near } \Gamma \,.$$

- Rate of degeneracy:

$$\boxed{\rho_0 \sim \text{dist}(\Gamma)} \text{ for } \gamma = 2 \,.$$

- Sound speed $c \sim \sqrt{d}$

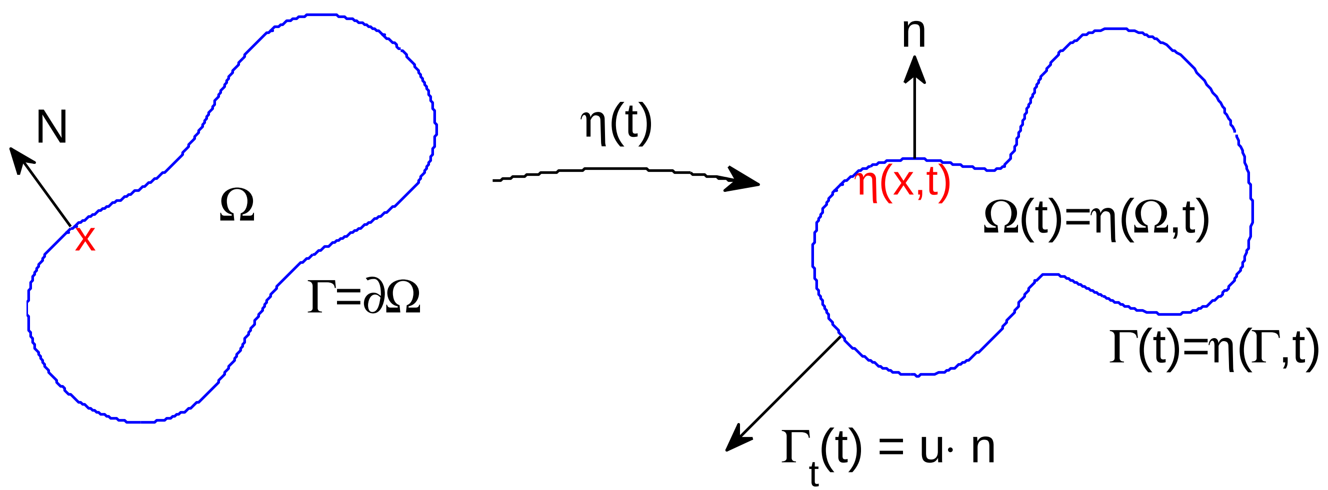
- Degenerate and characteristic hyperbolic-type, moving boundary

Prior Results – Moving Surfaces of Discontinuity

- Multi-D Shocks, non-characteristic hyperbolic system
  - Majda (1984), Glimm and Majda (1991), Francheteau and Métivier (2000), Métivier (2001), Gués, Métivier, Williams, and Zumbrun (2005)
- Multi-D vortex or current-vortex sheets, characteristic hyperbolic system
  - Linearization tends to lose derivatives – Nash-Moser typically used
  - G.-Q. Chen and Wang (2007), Coulombel and P. Secchi (2008,2009), Trakhinin (2005,2009)
- Multi-D free-boundary compressible liquid, characteristic, uniformly hyperbolic $\rho \geq > \lambda > 0$ on Γ
  - Derivative loss, Lindblad (2005), Trakhinin (2009)
  - No derivative loss, Coutand, Hole, & Shkoller (201)

- 1-D compressible Euler with physical vacuum is characteristic and degenerate
  - Early results due to T.P. Liu and Smoller (1980) and Lin (1987)
  - Damped case, Liu (1996)
  - Sound speed $c \sim d$ case studied by Makino (1986) and Liu and Yang (1997, 2000)
  - Jang and Masmoudi (2009), well-posedness in weighted Sobolev spaces – no regularity in standard Sobolev spaces
  - Coutand and Shkoller (2009), well-posedness and regularity up to the boundary in H^2
- 3-D compressible Euler with physical vacuum is characteristic and degenerate
  - A priori estimates, Coutand, Lindblad, and Shkoller (2009)
  - Well-posedness and regularity up to the boundary (degenerate parabolic approximation), Coutand and Shkoller (2011)
  - Existence/uniquess (normal derivative reformulation), Jang and Masmoudi (2013)

Fixing the domain – Lagrangian Variables



Let $\eta(\cdot, t) : \Omega \to \Omega(t)$ be the solution of

$$\eta_t = u \circ \eta, \quad \eta(x, 0) = \mathrm{Id} \quad \text{or} \quad \eta = \mathrm{Id} + \int_0^t v$$

$$v = u \circ \eta, \quad f := \rho \circ \eta$$

$$A(x,t) := [D\eta(x,t)]^{-1}, \quad J = \det D\eta\,, \quad a(x,t) := \mathrm{Cof}(D\eta) = JA$$

Euler equations – Lagrangian Variables, $\gamma = 2$

- Density-Jacobian relation: $f := \rho(\eta) = \rho_0 J^{-1}$.

$$\rho_t + (u \cdot D)\rho = -\rho \operatorname{div} u \longrightarrow f_t = -f A^j_i v^i,_j \longrightarrow f(t) = \rho_0 e^{-\int_0^t A^j_i v^i,_j} \text{ and } J(t) = e^{\int_0^t A^j_i v^i,_j}$$

$$\begin{aligned} v^i_t + 2A^k_i[\overbrace{\rho_0 J^{-1}}^{f}],_k &= 0 && \text{in } \Omega \times (0,T]\,, \\ \rho_0 &= 0 && \text{on } \Gamma \times (0,T]\,, \\ (\eta, v, f) &= (\text{Id}, u_0, \rho_0) && \text{on } \Omega \times \{t=0\}\,. \end{aligned}$$

- Lagrangian divergence and curl

$$\operatorname{div}_\eta(v) := \operatorname{div} u \circ \eta = A^j_i v^i, j$$
$$\operatorname{curl}_\eta(v) := \operatorname{curl} u \circ \eta = \varepsilon_{ijk} A^r_j v^k,_r$$

- Three forms of Euler used for the analysis:

$$v^i_t + 2A^k_i(\rho_0 J^{-1}),_k = 0 \quad \text{(curl estimates)}$$
$$\rho_0 v^i_t + a^k_i(\rho_0^2 J^{-2}),_k = 0 \quad \text{(energy estimates)}$$
$$v^i_t + \rho_0 a^k_i J^{-2},_k + 2a^3_i J^{-2} = 0 \quad \text{(elliptic-type estimates)}\,.$$

The reference domain Ω

Initial domain $\Omega \subset \mathbb{R}^3$ at time $t = 0$ is given by

$$\Omega = \mathbb{T}^2 \times (0, 1)$$

- *One* global Cartesian coordinate system
- At $t = 0$, the reference *vacuum* boundary is

$$\Gamma = \{x_3 = 0\} \cup \Gamma = \{x_3 = 1\},$$

- All functions are 1-periodic in x_1 and x_2
- The moving vacuum boundary is then given by

$$\Gamma(t) = \eta(t)(\Gamma).$$

The higher-order energy function

- Physical energy $\int_\Omega \left[\frac{1}{2}\rho_0|v|^2 + \rho_0^2 J^{-1}\right] dx$ is a conserved quantity
- Notation: $\|F\|_s := \|F\|_{H^s(\Omega)}$ and $|F|_s := \|F\|_{H^s(\Gamma)}$
- Define higher-order energy function

$$E(t) = \|\eta(t)\|_4^2 + \|v_t(t)\|_3^2 + \|v_{ttt}(t)\|_2^2 + \|\partial_t^5 v(t)\|_1^2 + \|\partial_t^7 v(t)\|_0^2$$
$$+ \|\rho_0 D\eta(t)\|_4^2 + \|\rho_0 Dv_t(t)\|_3^2 + \|\rho_0 Dv_{ttt}(t)\|_2^2 + \|\rho_0 \ D\partial_t^5 v(t)\|_1^2 + \|\rho_0 \partial_t^7 Dv(t)\|_0^2$$

- Set $M_0 = P(E(0))$.

Theorem 1 (Existence and uniqueness for the case $\gamma = 2$) *Suppose that $\rho_0 \in H^4(\Omega)$, $\rho_0(x) > 0$ for $x \in \Omega$, and $\rho_0 \sim d$. Furthermore, suppose that u_0 is given such that $M_0 < \infty$. Then there exists a solution on $[0,T]$ for $T > 0$ taken sufficiently small, such that*

$$\sup_{t \in [0,T]} E(t) \leq 2M_0 \,.$$

In particular, the flow map $\eta \in L^\infty(0,T; H^4(\Omega))$ and the moving vacuum boundary $\Gamma(t)$ is of Sobolev class $H^{3.5}$. Existence and uniqueness if we take one derivative better regularity on data.

Estimates: A standard inequality

For a constant $M_0 \geq 0$, suppose that $f(t) \geq 0$, $t \mapsto f(t)$ is continuous, and

$$f(t) \leq M_0 + t\, P(f(t)) \,,$$

where P denotes a polynomial function. Then for t taken sufficiently small, we have the bound

$$f(t) \leq 2M_0 \,.$$

Objective: establish the following estimate

$$\boxed{\sup_{t\in[0,T]} E(t) \leq M_0 + T P(\sup_{t\in[0,T]} E(t))}$$

The embedding of a weighted Sobolev space

- d = distance function to the boundary Γ
- let $p = 1$ or 2
- weighted Sobolev space $H^1_{d^p}(\Omega)$ with norm

$$\int_\Omega \left(|F(x)|^2 + |DF(x)|^2\right) \underbrace{d(x)^p\, dx}_{\text{weighted measure}}$$

satisfies the following embedding: $H^1_{d^p}(\Omega) \hookrightarrow H^{1-\frac{p}{2}}(\Omega)$

- Since $\rho_0 \sim d$,

$$\boxed{\|F\|_0^2 \leq C \int_\Omega \left(|F(x)|^2 + |DF(x)|^2\right) \rho_0^2\, dx}$$

Standard Energy Method Very Problematic

Consider non-linear wave equation on $(0,1) \times (0,T]$:

$$x\eta_{tt} + [x^2 \frac{1}{(\eta_x)^2}]_x = f$$

- Scaling is $\eta(t) \in H^k(0,1), \eta_t(t) \in H^{k-1/2}(0,1), \eta_{tt}(0,1) \in H^{k-1}(0,1), \ldots.$
- Differentiate:

$$x\eta_{xtt} - 2[x^2 \frac{1}{(\eta_x)^3} \eta_{xx}]_x = -\eta_{tt} - 2[x \frac{1}{(\eta_x)^2}]_x$$

- RHS same order as LHS, but loses the exact derivatives.
- Build space regularity with time-derivatives and Hardy-type elliptic estimates.
- This requires isolating certain nonlinear structure and boot-strapping in a certain order.

Properties of J, a and nonlinear interaction

Differentiating the Jacobian.

$$\bar\partial J = a^s_r \bar\partial \frac{\partial \eta^r}{\partial x^s} \text{ horizontal differentiation}\,, \quad \bar\partial = (\partial_{x_1}, \partial_{x_2})$$

$$\partial_t J = a^s_r \frac{\partial v^r}{\partial x^s} \quad \text{time differentiation (using } v = \eta_t)\,.$$

Differentiating the cofactor matrix.

$$\bar\partial a^k_i = \bar\partial \frac{\partial \eta^r}{\partial x^s} J^{-1}(a^s_r a^k_i - a^s_i a^k_r) \text{ horizontal differentiation}$$

$$\partial_t a^k_i = \frac{\partial v^r}{\partial x^s} J^{-1}(a^s_r a^k_i - a^s_i a^k_r) \quad \text{time differentiation } (v = \eta_t)$$

How to view this.

$$\bar\partial J \sim \bar\partial \operatorname{div} \eta\,, \quad \partial_t J \sim \operatorname{div} v\,,$$

$$\boxed{\bar\partial a^k_i \bar\partial v^i,_k = \underbrace{\bar\partial D\eta \bar\partial Dv}_{\text{good sign}} - \underbrace{\bar\partial \operatorname{div} \eta \bar\partial \operatorname{div} v}_{\text{bad sign}} - \underbrace{\bar\partial \operatorname{curl} \eta \bar\partial \operatorname{curl} v}_{\text{bad sign}}}$$

Piola and geometric identities

- Piola identity: $a_i^k{,_k} = 0$ for $i = 1, 2, 3$.
- The vectors $\eta{,_\alpha}$ for $\alpha = 1, 2$ span the tangent plane to the surface Γ in $\mathbb{R}^3$, and

$$n(t) := \frac{\eta{,_1} \times \eta{,_2}}{|\eta{,_1} \times \eta{,_2}|} = \frac{\eta{,_1} \times \eta{,_2}}{\sqrt{g}}$$

  is the outward-pointing unit normal vector to the moving boundary $\Gamma(t)$.
- The third row of the cofactor matrix is a normal vector to $\Gamma(t)$:

$$a_i^3 = \begin{bmatrix} \eta^2{,_1} \eta^3{,_2} - \eta^3{,_1} \eta^2{,_2} \\ \eta^3{,_1} \eta^1{,_2} - \eta^1{,_1} \eta^3{,_2} \\ \eta^1{,_1} \eta^2{,_2} - \eta^1{,_2} \eta^2{,_1} \end{bmatrix} = \eta{,_1} \times \eta{,_2} \quad \longrightarrow \quad \boxed{\boxed{a_i^3 = P(\bar{\partial}\eta)}}.$$

It follows that

$$\boxed{n(t) = a_i^3 / \sqrt{g}}.$$

Trace and Elliptic Estimates

- Normal trace: N outward unit normal to Γ

$$\|\bar{\partial} w \cdot N\|^2_{H^{-0.5}(\Gamma)} \leq C\Big[\|\bar{\partial} w\|^2_{L^2(\Omega)} + \|\mathrm{div} w\|^2_{L^2(\Omega)}\Big] .$$

- Tangential trace: τ_1 and τ_2 span the tangent space to Γ

  For $\alpha = 1, 2$

$$\|\bar{\partial} w \cdot \tau_\alpha\|^2_{H^{-0.5}(\Gamma)} \leq C\Big[\|\bar{\partial} w\|^2_{L^2(\Omega)} + \|\mathrm{curl} w\|^2_{L^2(\Omega)}\Big] .$$

- Hodge-type elliptic estimate: For $s \geq 1$, $\alpha = 1, 2$,

$$\|F\|_s \leq \bar{C}\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + |\bar{\partial} F \cdot N|_{s-\frac{1}{2}}\right)$$

$$\|F\|_s \leq \bar{C}\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + \sum_{\alpha=1}^{2} |\bar{\partial} F \cdot \tau_\alpha|_{s-\frac{1}{2}}\right) .$$

Formal Curl estimates

- Use $v^i_t + \underbrace{2A^k_i(\rho_0 J^{-1}),_k}_{\text{Lag. Gradient}} = 0$; recall that $\boxed{2A^k_i(\rho_0 J^{-1}),_k = 2D\rho \circ \eta}$.
- Notation $\operatorname{curl}_\eta v = \operatorname{curl} u \circ \eta = \varepsilon_{\cdot jk} A^r_j v^k,_r$
- Take curl_η of this equation: $\operatorname{curl}_\eta v_t = 0$.
- Time integrate:

$$\operatorname{curl}_\eta v(t) \sim \operatorname{curl} u_0 + \int_0^t P(Dv(t'), A(t'))dt'$$

- This yields the a priori estimate: $\forall t \in (0,T)$,

$$\sum_{a=0}^{3} \|\operatorname{curl} \partial_t^{2a} \eta(t)\|^2_{3-a} + \sum_{l=0}^{4} \|\rho_0 \, \bar\partial^{4-l} \operatorname{curl} \partial_t^{2l} \eta(t)\|^2_0 \le M_0 + C\, T\, P(\sup_{t\in[0,T]} E(t))\,.$$

Energy estimates

- Use $\rho_0 v_t^i + a_i^k(\rho_0^2 J^{-2}),_k = 0$ – cofactor a_i^k is crucial for energy estimates
- Start with the $\bar\partial^4$-problem

$$
\begin{aligned}
0 &= \int_\Omega \bar\partial^4(\rho_0 v_t^i + a_i^k(\rho_0^2 J^{-2}),_k)\,\bar\partial^4 v^i\ dx \\
&= \frac{1}{2}\|\sqrt{\rho_0}\bar\partial^4 v(t)\|_0^2 + \int_\Omega \underline{\bar\partial^4 a_i^k}(\rho_0^2 J^{-2}),_k\,\bar\partial^4 v^i dx + \int_\Omega a_i^k(\rho_0^2\underline{\bar\partial^4 J^{-2}}),_k\,\bar\partial^4 v^i dx + \text{l.o.t.}
\end{aligned}
$$

- Main energy contribution comes from

$$
\int_\Omega \underline{\bar\partial^4 a_i^k}(\rho_0^2 J^{-2}),_k\,\bar\partial^4 v^i dx = -\int_\Omega \underline{\bar\partial^4 a_i^k}(\rho_0^2 J^{-2})\bar\partial^4 v^i,_k\,dx
$$

where we crucially use

$$
a_i^k,_k = 0
$$

and the identity

$$
\boxed{-\bar\partial^4 a_i^k \bar\partial^4 v^i,_k = \bar\partial^4 D\eta\bar\partial^4 Dv - \bar\partial^4 \operatorname{div}\eta\bar\partial^4 \operatorname{div} v - \bar\partial^4 \operatorname{curl}\eta\bar\partial^4 \operatorname{curl} v}
$$

- The term $\int_\Omega a_i^k(\rho_0^2 \underline{\bar\partial^4 J^{-2}}),_k \bar\partial^4 v^i dx$ controls divergence:

$$\int_\Omega a_i^k(\rho_0^2 \underline{\bar\partial^4 J^{-2}}),_k \bar\partial^4 v^i dx = -\int_\Omega (\rho_0^2 \underline{\bar\partial^4 J^{-2}})\bar\partial^4 v^i,_k a_i^k dx$$
$$\sim 2\int_\Omega \rho_0^2 \bar\partial^4 \operatorname{div}\eta\, \bar\partial^4 \operatorname{div} v\, dx$$

and with curl estimates

$$\sup_{t\in[0,T]} \int_\Omega \rho_0^2(x)|\bar\partial^4 D\eta(x,t)|^2 dx \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- Using the weighted embedding

$$\sup_{t\in[0,T]} \|\bar\partial^4 \eta(t)\|_0^2 \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- Control GEOMETRY of moving hypersurface: for $\alpha = 1, 2,$

$$\underbrace{\sup_{t\in[0,T]} |\eta^\alpha(t)|_{3.5}^2}_{\text{boundary regularity}} \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- The $\bar{\partial}^3 \partial_t^2$-problem:

$$\sup_{t\in[0,T]} \left(\|\bar{\partial}^3 v_t(t)\|_0^2 + \|\rho_0 \bar{\partial}^3 D v_t(t)\|_0^2 + |v^\alpha{}_t|_{2.5}^2 \right) \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- The $\bar{\partial}^2 \partial_t^4$-problem:

$$\sup_{t\in[0,T]} \left(\|\bar{\partial}^2 v_{ttt}(t)\|_0^2 + \|\rho_0 \bar{\partial}^2 D v_{ttt}(t)\|_0^2 + |v^\alpha{}_{ttt}|_{1.5}^2 \right) \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- The $\bar{\partial} \partial_t^6$-problem:

$$\sup_{t\in[0,T]} \left(\|\bar{\partial} \partial_t^5 v(t)\|_0^2 + \|\rho_0 \bar{\partial} D \partial_t^5 v(t)\|_0^2 + |\partial_t^5 v^\alpha|_{0.5}^2 \right) \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- Finally, the ∂_t^8-problem: (full regularity)

$$\sup_{t\in[0,T]} \left(\|\partial_t^7 v(t)\|_0^2 + \|\rho_0 D \partial_t^7 v(t)\|_0^2 \right) \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

OBJECTIVE: $\partial_t^7 v \in L^2(\Omega)$ to $\partial_t^5 v \in H^1(\Omega)$

- Energy plus curl estimates show that

$$\| \operatorname{curl} \partial_t^5 v\|_0^2 + |\partial_t^5 v^\alpha|_{0.5}^2 \leq M_0 + C\, T\, P(\sup_{t\in[0,T]} E(t))$$

- In order to prove that $\partial_t^5 v(t)$ is bounded, we need

$$\|\operatorname{div} \partial_t^5 v\|_0^2 \leq M_0 + C\, T\, P(\sup_{t\in[0,T]} E(t))$$

- Elliptic-type estimates with the form

$$\boxed{v_t^i + \rho_0 a_i^k J^{-2},_k + 2\rho_{0,3}\, a_i^3 J^{-2} = 0}$$

- For notational simplicity, suppose that $\rho_{0,3} = 1$ and recall $J_t \sim \operatorname{div} v$

Building regularity $\partial_t^5 v \in H^1(\Omega)$

- Using that $\partial_t^7 v(t) \in L^2(\Omega)$ bounded, we take six time-derivatives of Euler

$$\rho_0 a_i^3 \partial_t^6 J^{-2},_3 + 2 a_i^3 \partial_t^6 J^{-2} = -\partial_t^7 v + \text{l. o. t}$$

- This is equivalent to

$$\rho_0 \operatorname{div} \partial_t^5 v,_3 + 2 \operatorname{div} \partial_t^5 v = -\partial_t^7 v + \text{l. o. t}$$

- Using the energy control that we already have, we see that

$$4\| \operatorname{div} \partial_t^5 v\|_0^2 + \|\rho_0 \operatorname{div} \partial_t^5 v,_3 \|_0^2 + 2 \int_\Omega \rho_0 |\operatorname{div} \partial_t^5 v|^2,_3 \, dx \le M_0 + C\, T\, P(\sup_{t\in[0,T]} E(t))$$

- Cross-term

$$2 \int_\Omega \rho_0 |\operatorname{div} \partial_t^5 v|^2,_3 \, dx = -2 \int_\Omega |\operatorname{div} \partial_t^5 v|^2 dx$$

controlled by integration-by-parts and $\rho_0 = 0$ on Γ ONLY when $\underline{\operatorname{div} \partial_t^5 v(t)}$ is *sufficiently smooth*!!

A simple ODE example

- Consider the ODE

$$\boxed{xf' + 2f = 0}.$$

- The sum $xf' + 2f \in L^2$
- The solution $f = \frac{1}{x^2} \notin L^2$
- If we rule-out any singular behavior then the integration-by-parts argument will give a bound for $f \in L^2$

$$\begin{aligned}\int_0^\infty (xf' + 2f)^2 dx &= \int_0^\infty (x^2 f'^2 + 4f^2 + 4xff')dx \\ &= \int_0^\infty (x^2 f'^2 + 4f^2 + 2x(f^2)')dx = \int_0^\infty (x^2 f'^2 + 2f^2)dx\end{aligned}$$

Completing the regularity argument

- As long as the assumed solution is *sufficiently smooth*, then

$$\operatorname{div} \partial_t^5 v(t) \in L^2(\Omega)$$

- We have already proven that

$$\operatorname{curl} \partial_t^5 v(t) \in L^2(\Omega)$$

and for $\alpha = 1, 2$

$$\partial_t^5 v^\alpha(t) \in H^{0.5}(\Gamma)$$

- Our elliptic estimate then shows that $\boxed{\partial_t^5 v(t) \in H^1(\Omega)}$
- Then $\rho_0 \operatorname{div} v_{ttt},_3 + 2 \operatorname{div} v_{ttt} = -\partial_t^5 v + \mathrm{l.\,o.\,t} \in H^1(\Omega)$
- Same argument shows $v_{ttt}(t) \in H^2(\Omega) \longrightarrow v_t(t) \in H^3(\Omega) \longrightarrow \eta(t) \in H^4(\Omega)$

Constructing Solutions: The approximate κ-problem

- Degenerate parabolic regularization, the κ-problem: For $\kappa > 0$,

$$\begin{aligned} \rho_0 v_t^i + a_i^k(\rho_0^2 J^{-2}),_k + \kappa\partial_t\left[a_i^k(\rho_0^2 J^{-2}),_k\right] &= 0 && \text{in } \Omega \times (0, T_\kappa]\,, \\ (\eta, v) &= (e, u_0) && \text{on } \Omega \times \{t=0\}\,, \\ \rho_0 &= 0 && \text{on } \Gamma\,. \end{aligned}$$

- Initial data is smoothed
- Preserves the geometric (determinant) structure of energy estimates
- Preserves the elliptic-type estimates for normal derivatives
- Does not quite preserve the structure of vorticity, but we can nevertheless asymptotically control curl estimates

Two Lemmas

Lemma 0.1 (Higher-order Hardy's inequality) *Let $s \geq 1$ be a given integer, and suppose that*

$$u \in H^s(\Omega) \cap \dot{H}^1_0(\Omega)\,.$$

If $d(x) > 0$ for $x \in \Omega$, $d \in H^{s-1}(\Omega)$, and d is dist function near Γ, then

$$\left\| \frac{u}{d} \right\|_{s-1} \leq C \|u\|_s.$$

Lemma 0.2 *Let $\kappa > 0$ and $g \in L^\infty(0,T; H^s(\Omega)))$ be given, and let $f \in H^1(0,T; H^s(\Omega))$ be such that*

$$f + \kappa f_t = g \quad \textit{in } (0,T) \times I.$$

Then,

$$\|f\|_{L^\infty(0,T;H^s(\Omega))} \leq C \max\{\|f(0)\|_s, \|g\|_{L^\infty(0,T;H^s(\Omega))}\}.$$

Energy function for the asymptotics $\kappa \to 0$.

THE NORM: We set on $[0, T_\kappa]$

$$\tilde{E}(t) = 1 + \sum_{a=0}^{4} \Big[\underbrace{\|\rho_0\, \partial_t^{2a} D\eta(t)\|_{4-a}^2}_{\text{nonlinear det structure}} + \underbrace{\|\partial_t^{2a}\eta(t)\|_{4-a}^2}_{\text{weighted embedding}} \Big] + \sum_{a=0}^{4} \underbrace{\|\sqrt{\rho_0}\, \bar\partial^{4-a}\partial_t^{2a} v(t)\|_0^2}_{\text{inertia}}$$

$$+ \sum_{a=0}^{4} \underbrace{\int_0^t \|\sqrt{\kappa}\rho_0\, \partial_t^{2a} Dv(s)\|_{4-a}^2\, ds}_{\text{artificial viscosity}}$$

TO PROVE:

- $\sup_{t\in[0,T]} \tilde{E}(t) \le 2M_0$ on $[0, T]$, T is independent of κ
- Curl estimates are κ-independent
- Energy estimates for time and horizontal derivatives are κ-indepent
- Elliptic-type estimates for vertical derivatives are κ-independent

Curl estimates for κ-problem

- Vorticity not exactly transported: $\boxed{\operatorname{curl}_\eta v_t = 2\kappa\Big([Du]^T D^2\rho\Big)\circ\eta}$
- Estimates for κ-problem:

$$\sum_{a=0}^{3}\|\operatorname{curl}\partial_t^{2a}\eta(t)\|_{3-a}^2 + \sum_{l=0}^{4}\|\rho_0\,\bar\partial^{4-l}\operatorname{curl}\partial_t^{2l}\eta(t)\|_0^2 + \sum_{l=0}^{4}\int_0^t \|\sqrt{\kappa}\rho_0\operatorname{curl}_\eta\bar\partial^{4-l}\partial_t^{2l}v(s)\|_0^2 ds$$
$$\le M_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde E(t))\,.$$

$$D^3\operatorname{curl}\eta(t) = \text{ good terms } + \kappa\int_0^t\int_0^{t'} Dv\,A\,D^4\big[D\rho(\eta)\big]\,dt''dt'\,.$$

- Key observation to estimate $\|\operatorname{curl}\eta(t)\|_3$:

$$v_t + 2D\rho\circ\eta + 2\kappa[D\rho\circ\eta]_t = 0$$

so that

$$2\int_0^t\int_0^{t'} D^4(D\rho\circ\eta)dt''dt' + 2\kappa\int_0^t D^4(D\rho\circ\eta)dt' = -D^4\eta(t) + tD^4u_0$$

Energy estimates and the reason for the κ-viscosity

- Expanding the time-derivative in the κ-term:

$$\partial_t[a_i^k(\rho_0^2 J^{-2}),_k] = \underbrace{\partial_t a_i^k(\rho_0^2 J^{-2}),_k}_{\text{main term}} + \underbrace{a_i^k(\rho_0^2 \partial_t J^{-2}),_k}_{\text{div term}}$$

- Same nonlinear interaction between cofactor a_i^k and $v^i,_k$ (recall $v = \eta_t$):

$$-\kappa\rho_0^2\bar\partial^4\partial_t a_i^k\bar\partial^4 v^i,_k = \kappa|\rho_0\bar\partial^4 Dv|^2 - \kappa|\rho_0\bar\partial^4 \operatorname{div} v|^2 - \kappa|\rho_0\bar\partial^4 \operatorname{curl} v|^2$$

Energy estimates for κ-problem

- The $\bar\partial^4$-problem:

$$\sup_{t\in[0,T]} \left(\|\bar\partial^4\eta(t)\|_0^2 + \|\rho_0\bar\partial^4 D\eta(t)\|_0^2 + |\eta^\alpha(t)|_{2.5}^2\right) + \sqrt{\kappa}\int_0^t \|\rho_0\bar\partial^4 Dv\|_0^2 \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- The $\bar\partial^3\partial_t^2$-problem:

$$\sup_{t\in[0,T]} \left(\|\bar\partial^3 v_t(t)\|_0^2 + \|\rho_0\bar\partial^3 Dv_t(t)\|_0^2 + |v^\alpha{}_t|_{2.5}^2\right) + \sqrt{\kappa}\int_0^t \|\rho_0\bar\partial^3 Dv_{tt})\|_0^2 \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- The $\bar\partial^2\partial_t^4$-problem:

$$\sup_{t\in[0,T]} \left(\|\bar\partial^2 v_{ttt}(t)\|_0^2 + \|\rho_0\bar\partial^2 Dv_{ttt}(t)\|_0^2 + |v^\alpha{}_{ttt}|_{1.5}^2\right) + \sqrt{\kappa}\int_0^t \|\rho_0\bar\partial^2 D\partial_t^4 v)\|_0^2 \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- The $\bar\partial\partial_t^6$-problem:

$$\sup_{t\in[0,T]} \left(\|\bar\partial\partial_t^5 v(t)\|_0^2 + \|\rho_0\bar\partial D\partial_t^5 v(t)\|_0^2 + |\partial_t^5 v^\alpha|_{0.5}^2\right) + \sqrt{\kappa}\int_0^t \|\rho_0\bar\partial D\partial_t^6 v)\|_0^2 \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

- Finally, the ∂_t^8-problem: (full regularity)

$$\sup_{t\in[0,T]} \left(\|\partial_t^7 v(t)\|_0^2 + \|\rho_0 D\partial_t^7 v(t)\|_0^2\right) + \sqrt{\kappa}\int_0^t \|\rho_0 D\partial_t^8 v)\|_0^2 \le M_0 + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t))\,.$$

Regularity via Elliptic-type estimates for κ-problem

- Getting $\partial_t^5 v(t) \in H^1(\Omega)$ from $\partial_t^7 v(t) \in L^2(\Omega)$

$$\kappa \partial_t^7 \left[A_i^k (\rho_0 J^{-1}),_k \right] + 2\partial_t^6 \underbrace{\left[A_i^k (\rho_0 J^{-1}),_k \right]}_{Dp\circ\eta} = -\partial_t^7 v^i \,.$$

According to Lemma 2, independent of $\kappa > 0$,

$$\left\| \partial_t^6 \left[2 A_i^k (\rho_0 J^{-1}),_k \right] \right\|_0^2 \leq \|\partial_t^7 v(t)\|_0^2 \leq M_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\, T\, P(\sup_{t\in[0,T]} \tilde{E}(t)) \,.$$

- Just as in formal estimate,

$$\|\rho_0 \operatorname{div} \partial_t^5 v,_3 + 2 \operatorname{div} \partial_t^5 v\|_0^2 \leq M_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\, T\, P(\sup_{t\in[0,T]} \tilde{E}(t))$$

- As before, $\partial_t^5 v(t) \in H^1(\Omega) \longrightarrow v_{ttt}(t) \in H^2(\Omega) \longrightarrow v_t(t) \in H^3(\Omega) \longrightarrow \eta(t) \in H^4(\Omega)$

Construction of solutions for the κ-problem

- κ-approximation to Euler $\rho_0 v_t^i + a_i^k(\rho_0^2 J^{-2}),_k + \kappa\partial_t\left[a_i^k(\rho_0^2 J^{-2}),_k\right] = 0$
- Compute div_η of κ-problem:

$$a_i^j v_t^i,_j - 2\kappa\left[a_i^j a_i^k \frac{1}{\rho_0}(\rho_0^2 J^{-3} J_t),_k\right],_j = -\kappa[a_i^j \partial_t a_i^k \frac{1}{\rho_0}(\rho_0^2 J^{-2}),_k],_j - 2[a_i^j A_i^k(\rho_0 J^{-1}),_k],_j \ .$$

- Introduce $X = \rho_0 J^{-2}\,\text{div}_\eta\, v$ — degenerate parabolic equation

$$\frac{J^3 X_t}{\rho_0} - 2\kappa\left[a_i^j a_i^k \frac{1}{\rho_0}(\rho_0 X),_k\right],_j = -\kappa[a_i^j \partial_t a_i^k \frac{1}{\rho_0}(\rho_0^2 J^{-2}),_k],_j - 2[a_i^j A_i^k(\rho_0 J^{-1}),_k],_j - 3J^{-1}(J_t)^2 + \partial_t a_i^j\, v^i,_j \ .$$

- We will use our higher-order Hardy-type inequality to solve and build regularity for X.

Defining vector field v given X

- Divergence of v_t

$$\operatorname{div}_\eta v_t = \frac{(XJ^2)_t}{\rho_0} - \partial_t A^j_i v^i,_j \ .$$

- Curl of v_t

$$\operatorname{curl}_\eta v_t = 2\kappa\Big([Du]^T D^2\rho\Big) \circ \eta$$

- Normal component of v_t on Γ

$$v^3_t = -2\kappa J^{-2}\partial_t a^3_3 \rho_0,_3 - 2\kappa \partial_t J^{-2}\, a^3_3 \rho_0,_3 - 2J^{-2} a^3_3 \rho_0,_3$$

- $\partial_t a^3_3 \sim div_\Gamma v = v^1,_2 + v^2,_2$ –amazing estimates for this PDE on Γ

Solution via a fixed-point scheme

$$
\begin{aligned}
\operatorname{div} v_t &= \operatorname{div} \bar v_t - \operatorname{div}_{\bar\eta} \bar v_t + \frac{[\bar X \bar J^2]_t}{\rho_0} - \partial_t \bar A^j_i \bar v^i,_j && \text{in } \Omega\,,\\
\operatorname{curl} v_t &= \operatorname{curl} \bar v_t - \operatorname{curl}_{\bar\eta} \bar v_t + 2\kappa \varepsilon_{\cdot ji} \bar v,^r_s \bar A^s_j\, \bar\Xi,^i_r (\bar\eta) && \text{in } \Omega\,,\\
v^3_t + 2\kappa\rho_{0,3} \operatorname{div}_\Gamma v &= 2\kappa\rho_{0,3} \operatorname{div}_\Gamma \bar v - 2\rho_{0,3}\, \bar J^{-2} \bar a^3_3 \\
&\quad - 2\kappa\rho_{0,3}\, \bar J^{-2} \partial_t \bar a^3_3 - 2\kappa\rho_{0,3}\, \bar a^3_3 \partial_t \bar J^{-2} + \bar c(t) N^3\,, && \text{on } \Gamma\\
\int_\Omega v^\alpha_t dx &= -2 \int_\Omega \bar A^k_\alpha (\rho_0 \bar J^{-1}),_k dx - 2\kappa \int_\Omega \partial_t [\bar A^k_\alpha (\rho_0 \bar J^{-1}),_k] dx\,,\\
&(x_1, x_2) \mapsto v_t(x_1, x_2, x_3, t) \text{ is 1-periodic}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar c(t) = &\frac{1}{2} \int_\Omega (\operatorname{div} \bar v_t - \operatorname{div}_{\bar\eta} \bar v_t) dx + \frac{1}{2} \int_\Omega \frac{[\bar X \bar J^2]_t}{\rho_0} dx - \frac{1}{2} \int_\Omega \partial_t \bar A^j_i \bar v^i,_j dx + \int_\Gamma \bar J^{-2} \bar a^3_3 \rho_{0,3}\, N^3 dS \\
&+ \kappa \int_\Gamma \bar J^{-2} \partial_t \bar a^3_3 \rho_{0,3}\, N^3 dS + \kappa \int_\Gamma \partial_t \bar J^{-2} \bar a^3_3 \rho_{0,3}\, N^3 dS + \kappa \int_\Gamma \operatorname{div}_\Gamma (v - \bar v) \rho_{0,3}\, N^3 dS\,,
\end{aligned}
$$

and the vector field $\Xi(\bar\eta)$ solves

$$
\begin{aligned}
\bar v_t + 2\, \Xi(\bar\eta) + 2\kappa [\Xi(\bar\eta)]_t &= 0\,,\\
\Xi(0) &= D\rho_0\,.
\end{aligned}
$$

$$
\Xi(\bar\eta)(t, \cdot) = e^{-\frac{t}{2\kappa}} D\rho_0(\cdot) + \int_0^t \frac{e^{\frac{t'-t}{\kappa}}}{2\kappa^2} \bar v(t', \cdot) dt' - \frac{1}{2\kappa} \bar v(t, \cdot) + \frac{e^{-\frac{t}{\kappa}}}{2\kappa} u_0(\cdot)\,.
$$

Constructing solutions $\bar{X}$, $\bar{X}_t$, etc.

Definition 1 (Weak Solutions for $\bar{X}$) *A function $\bar{X} \in L^2(0,T;\dot{H}^1_0(\Omega))$ with $\frac{\bar{X}_t}{\rho_0} \in H^{-1}(\Omega)$ is a weak solution if*

(i) for all $\mathcal{W} \in \dot{H}^1_0(\Omega)$,

$$\langle \frac{\bar{J}^3 \bar{X}_t}{\rho_0}, \mathcal{W}\rangle + 2\kappa \int_\Omega \frac{\bar{B}^{jk}}{\rho_0}(\rho_0 \bar{X}),_k \mathcal{W},_j \, dx = \langle \bar{G}, \mathcal{W}\rangle \quad a.e.\ [0,T]\,,$$

(ii) $\bar{X}(0) = X_0$.

The duality pairing between $\dot{H}^1_0(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$, and $\bar{G} \in L^2(0,T;H^{-1}(\Omega))$.

Estimates for the integral $2\kappa \int_\Omega \frac{\bar{B}^{jk}}{\rho_0}(\rho_0 \bar{X}),_k \bar{X},_j \, dx$

$$2\kappa \int_\Omega \frac{\bar{B}^{jk}}{\rho_0}(\rho_0 \bar{X}),_k \bar{X},_j \, dx = 2\kappa \int_\Omega \bar{B}^{jk} \bar{X},_k \bar{X},_j \, dx + 2\kappa \int_\Omega \frac{\bar{B}^{jk}}{\rho_0} \rho_0,_k \bar{X}\, \bar{X},_j \, dx$$

and

$$2\kappa \int_\Omega \frac{\bar{B}^{jk}}{\rho_0} \rho_0,_k \bar{X}\, \bar{X},_j \, dx = -\kappa \int_\Omega \frac{\rho_0,_{jk}}{\rho_0} \bar{B}^{jk} |\bar{X}|^2 \, dx + \kappa \int_\Omega \frac{\rho_0,_k\, \rho_0,_j}{\rho_0^2} \bar{B}^{jk} |\bar{X}|^2 \, dx$$
$$- \kappa \int_\Omega \frac{\rho_0,_k}{\rho_0} \bar{B}^{jk},_j |\bar{X}|^2 \, dx\,,$$

- Using a Galerkin approximation, we can show that

$$\sup_{t\in[0,T]} C \left\| \frac{\bar{X}(t)}{\sqrt{\rho_0}} \right\|_0^2 + C_p \int_0^T \|\bar{X}(t)\|_1^2 \leq \left\| \frac{X(0)}{\sqrt{\rho_0}} \right\|_0^2 + C_{\kappa\lambda} \int_0^T \|\bar{G}(t)\|_{H^{-1}(\Omega)}^2 \ .$$

- We can do this for time-differentiated versions of the $\bar{X}$ equation

$$\sup_{t\in[0,T]} C \left\| \frac{\bar{X}_t(t)}{\sqrt{\rho_0}} \right\|_0^2 + C_p \int_0^T \|\bar{X}_t(t)\|_1^2 \leq \left\| \frac{\partial_t X(0)}{\sqrt{\rho_0}} \right\|_0^2 + C_{\kappa\lambda} \int_0^T \|\partial_t \bar{G}(t)\|_{H^{-1}(\Omega)}^2 \ .$$

- How do we build regularity: Suppose that $\bar{X}_t \in L_t^2 H_x^1$. Prove that $X \in L_t^2 H_x^2$

- We see that

$$4\kappa^2\Big\|\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}),_k\Big],_j\Big\|^2_{L^2(0,T;L^2(\Omega))}=\Big\|-\frac{\bar{J}^3\bar{X}_t}{\rho_0}+\mathcal{G}_2\Big\|^2_{L^2(0,T;L^2(\Omega))}\le C$$

- Explanation of regularity in 1-D: $X=\rho_0 v'$

$$\Big\|\Big[\frac{1}{\rho_0}(\rho_0 X')\Big]'\Big\|^2_0\le C$$

- From this bound, we want to infer that $\|X\|_2\le C$
- We write $\hat{v}(x)=\int_0^x\frac{X}{\rho_0}$ so $\hat{v}$ satisfies Poincaré inequality, and $X=\rho_0\hat{v}'$
- We see that

$$\frac{1}{\rho_0}(\rho_0\;X)'=\rho_0\hat{v}''+2\rho_0'\hat{v}'$$

- So that

$$\|\rho_0\hat{v}'''+3\rho_0'\hat{v}''+2\rho_0''\hat{v}'\|\le C$$

- Hence

$$\|(\rho_0\hat{v})'''\|_0 \leq C$$

- The terms $(\rho_0\hat{v})''$, $(\rho_0\hat{v})'$, and $\rho_0\hat{v}$ are also bounded in L^2, so that

$$\|\rho_0\hat{v}\|_3 \leq C \tag{5}$$

- By our higher-order Hardy inequality,

$$\|\hat{v}\|_2 \leq C \tag{6}$$

- From (5), we see that

$$\|\rho_0\hat{v}' + \rho_0'\hat{v}\|_2 \leq C$$

- From (6), we see that

$$\|X\|_2 \leq C$$