Groups of Automorphisms in Birational and Affine Geometry

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CONFIRMED SPEAKERS

Hamid Ahmadinezhad
Klaus Altmann
Marco Andreatta
Ivan V. Arzhantsev
Jérémi Blanc
Serge Cantat
Alessio Corti
Tommaso de Fernex
Adrien Dubouloz
Jürgen Hausen
Alexei Kanel-Belov
Takashi Kishimoto
Hanspeter Kraft
Shigeru Kuroda
Frank Kutzschebauch
Stéphane Lamy
Vladimir Lazič
Frédéric Mangolte
Masayoshi Miyanishi
Lucy Moser-Jauslin
Vladimir L. Popov
Alexandr Pukhlikov
Peter Russell
Francesco Russo
Constantin Shramov
Hendrik Süss
Jörg Winkelmann
David Wright
De-Qi Zhang

ABSTRACTS

Hamid Ahmadinezhad: Birational nonrigidity of Fano fibre spaces

Abstract: Minimal model program of Mori provides a starting point for birational classification in dimension three. One possible outcome of this machinary is a Mori fibre space. The relations between Mori fibre spaces, as a crucial point in the classification, is explained via Sarkisov links. I explore different Mori fibre space structures, in the framework of Sarkisov program, for a nonrational fibration of degree four del Pezzo surfaces over the projective line. The models are obtained by means of 2-ray game played on Cox rings.

Klaus Altmann: Merging divisorial with colored fans

Abstract: By Luna and Vust, embeddings of spherical homogeneous spaces can be represented by so-called colored fans. Generalizing the language of toric varieties, these combinatorial objects characterize the spherical embedding uniquely. Moreover, they describe exactly the orbit structure with respect to several automorphism groups.

Another generalization of the toric concept is the notion of polyhedral divisors and divisorial fans (done with J. Hausen and H. Süss). They were designed to describe actions of k-dimensional tori on n-dimensional varieties.

The number n-k is called the complexity of the torus action. These divisorial fans are hybrid objects consisting of an (n-k)-dimensional algebraic variety forming the base of a k-dimensional polyhedral structure.

In the talk we are going to merge both concepts. On spherical varieties, there is a special torus action such that the corresponding divisorial fan is strongly related to the colored fan. The base variety is a modification of a wonderful compactification. Moreover, these data allow an explicit recovering of the original spherical embedding.

This is joint work with Valentina Kiritchenko and Lars Petersen.

Marco Andreatta: Minimal Model Program and Adjunction Theory

A quasi polarized variety is a pair (X, L), where X is a normal projective variety, with at most terminal singularities, and L is a nef and big Cartier divisor. In order to classify these pairs a useful and classical instrument is the study of the positivity (nefness) of the adjoint divisors, $K_X + rL$, for a positive rational number r. After the celebrated proof of the existence of the Minimal Model Program with scaling, this could be done via such a Program, with scaling rL. The Program can be described explicitly if $r \geq (n-2)$; some further steps were done recently in the case $r \geq (n-3)$. A series of classical results can then be proved in a simpler and more general form.

Ivan Arzhantsev: The automorphism group of a variety with torus action of complexity one

Abstract: In 1970, Demazure gave a combinatorial description of the automorphism group Aut(X) of a complete smooth toric variety X as a linear algebraic group. The central concept is a root system associated with a complete fan. Later Cox interprated and generatlized these results in terms of the homogeneous coordinate ring.

We describe the automoprhism group of a complete rational variety X with torus action of complexity one. Our description is based on a presentation of the Cox ring R(X) in terms of trinomials and on an interpretation of Demazure's roots as homogeneous locally nilpotent derivations of R(X). Also we obtain an explicit description of the root system of the semisimple part of Aut(X). The results are applied to the study of almost homogeneous varieties.

This is a joint work with Jürgen Hausen, Elaine Herppich and Alvaro Liendo.

Jérémy Blanc: On the structure of the Cremona groups

Abstract: In this talk, I will describe the structure of the n-dimensional Cremona group, which is the group of birational transformation of the projective space of dimension n, showing that it is not an algebraic group of infinite dimension if $n \geq 2$, and describe the obstruction for this.

To the converse, we can show the existence of a transcendental topology on the Cremona group which extends the one of its classical subgroups and makes it a topological group.

Serge Cantat: Abelian subgroup of the Cremona group

Abstract: I'll survey the classification of abelian subgroups of the Cremona group in two variables, and explain one or two important arguments that lead to this classification.

Alessio Corti: Some open questions on quartic 3-folds

Abstract: I start by recalling the Sarkisov program, some general conjectures on the birational geometry of 3-folds, and my work with M. Mella from many years ago on the birational geometry of singular quartic 3-folds. I then discuss some possible ways to continue that work and describe a joint project with A.S. Kaloghiros.

Tommaso de Fernex: Birational automorphisms of Fano hypersurfaces of index one

Abstract: The study of the birational group of Fano hypersurfaces was undertaken by Fano in dimension three, and has lead to the proof, by Iskovskikh and Manin, that smooth quartic threefolds are not rational. Their work eventually led to the notion of birational rigidity, and higher dimensional Fano hypersurfaces of index one have since been investigated from this point of view. In this talk I will overview the problem and the latest results in this direction.

Adrien Dubouloz: Exotic affine spheres

Abstract: Exotic complex algebraic structures on even dimensional Euclidean spaces , which first arose as a byproduct of a topological characterization of the complex affine plane due to Ramanujam, have been ubiquitous in affine geometry during the last two decades. In this talk, we will consider the existence of such exotic structures on another class of familiar objects, namely affine quadrics $\mathbb{S}^n = \{x_1^2 + \cdots + x_{n+1}^2 = 1\}$ in the usual complex affine space of dimension $n+1$, which we call also affine n-spheres. We will first see that similarly as in the case of Euclidean spaces, the algebraic structure of the affine 2-sphere is uniquely determined by the homotopy type of its underlying differentiable manifold. Then we will discuss exotic structures on the affine 3-sphere which arise from certain principal $\mathbb{G}_a$-bundles over the punctured affine plane and give some partial classification results for such objects. We will show in particular that there exists exotic affine 3-spheres which are stably algebraically isomorphic to $\mathbb{S}^3$ and whose underlying analytic spaces are biholomorphic to $\mathbb{S}^3$.

This is a joint work with David Finston.

Jürgen Hausen: Explicit methods for Mori dream spaces

Abstract: Mori dream spaces are varieties with a finitely generated Cox ring. Well known examples are toric varieties. The strong finiteness properties of Mori dream spaces allow to extend many features of toric geometry to this class. This leads to an explicit understanding of many aspects of their geometry in terms of combinatorial and algebraic properties of the Cox ring. The talk begins with a brief general survey on this approach. Then we focus on particular classes of Mori dream spaces, for example rational complete varieties with a torus action of complexity one. We discuss (global) resolution of singularities and applications to Fano varieties.

Alexei Ya. Kanel-Belov: Aut($K[x_1, ..., x_n]$) as ind-scheme and Kontsevich conjecture

Abstract: Regarding the automorphism group of the group Aut($K[x_1, ..., x_n]$) ($n > 2$) with its *ind*-scheme structure we discuss relations with the Kontsevich-Belov conjecture which says that the automorphism group of the Weil algebra W_n is naturally isomorphic to the group of polynomial symplectomorphisms Sympl ($\mathbb{C}^{2n}$).

Note that such Ind-schemes are singular varieties in some reasonable sense.

Takashi Kishimoto: An application of the minimal model program in studies on affine algebraic threefolds

Abstract: The theory of affine algebraic surfaces is highly developed since the end of 1970's due to efforts of many mathematicians by making use of the classification of projective algebraic surfaces. Regarding the recent progress of the minimal model program (MMP) it seems to be reasonable to investigate the structure of affine algebraic varieties of higher dimension from the viewpoint of the MMP via compactifications. However, this attempt is not successfully realized so far mainly because of the existence of log flips. In this talk, I will devote myself to the case of affine algebraic threefolds, and I will mention more precisely possible difficulties and the conditions on compactifications under which we are able to avoid such difficulties.

Hanspeter Kraft: Automorphism groups of affine varieties and vector fields

Abstract: The group $Aut(X)$ of automorphisms of an affine variety X has a natural structure of an ind-group (sometimes called "infinite dimensional algebraic group"). This allows to use tools from algebraic geometry and algebraic group actions. The group $Aut(X)$ admits, like every ind-group, a Lie algebra $L(X) := LieAut(X)$ which comes with a natural Lie algebra homomorphism $L(X) \to Vec(X)$ to the vector fields on X. E.g. in case $X = A^n$, the complex affine space, this is an isomorphism of $LieAut(C^n)$ with the vector fields $\delta = \sum_i f_i \partial_{x_i}$ with constant divergent $div\delta = \sum_i \frac{\partial f_i}{\partial x_i}$.

For a connected algebraic group G acting on X we know that the image of $LieG$ in $Vec(X)$ completely determines the action. The basic question is if there are similar results for ind-groups. How much information about $Aut(X)$ can we deduce from the $L(X)$ and its image in $Vec(X)$?

Here is one result.

Theorem: Let $G \subset \mathrm{Aut}(X)$ be a subgroup generated by connected algebraic groups G_i. Assume that the Lie algebra $L \subset Vec(X)$ generated by the Lie algebras $Lie\, G_i$ is finite dimensional. Then the closure of G is an algebraic group.

(Joint work with Mikhail Zaidenberg)

Shigeru Kuroda: Wild automorphisms of a polynomial ring in three variables

Abstract: It was a longstanding open question whether every automorphism of a polynomial ring over a field is tame, i.e., a composite of so-called elementary automorphisms. In 1970s, Nagata conjectured the existence of non-tame (wild) automorphisms, and constructed a candidate in the case of three variables. In 2003, Shestakov-Umirbaev gave a criterion for deciding whether a given automorphism in three variables is tame, and solved Nagata's conjecture in the affirmative. Recently, the theory of Shestakov-Umirbaev was modified by us and became very useful in application. In this talk, we discuss the Shestakov-Umirbaev theory and its various applications.

Frank Kutzschebauch: Holomorphic factorization of maps into the special linear group

Abstract: It is standard material in a Linear Algebra course that the group $\mathrm{SL}_m(\mathbb{C})$ is generated by elementary matrices $E + \alpha e_{ij}$ $i \neq j$, i.e., matrices with 1's on the diagonal and all entries outside the diagonal are zero, except one entry.

The same question for matrices in $\mathrm{SL}_m(R)$ where R is a commutative ring instead of the field $\mathbb{C}$ is much more delicate, interesting is the case that R is the ring of complex valued functions (continuous, smooth, algebraic or holomorphic) from a space X.

For $m \geq 3$ (and any n) it is a deep result of Suslin that any matrix in $\mathrm{SL}_m(\mathbb{C}[\mathbb{C}^n])$ decomposes as a finite product of unipotent (and equivalently elementary) matrices.

In the case of continuous complex valued functions on a topological space X the problem was studied and solved by Thurston and Vaserstein. For rings of holomorphic functions on Stein spaces, in particular on $\mathbb{C}^n$, this problem was explicitly posed as the **Vaserstein problem** by Gromov in the 1980's.

In the talk we explain a complete solution to Gromov's *Vaserstein Problem* from a joint work with B. Ivarsson. The proof uses a very advanced version of the Oka-principle proposed by Gromov and proved in recent years by Forstnerič: An elliptic stratified submersion over a Stein space admits a holomorphic section iff it admits a continuous section.

Stéphane Lamy: Automorphisms of some affine threefolds

Abstract: Since the automorphism group of $\mathbb{A}^3 = \mathbb{P}^3 \setminus \mathbb{P}^2$ seems so difficult to handle, one is tempted to study similar but hopefully easier situations in order to gain insight into the problem. In this talk I shall consider affine threefolds of the form $V = X \setminus B$ where X is a smooth hypersurface of degree n in $\mathbb{P}^4$ and B is a smooth intersection of X with a hypersurface of degree d. It is easy to see that the automorphism group $\mathrm{Aut}(V)$ embeds in $\mathrm{Aut}(X)$ when $d > 3$ or $n > 3$. I will mention partial results and open problems for $n = 1$, $d = 3$ (complement of a smooth cubic surface in $\mathbb{P}^3$) and $n = 2$, $d = 2$ (complement of a del Pezzo surface of degree 4 in a quadric

threefold). Then I will describe a joint work with S. Vénéreau where we develop an analogue of the theory of Shestakov-Umirbaev and Kuroda in the context of $n = 2$, $d = 1$ (affine quadric threefold, or in other words the underlying variety of $\mathbf{SL}(2, \mathbb{C})$).

Vladimir Lazič: Birational automorphisms of Calabi-Yau manifolds

Abstract: I will first survey results related to the finite generation of the canonical ring and applications to Calabi-Yaus, and then present some very recent work on the Cone conjecture of Morrison and Kawamata, which predicts how birational automorphisms act on the movable cone.

Frédéric Mangolte: Cremona transformations, diffeomorphisms of surfaces and approximation by (-1)-curves

Abstract: In 2009, we showed that the action of Cremona transformations on the real points of quadrics exhibits the full complexity of the diffeomorphisms of the sphere, the torus, and of all non-orientable surfaces. The main result says that if X is rational, then Aut(X), the group of algebraic automorphisms, is dense in Diff(X), the group of self-diffeomorphisms of X. In this talk, I will recall the main part of the proof and explain how this result is applied in recent developments. In particular, we obtain necessary and sufficient topological conditions for a simple closed curve on a rational surface to be approximated by a (-1)-curve. Note that (-1)-curves are quite rigid objects, hence approximating by (-1)-curves is a subtle problem.

(Joint work with János Kollár)

Masayoshi Miyanishi: Surjective vector fields on affine algebraic varieties

Abstract: Inspired by a paper of D. Cerveau [J. Algebra **195** (1997), 320–335] which determines a surjective derivation on a polynomial ring $\mathbb{C}[x_1, x_2]$ (his proof contains unfortunately an error and thereby suspends the validity of the result), we consider a vector field θ on an affine algebraic variety Spec B such that the associated derivation D on B is surjective. The surjectivity implies that any differential 1-form ω is decomposed as $\omega = dh + \eta$, where $\delta(\eta) = 0$ for the B-homomorphism $\delta : \Omega^1_{B/\mathbb{C}} \to B$ such that $D = \delta \cdot d$ with the standard differentiation $d : B \to \Omega^1_{B/\mathbb{C}}$. The interesting idea of Cerveau to use a result of Dimca-Saito [Compositio Math. **85** (1993), 299–309] suggests a possibility of considering a surjective derivation on a smooth affine variety in general.

The present talk reports a recent joint work with R.V. Gurjar and K. Masuda on this subject.

Lucy Moser-Jauslin: Automorphism groups of certain rational hypersurfaces in complex affine four-space

Abstract: The Russell cubic is a smooth contractible affine complex threefold which is not isomorphic to affine three-space. It can be constructed as the hypersurface in affine four-space defined by the equation $x^2y + z^2 + t^3 + x = 0$. Previously, in collaboration with A. Dubouloz and P.-M. Poloni, we discussed the structure of the automorphism group of this variety. One main technique is to use the fact that the variety is obtained as an affine modification of affine three-space. In the present talk, we will generalize the construction to other hypersurfaces and study properties of the automorphism groups obtained. We are particularly interested in the question of extending automorphisms to the ambient four-space.

Vladimir L. Popov: Jordan groups and automorphism groups of algebraic varieties

Abstract: I shall define the general notion of abstract Jordan group and, after considering examples and some general properties, discuss the problem of classifying algebraic varieties whose group of automorphisms (biregular or birational) is Jordan.

Alexandr Pukhlikov: Birational rigidity of Fano varieties and fibre spaces

Abstract: I will explain the basics of birational rigidity and super-rigidity of Fano varieties and Fano fibre spaces, with some emphasis on the connection with the groups of birational automorphisms. Some very recent results on birational (super)rigidity will be presented.

Peter Russell: Relative linearization, particularly in relative dimension 2

Abstract: My main objective will be to explain the ingredients for a proof of:
Theorem: An action of a reductive group G on $\mathbb{A}^3$ that admits a variable as a semi-invariant is linearizable.
The most interesting case is that of a finite G, where linearizability in general is an open problem.
Along the way I will explain a number of results in the more general setting of (locally) trivial bundles $\pi : Y \to X$ with affine fiber F, equivariant for the action of an algebraic group G, particularly when $F \simeq \mathbb{A}^2$.

This is a joint work with H. Kraft and G. Schwarz.

Francesco Russo: Special birational maps defined by quadratic equations

Abstract: A birational map from a projective space onto a projective manifold with a single irreducible non-singular base locus scheme (*special birational map*) is a rare enough phenomenon to allow meaningful and concise classification results.

We shall concentrate on maps defined by quadratic equations onto some Fano manifolds (especially projective hypersurfaces of small degree), where quite surprisingly the base loci are interesting projective manifolds appearing in other contexts (exceptions for adjunction theory, small degree or small codimensional manifolds, Severi or more generally homogeneous varieties, OADP-varieties, etc, etc).

This is a joint work with Giovanni Staglianò.

Constantin Shramov: Jordan property for Cremona groups

Abstract: An (infinite) group F is called Jordan if there is a constant J such that for any finite subgroup G in F there is a normal abelian subgroup H in G of index $[G : H] < J$. Classical examples of groups with this property include $GL_n(C)$, and thus all affine algebraic groups over fields of zero characteristic. I will prove that a group of birational self-maps of any n-dimensional rationally connected variety is Jordan for $n = 3$, and the same result holds for any $n > 3$ modulo Borisov-Alexeev-Borisov conjecture.

The talk is based on a joint work with Yuri Prokhorov.

Hendrik Süss: Log-canonical thresholds under torus quotients

Abstract: The log-canonical threshold of a Fano variety X is an invariant with applications in birational geometry as well as in Kähler geometry. It is defined with respect to a certain finite subgroup G of $\mathrm{Aut}(X)$. After choosing a maximal torus T in the automorphism group of our Fano variety X we would like to reduce the computation of the log-canonical threshold on X to that of a log-canonical threshold on some torus quotient $X = X/T$. As it turns out, this works well if the following conditions are fulfilled: (i) G is contained in the normalizer of the maximal torus, (ii) the G-action on the characters given by conjugation has the trivial character as its unique fixed point.

In this situation the T-variety is called symmetric. As an application we provide a criterion for the existence of Kähler-Einstein metrics on symmetric Fano T-varieties of complexity one.

Jörg Winkelmann: Non-linearizable actions of reductive groups

Abstract: Since most general actions of reductive groups on the affine space are linear, there is an interest in those which are not. There have been examples in particular for the field of real numbers. We show that in fact there are non-linearizable actions for all fields which admit a quadratic extension.

David Wright: Survey on Polynomial Automorphism Groups

Abstract: We will review the known results on the group of automorphisms of affine n-space, beginning with the classical results. We will discuss, as time permits, special subgroups, theorems on generation, theorems on structure, combinatorial group theory techniques, base rings other than a field, tameness, stabilization techniques, recent results on stable tameness, automorphisms arising from $\mathbb{G}_a$ and $\mathbb{G}_m$ actions, relation to the full Cremona group, coordinate-like polynomials, exotic examples, and open questions.

De-Qi Zhang: Pseudo-automorphisms of positive entropy on the blowups of products of projective spaces

Abstract: We use a concise method to construct pseudo-automorphisms f_n of the first dynamical degree $d_1(f_n) > 1$ on the blowups of the projective n-space for all $n > 1$ and more generally on the blowups of products of projective spaces. These f_n are the geometric realization of Coxeter elements of T-shaped hyperbolic systems. These f_n, for $n = 3$, have positive entropy. In particular, we realize as $d_1(f_n)$ for some f_n, the smallest Salem numbers of degrees 10, 8 and 6, respectively.

This is a joint work with F. Perroni.

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