Carsten Thomassen:

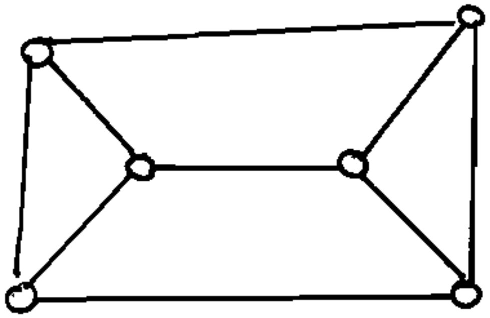
The number of cycles in graphs

Institutional lecture at

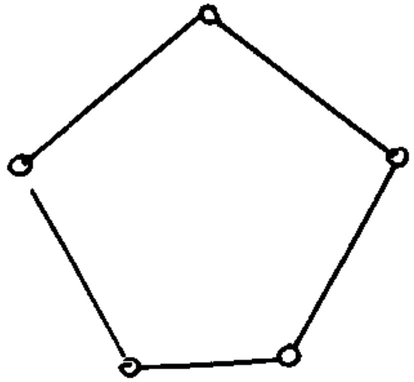
ADONET-CIRM School of
Graphs and Algorithms

Oct. 22, 2007

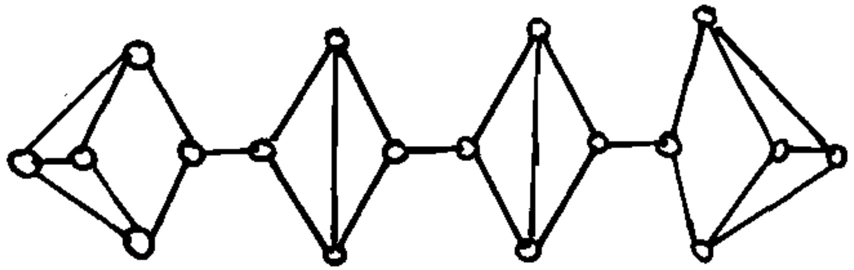
How many cycles must a cubic graph have?



A cubic graph

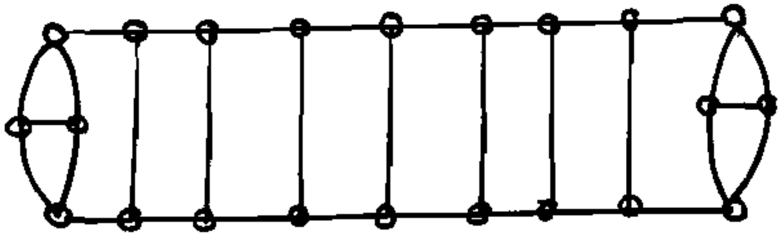


5-cycle



n vertices

$\frac{3}{4}n + \frac{13}{2}$ cycles



n vertices

$(n^2+14n)/8$ cycles

Barefoot, Clark, Entringer (1986)

Every 2-conn. cubic graph on n vertices has $\geq (n^2+14n)/8$ cycles.

$f_3(n)$ = the smallest no. of cycles in a 3-conn. cubic graph with n vertices

B, C, E proved

$$f_3(n) < 2^{n^{0.95}}$$

and conjectured

$f_3(n)$ superpolynomial

<u>Theorem</u> (Aldred + CT)

$$2^{n^{0.17}} < f_3(n) < 2^{n^{0.95}}$$

Upper bound:

<u>Theorem</u> (Bondy, Simonovits, 1980)

$\exists c>0\ \forall$ even n there exists a cubic 3-conn. graph on n vertices whose <u>longest cycles</u> has length <

$$cn^{\log 8/\log 9} = k$$

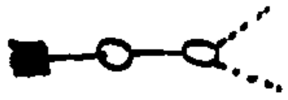
The no. of cycles = the no. of cycles with at most k vertices

$\leq$

the no. of <u>paths</u> with at most k vertices

$\leq$

$$n \cdot 2^k$$



$\leq$

$$n\, 2^{cn^{\log 8/\log 9}}$$

$\leq$

$$2^{n^{0.95}}$$

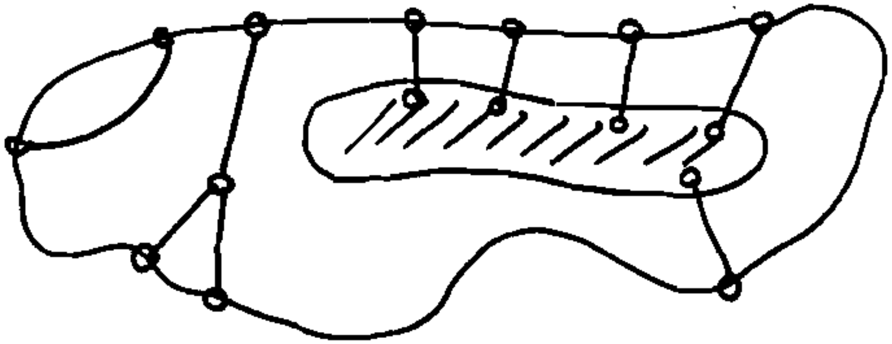
Lower bound on f_3, that is, existence of <u>many</u> cycles uses existence of a <u>long</u> cycle.

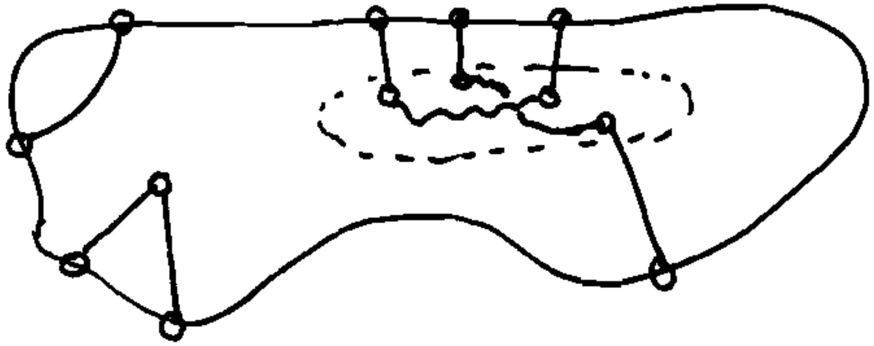
<u>B. Jackson</u> (1986):

A cubic 3-conn. graph with n vertices has a cycle of length at least

$$n^{0.69}$$

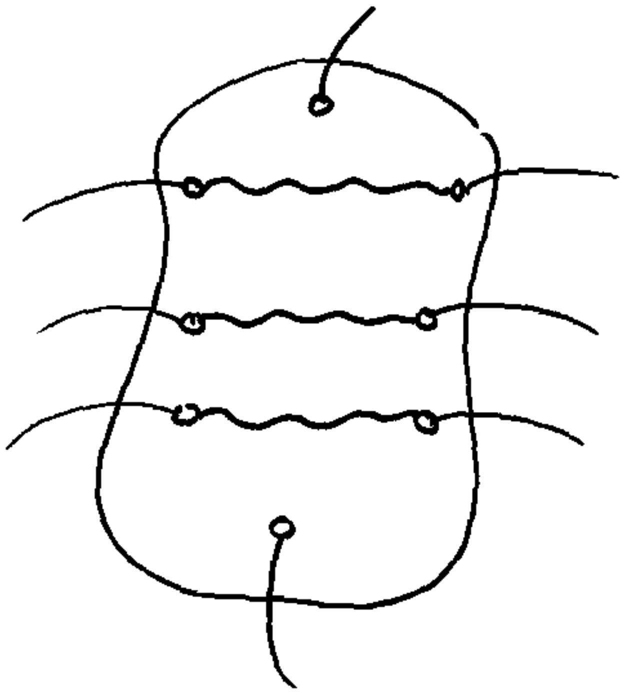
Focus on a longest cycle C





We loose many vertices on C, but less than half

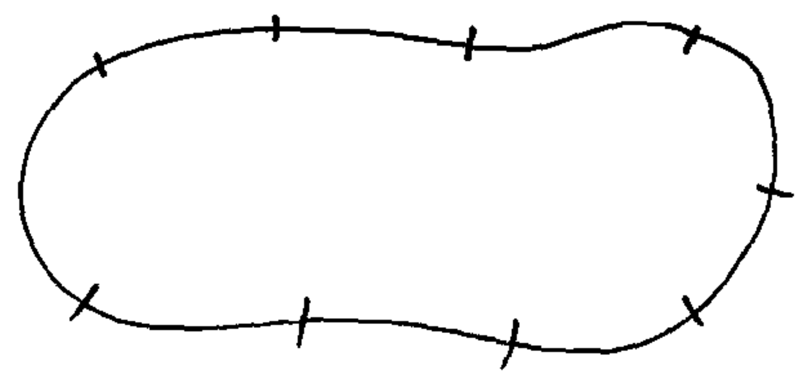
For every component of $G - V(C)$ we loose ≤ 1 vertex on C.

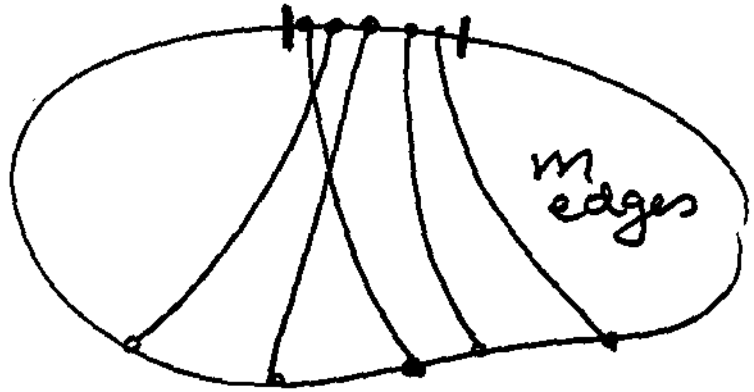


Now C induces a cubic "graph" with $n^{0.69}$ vertices

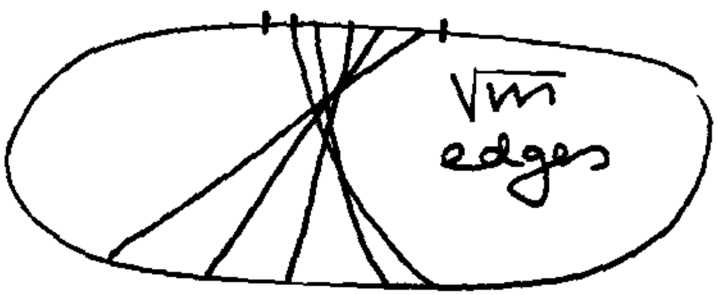
Put $m = (n^{0.69})^{1/4}$ and divide C into paths of length m

<u>Case 1:</u> Each has a chord

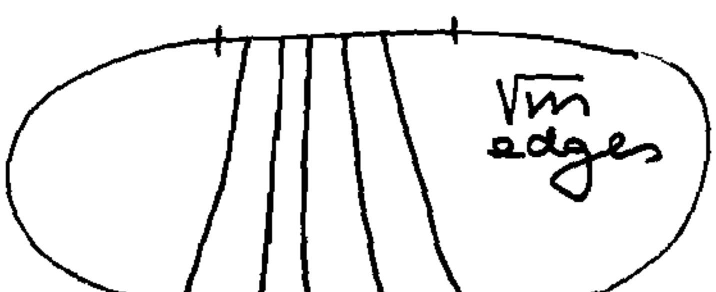




Erdös - Szekeres :

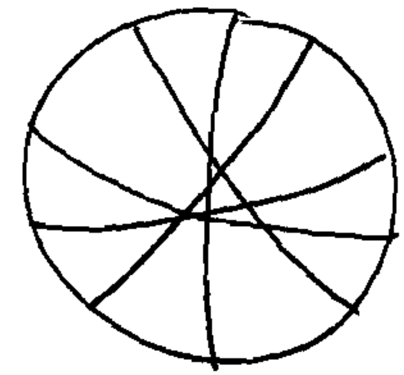


or

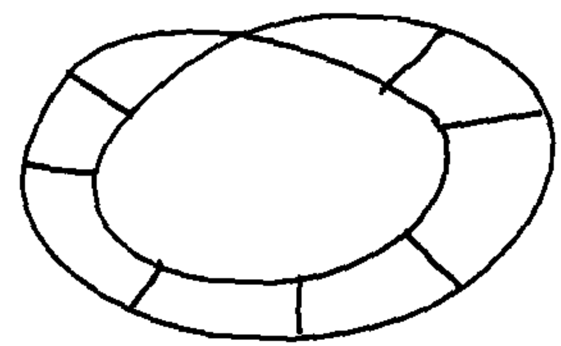


<u>Case 2</u> :

Big Möbius ladder

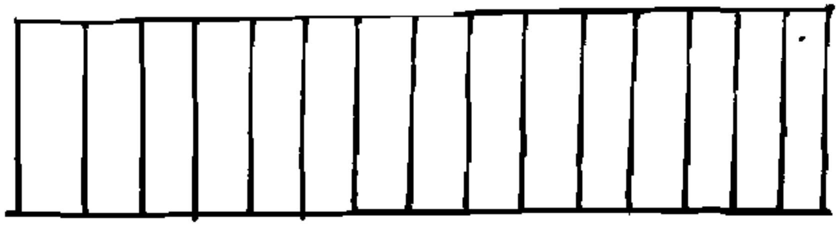


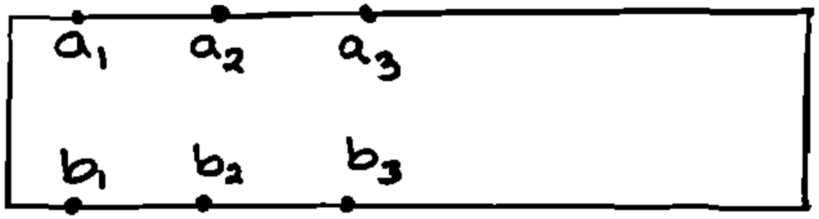
=



Many cycles

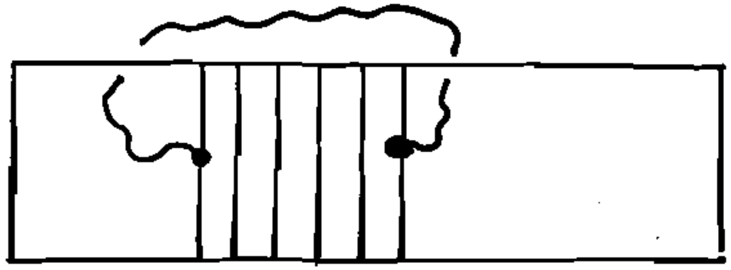
Case 3: Big ladder



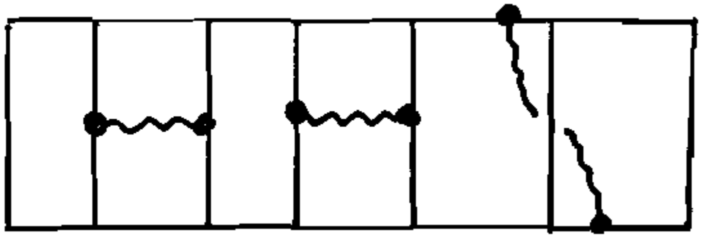


$G - \{a_i, b_i\}$ has a path from the left part to the right part.

Case 3_1 Some two intersect



Case 3_2 No two intersect



In each case:
Many cycles

Conjecture 1 $\exists \alpha > 1$ s.t. every Hamiltonian cubic graph has $> \alpha^n$ cycles

Conjecture 2 Every cubic cyclically 4-edge-conn. graph has $> \alpha^n$ cycles.

Conjecture 3 Every 4-connected line graph has a Hamiltonian cycle.

$1 + 3 \Rightarrow 2$

Conjecture 3

$\Downarrow$

Conjecture 4:

Every cyclically 4-edge conn. graph G has a cycle C s.t. $G - V(C)$ has no edge.

C is long, $|V(C)| \geq \frac{3}{4} n$

Conjectures 3, 4 are true for planar graphs by Tutte's theorem.

So perhaps the easiest unsolved case is the following

Conjecture 5 :

Every planar, cubic cyclically 4-edge-connected graph has $> \alpha^n$ cycles

How many cycles may a graph with n vertices and e edges have ?

Trivial upper bound :

$$2^{e-n+1}$$

How close can we get to this upper bound ?

It suffices to look at cubic graphs if we fix $e-n+1$

The no. of vertices is now $2n$

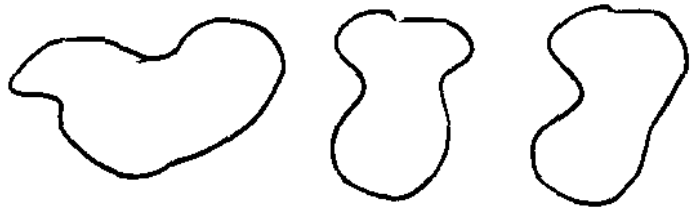
The *dimension* of the cycle space is

$$e-(2n)+1 = n+1$$

The *number of elements* in the cycle space is

$$2^{n+1}$$

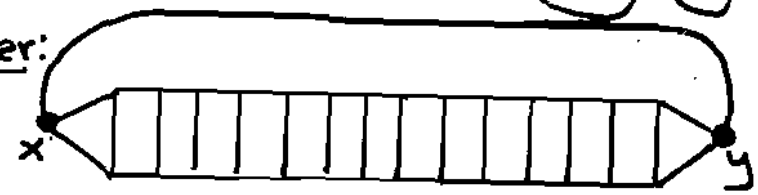
The elements are:

 ...

Collections of pairwise disjoint cycles

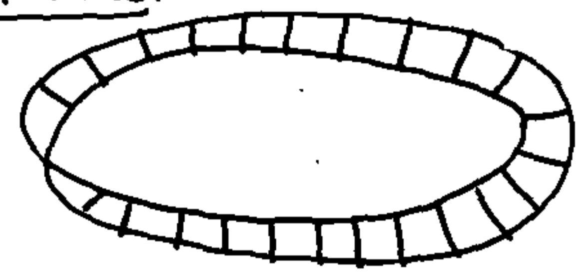
Cubic graphs with $2n$ vertices and many cycles

Ladder:



$$2^n + \binom{n+1}{2}$$

Möbius ladder:



$$2^n + n^2 + O(n)$$

Rautenbach, Stella (2005)

$$\geq 2^n + \frac{5}{2}n^2 + O(n)$$

cycles

Conjecture (Barefoot Clark, Entringer 1985)

The max. no. of cycles in cubic graphs with $2n$ vertices is

$$\approx 2^n$$

that is,

$$\frac{1}{2} \times (\text{the trivial upper bound})$$

Rautenbach and Stella improved the upper bound 2^{n+1} to

$$2^{n+1} - n\frac{\sqrt{n}}{\log n} - \cdots$$

Theorem (Aldred + CT)

A cubic graph with $2n$ vertices has at most

$$\frac{15}{16} 2^{n+1} \text{ cycles}$$

Delete an edge xy from a cubic graph on $2n$ vertices

Here the number of paths from x to y is

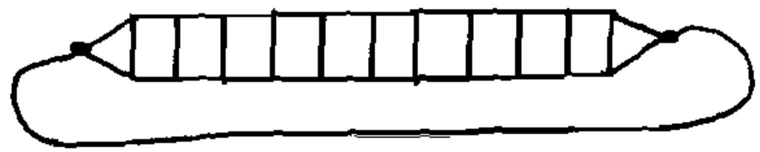
$$2^n$$

In all other cases it is

$$< \frac{15}{16} \cdot 2^n$$

<u>Theorem</u> (Aldred + CT)

A <u>planar</u> cubic graph on $2n$ vertices may have more than 2^n cycles



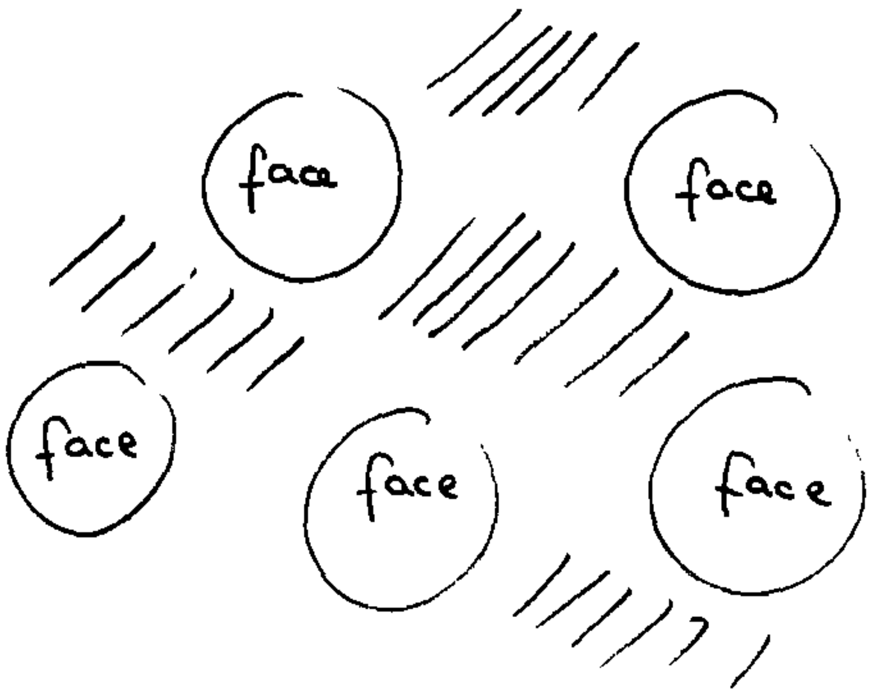
but it cannot have more than

$$2^n + o(2^n)$$

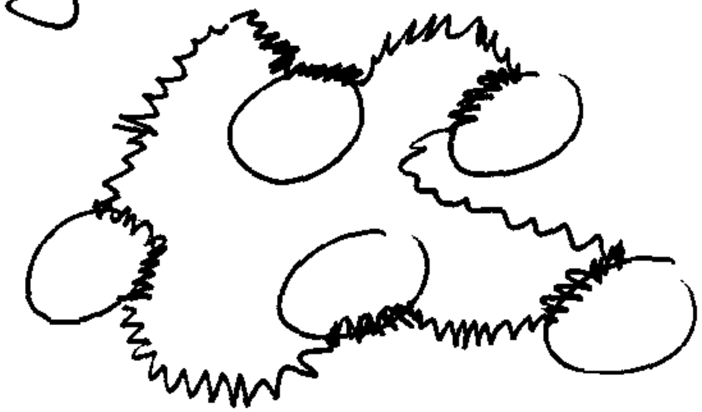
cycles.

Proof :

Fix a collection
of facial cycles
which are
pairwise disjoint
of length ≥ 6
of distance ≥ 5 apart
in the dual graph.



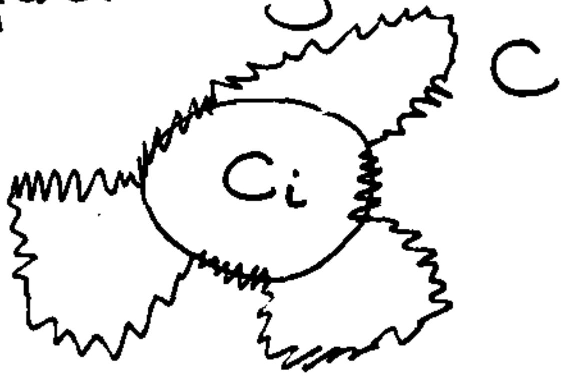
$\mathcal{C}_1$ = the set of cycles
with "nice" intersection
with the facial
cycles



$$|\mathcal{C}_1| \leq o(2^n)$$

which can also be
proved for non-planar
graphs using the
trivial upper bound.

$\mathcal{C}_2$ = the set of cycles with "not-nice" intersection with some of the facial cycles



$$C \to C + C_i$$

This gives a 1-1 map from $\mathcal{C}_2$ to

$\mathcal{C}_g$ = the collections of disjoint cycles with ≥ 2 components

The cycle space

$$Z(G) = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_g$$

$$|Z(G)| = 2^{n+1}$$

$$|\mathcal{C}_1| = o(2^n)$$

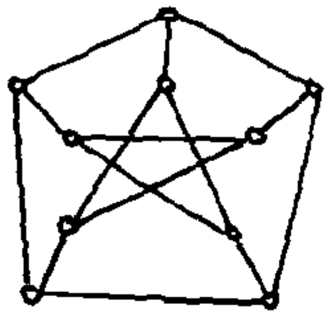
$$|\mathcal{C}_2| \leq |\mathcal{C}_g|$$

Hence the no. of cycles =

$$|\mathcal{C}_1| + |\mathcal{C}_2| \leq 2^n + o(2^n)$$

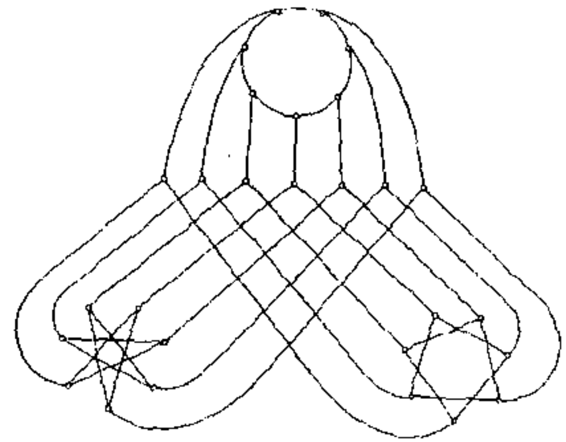
<u>Hamiltonian cycle</u> = cycle through all vertices.

The Petersen graph



is cubic
cyclically 5-edge-conn.
symmetric
(vertex-transitive)
and non-hamiltonian

The Coxeter graph



is cubic
cyclically 7-edge-conn.
vertex-transitive
and
non-Hamiltonian

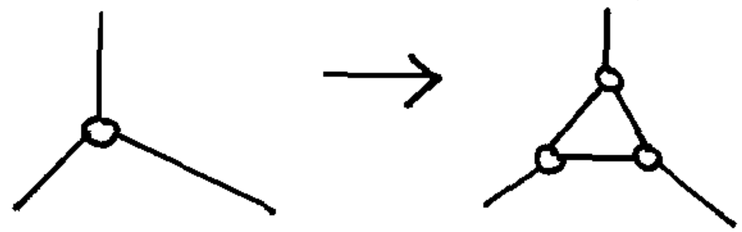
Conjecture

Every cubic
cyclically 8-edge-conn
graph has a
hamiltonian cycle.

Conjecture

Every (cubic) connected
vertex-transitive graph
has a hamiltonian
cycle
except
the four:

The Petersen graph
the Coxeter graph
and



Inspired by the
conjecture of Lovász:

Every connected
vertex-transitive
graph
has a hamiltonian
path

Smith's theorem:

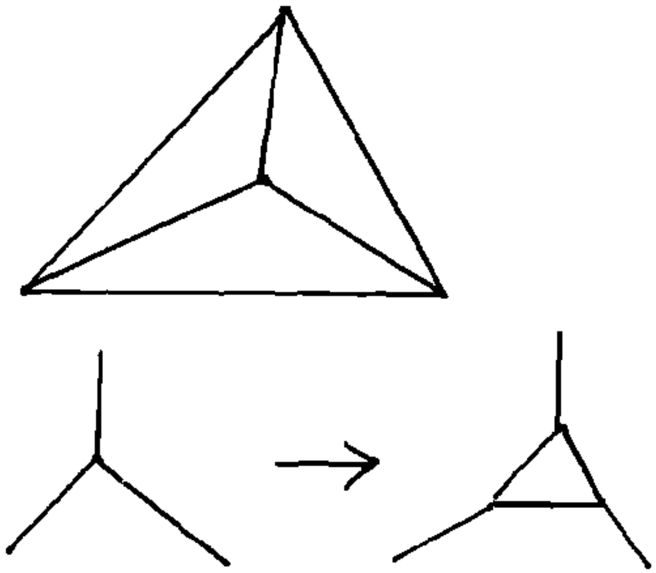
Let e be any edge in a cubic graph G.
The no. of Hamiltonian cycles through e is <u>even</u>

<u>Corollary</u> G has ≥ 3 Hamiltonian cycles (if it has one).

<u>Proof</u> C_1 Ham. cycle.
Pick $e \in E(C_1)$.
C_2 another Ham. cycle through e

Pick $e' \in E(C_2) \setminus E(C_1)$.

There are infinitely many cubic 3-conn. graphs with precisely 3 Hamiltonian cycles



What about a Hamiltonian, cubic cyclically 4-edge-conn. graph on n vertices?

Does it have > 100 Hamiltonian cycles (if n is large)?

Does it have $>$

$$(1+10^{-10})^n$$

Hamiltonian cycles?

Analogous question of Lovász

Let G be a cubic graph on n vertices in which every edge is in a perfect matching.

Does G contain

$$(1+10^{-10})^n$$

perfect matchings?

True for <u>bipartite</u> graphs.

(Van der Waerden's conjecture)

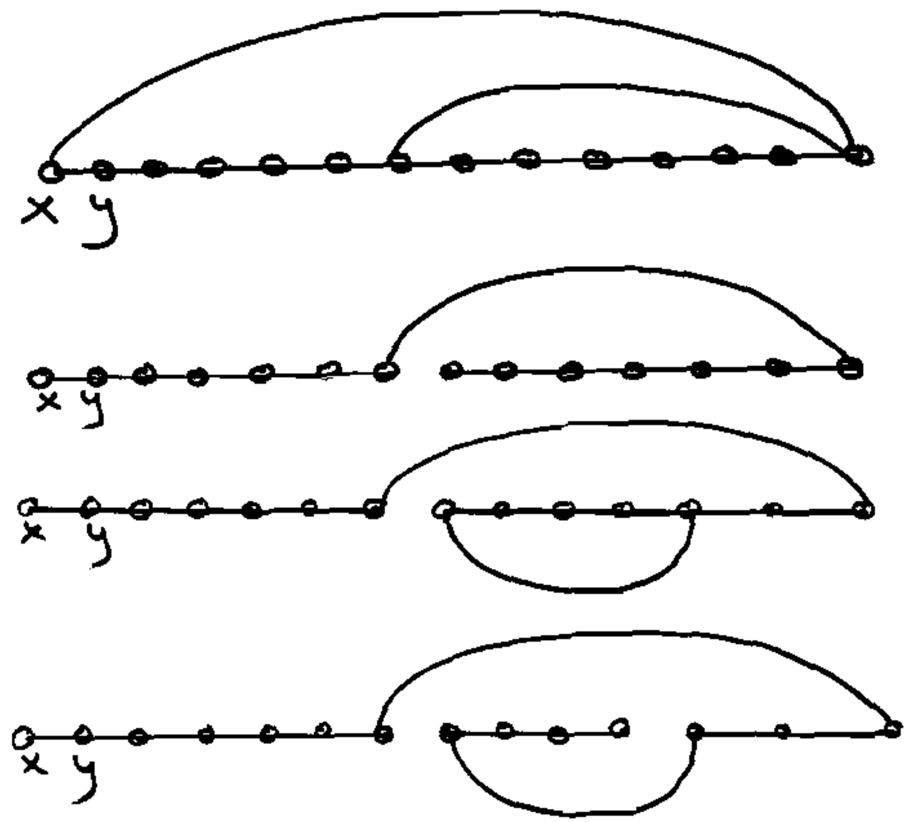
Unsolved problem:
Does there exist a polynomial time algorithm for finding a ~~Hamiltonic~~ <u>second</u> Hamiltonian cycle in a cubic graph ?

We know it <u>exists</u> (Smith)

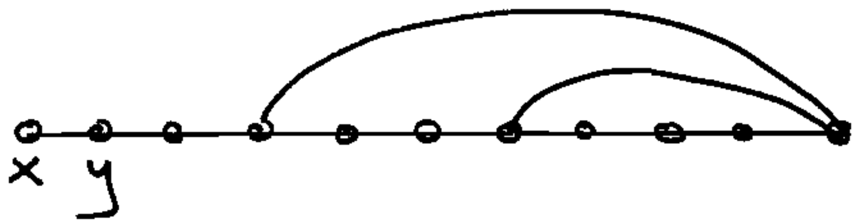
We know a <u>simple</u> algorithm (from Andrew Thomason's proof)

But we don't know a poly-time algo.

A simple algorithm starting with a Hamiltonian cycle.



The general step



There are two possibilities

for getting a new Hamiltonian path starting with xy.

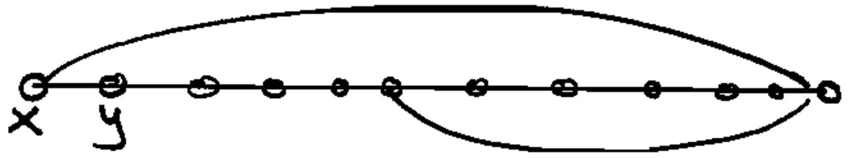
Choose the one which give a <u>new</u> Hamiltonian path

When the algorithm stops you have a second Hamiltonian cycle.

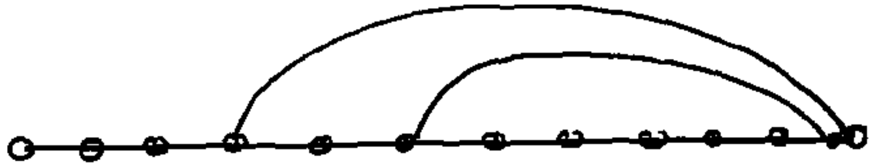
Why does the algorithm stop?

Why does it not stop at the original Ham. cycle?

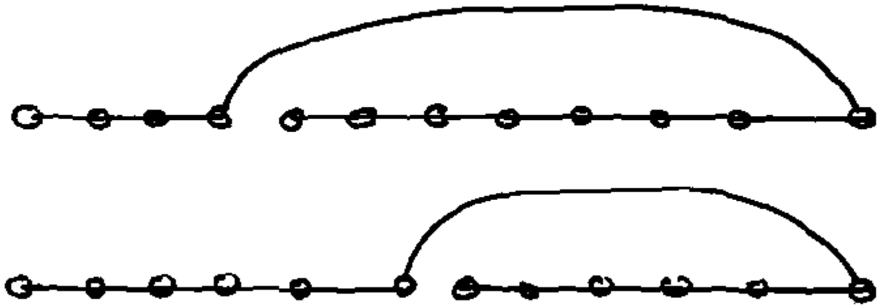
Form a new graph $H(G, x, y)$ whose vertices are the Hamiltonian paths starting at x and with first edge xy.



This has one neighbor:



This has <u>two</u> neighbors

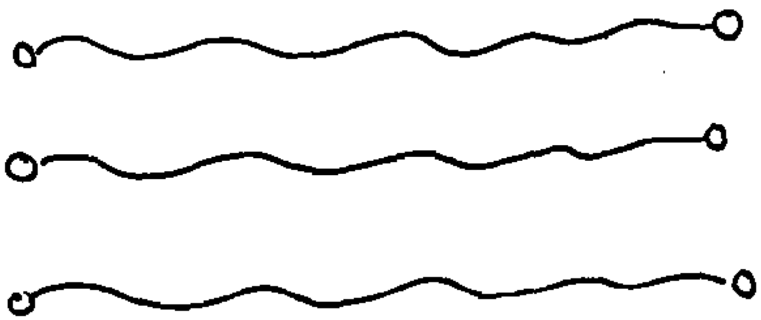


So all vertices of $H(G, x, y)$ have degree 1 or 2

Those of degree 1 are the Hamiltonian cycles starting with xy.

$H(G, x, y)$:

P_1



The algorithm starts in a vertex of degree 1 and proceeds to <u>another</u> vertex of degree 1.

Examples show that <u>some</u> path in $H(G, x, y)$ can have exponential length.

If $C: x_1 x_2 x_3 \dots x_n x_1$ we may also consider

$H(G, x_1, x_2)$, $H(G, x_2, x_1)$

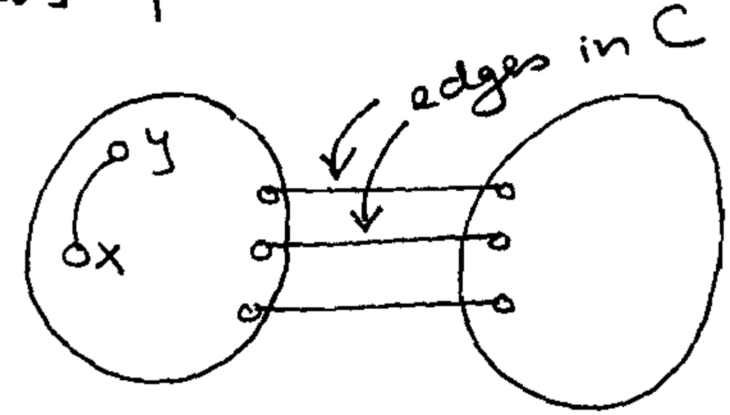
$H(G, x_2, x_3)$, $H(G, x_3, x_2$

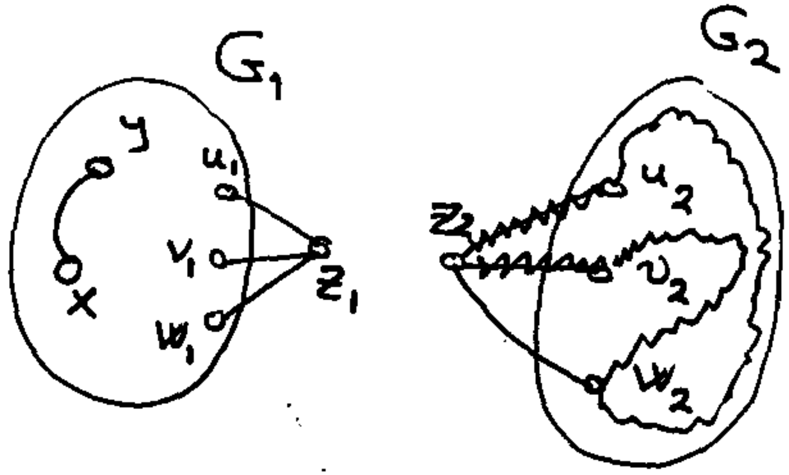
$\vdots$

Can <u>all</u> of these have exponential length?

What about cyclically 4-edge-conn. cubic graphs?

If the algorithm for a second Hamiltonian cycle is polynomial for those, then we get a polynomial algorithm in general as follows:





Find a new Ham. cycle through xy.

If it contains $u_1 z_1 v_1$, stop.

If it contains $u_1 z_1 w_1$, then find a new Ham. cycle through $z_2 u_2$

Andrew Thomason (1975):

If all vertices of G have odd degree, then the number of Ham. cycles containing the edge xy is even.

Proof: The Ham. cycles containing $e = xy$ are the vertices of odd degree in

$$H(G, x, y).$$

Unsolved problems.

Let G be a graph with a Hamiltonian cycle.
$\delta(G)$ = min. degree.

1. $\delta(G)$ large $\Rightarrow$ many Ham. cycles?

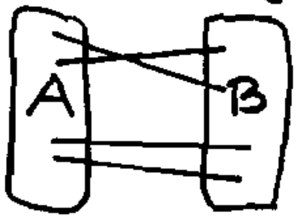
2. $\left.\begin{array}{l} G \text{ } r\text{-regular} \\ r \text{ large} \end{array}\right\} \Rightarrow$ many Ham. cycles?

3. $\delta(G) > 10^{10} \Rightarrow$ A second Ham. cycle?

4. Sheehan's conjecture (1976)

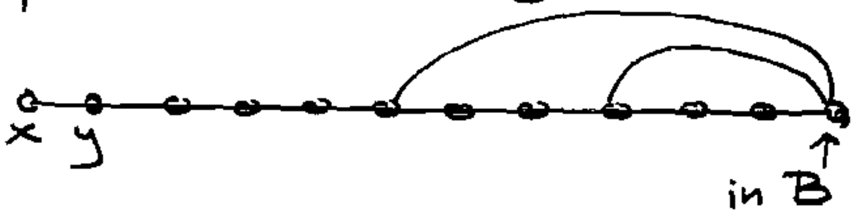
   G 4-regular $\Rightarrow$ a second Ham. cycle?

Bipartite graphs.



Theorem G bipartite Hamiltonian with $\delta \geq 3$
$\Downarrow$
A second Ham. cycle.

Proof Let xy be an edge of the Ham. cycle C. $x \in A$, $y \in B$.
Delete edges outside C s.t. all vertices of B have degree precisely 3
Look at Hamiltonian paths starting with xy



Theorem (CT, 1996)

Let G be bipartite, Hamiltonian.

If $\delta(G) = d$, then G has at least

$$2d!/2^d$$

Hamiltonian cycles.

If $\delta(G) \geq 4$, girth $= g$ then G has at least

$$\left(3/2\right)^{g/8}$$

Hamiltonian cycles.