Carsten Thomassen :

Decompositions and Grötzsch's theorem

Institutional lecture at ADONET-CIRM school of Graphs and Algorithms

Oct. 24, 2007

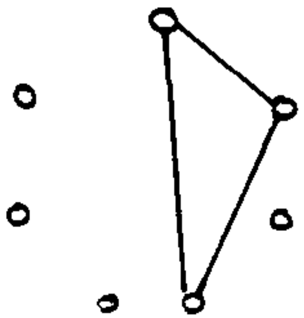
Steiner triple system on a set S is a collection of triples such that each pair in S is in precisely one triple

Theorem A STS on S exists iff n is odd and $\binom{n}{2} \equiv 0 \mod 3$

Set system generalization: Theory of block design

A STS on the set S is also a decomposition of $K_{|S|}$ into edge-disjoint triangles.

Example: A decomposition of K_7 into triangles



Rotate this triangle to get 7 edge-disjoint triangles.

Graph decomposition

H fixed graph
G arbitrary graph, for example a big complete graph.

An H-decomposition of G is a collection $H_1, H_2, \dots, H_m$ of pairwise edge-disjoint subgraphs in G such that

$$H_1 \cup H_2 \cup \dots \cup H_m = G$$

and each

$$H_i \simeq H$$

Simplest example:

H = the 2-path P_3

A connected graph G has a P_3-decomposition iff $|E(G)|$ is even.

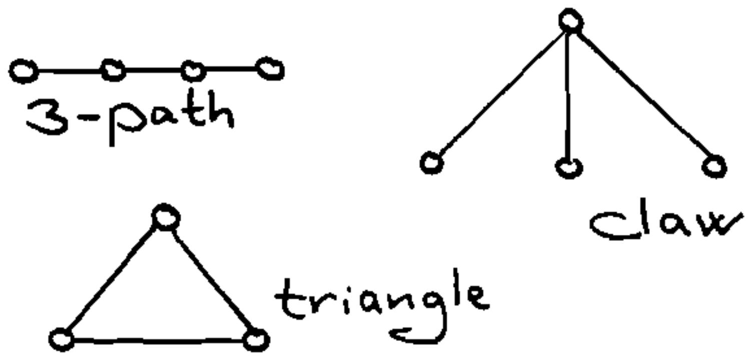
Proof: Delete from G the edges of a 2-path such that the resulting graph is connected (or conn + isolated vertex)

Dor and Tarsi (1997)

For each fixed connected graph with at least 3 edges, the H-decomposition problem is NP-complete.

There are several results when H is fixed and G is either very dense or of a very special structure

Smallest interesting cases



J.-C. Bermond + Schönheim (1977)

H graph with ≤ 4 vertices

Then the complete graph K_n has an H-decomposition if and only if n satisfies ...

This extends the theorem on Steiner triples.

R.M. Wilson (1975)

H any fixed graph
n sufficiently large natural no.

Then K_n has an H-decomposition if and only if

K_n has the right size

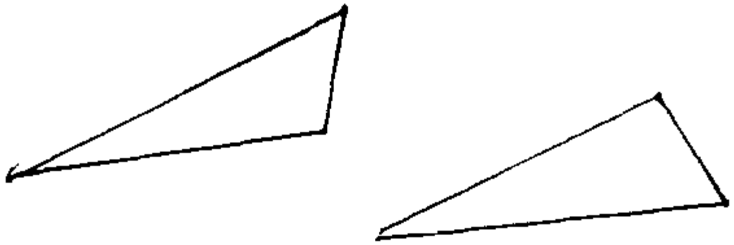
($\binom{n}{2}$ divisible by $|E(H)|$)

and the right degrees.

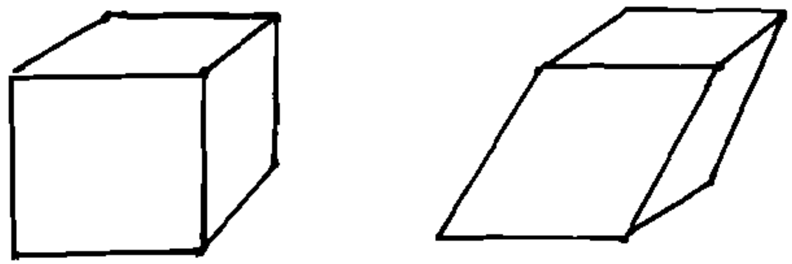
If you fix the sides of a triangle, the triangle is rigid.

More generally:

Two triangles with ~~the~~ same sides are congruent.



If you fix the edges of a polyhedron in $\mathbb{R}^3$ it need not be rigid.



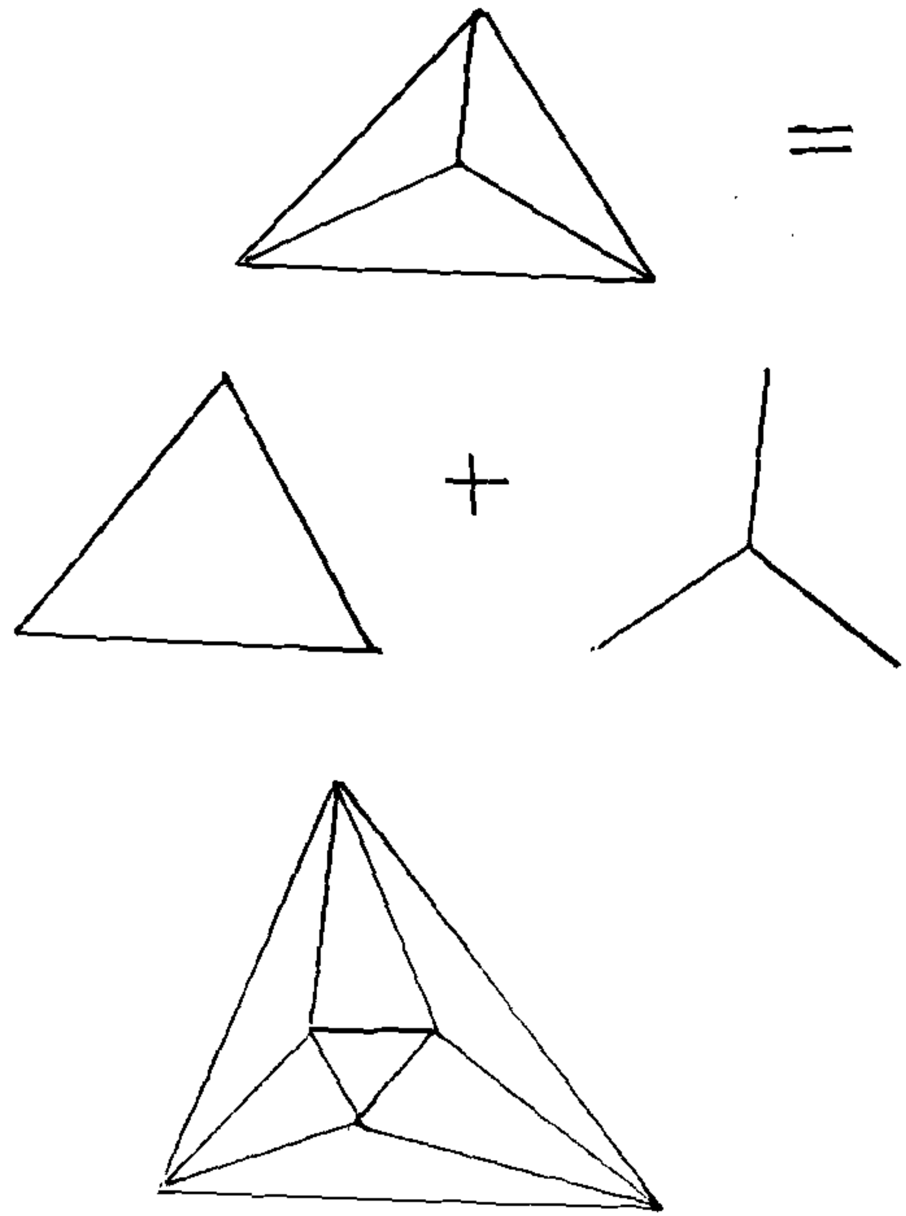
But, if you fix the sides (facets) it is rigid.

Caychy If two polytopes have congruent facets, they are congruent.

Motivated by Cauchy's result and rigidity of convex polyhedra
M. Dehn (1916) proved:

The graph of a triangulated polyhedron
that is,
a planar triangulation
has an edge-decomposition
into a

and a collection of
claws:



A proof of Dehn's result

G graph
$f: V(G) \to \{0, 1, 2, \dots\}$.
Then G has an orientation such that
$$d^+(v) \le f(v) \quad \forall v \in V(G)$$
iff, for every subgraph H of G,
$$|E(H)| \le \sum_{v \in V(H)} f(v).$$

First proved by Hakimi (1965).

Follows also from Edmonds' matroid partition theorem

Application to a planar graph G.

Put $f(v_1) = f(v_2) = 0$
and $f(v) = 3$ for all other v.

Then G has an oritentation such that
$$d^+(v_1) = d^+(v_2) = 0$$
and $d^+(v) \le 3$ for all other v.

If G is a triangulation equality holds because
$$|E(G)| = 3n - 6$$
which gives a claw-decomposition.

If G triangulates the sphere + g handles then $|E(G)| = 3n - 6 + 6g$

Define $f: V(G) \to N$ such that
$f(v) = 6$ for $2g-2$ vertices and
$f(v) = 3$ for all others.

If $d^+(v) \leq f(v)$ $\forall v$ then equality holds.

Can the $2g-2$ vertices be chosen such that the f-inequality holds?

We don't know, but:

by another method we prove:

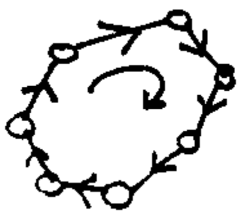
Barat + CT:

Every triangulation of every surface has a claw decomposition

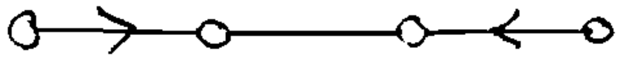
Observation of Jaeger:

Every cubic 2-conn. graph has a 3-path- decomposition

Proof: Let M be a perfect matching.
Then G-M is a collection of cycles. Make each to an oriented cycle:



For each edge in M, take



We have decomposition results for dense graphs (Wilson) and graphs withe special structure (Dehn, Jaeger).

Conjecture for general graphs (Barat + CT)

For every tree T, there exists a number k_T such that each k_T- edge-connected graph of size divisible by $|E(T)|$ has a T-decomposition

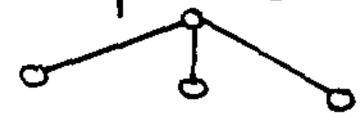
Important that T is a tree because there are 10^{10}- edge-conn. graphs of girth $> 10^{10}$.

When we made the conj. we could not prove it for one single tree of size ≥ 3.

Recently I proved it for one (but only one) tree for graphs of edge-conn.

$$10^{10^{10^{14}}}$$

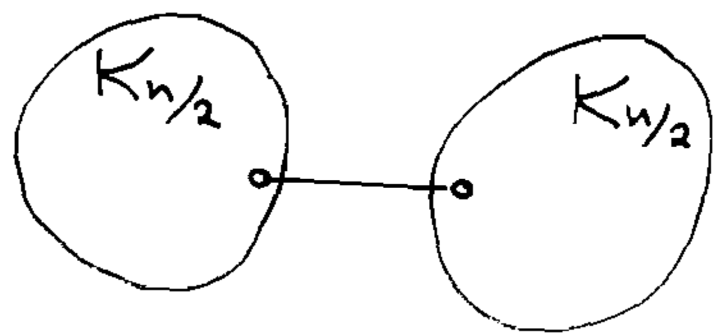
Now focus on the claw



Theorem (J. Barat + CT)

$$\left.\begin{array}{l} |V(G)| = n \\ \delta(G) > \frac{n}{2} \\ n \text{ large} \\ |E(G)| \equiv 0 \bmod 3 \end{array}\right\} \Longrightarrow G \text{ has a claw-decompos.}$$

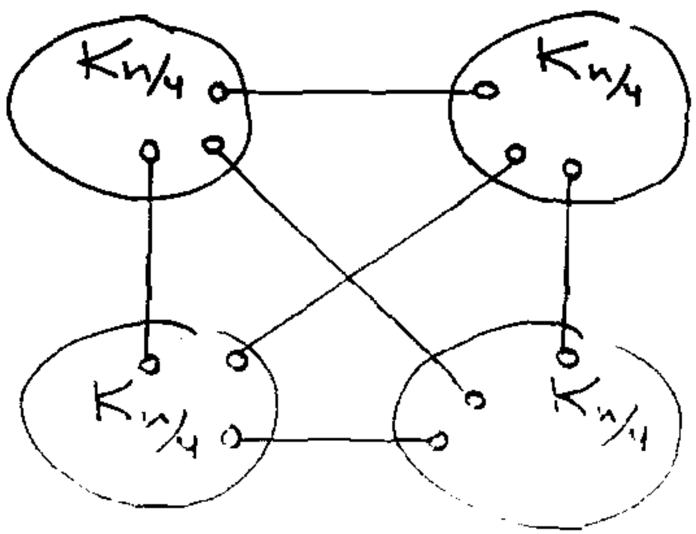
$n/2$ best possible



<u>Theorem</u> (JB+CT):

$$\left.\begin{array}{l} |V(G)| = n \\ |E(G)| \equiv 0 \bmod 3 \\ G \text{ 2-edge-conn.} \\ \delta(G) > \frac{n}{4} \\ n \text{ large} \end{array}\right\} \Rightarrow \begin{array}{l} G \text{ has a} \\ \text{claw-} \\ \text{decompos.} \end{array}$$

$n/4$ best possible



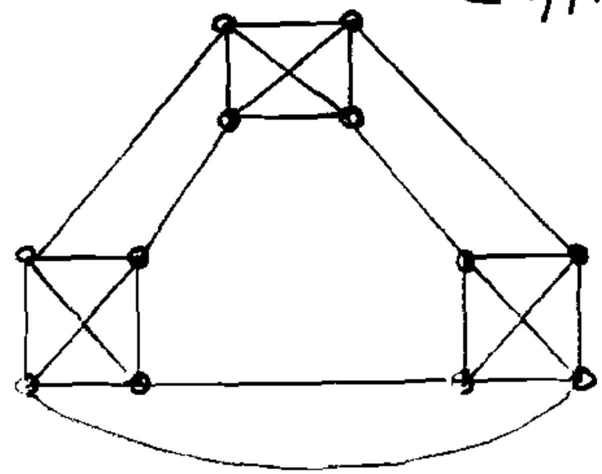
<u>Conjecture</u> (JB+CT)

$$\left.\begin{array}{l} G \text{ 10-edge-conn.} \\ |E(G)| \equiv 0 \pmod 3 \end{array}\right\} \Longrightarrow$$

G has a claw-decompos.

What about 9-edge-conn. ?
8 ?
7 ?
6 ?
5 ?

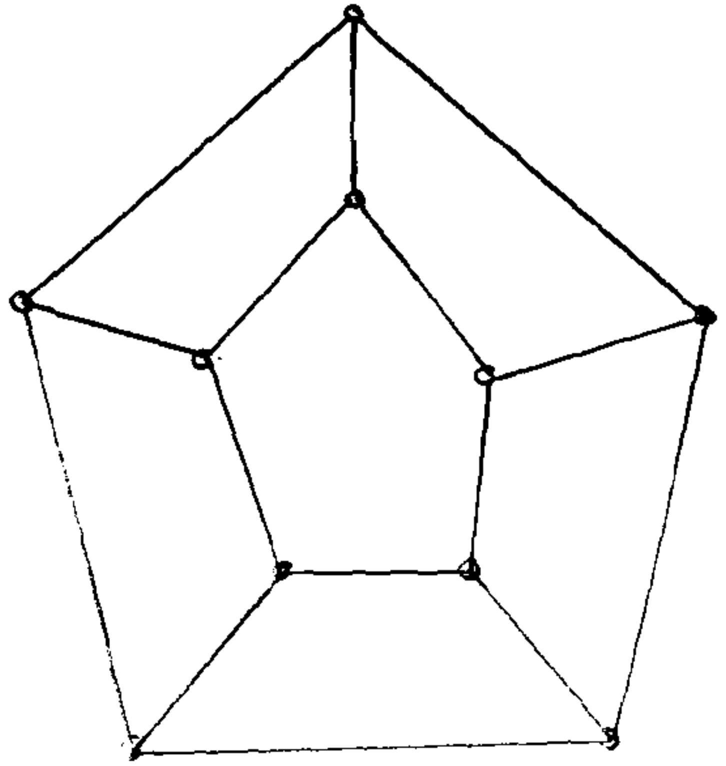
4-edge-conn. is not sufficient



Grötzsch's theorem (1959)

G planar, triangle-free graph $\Longrightarrow$

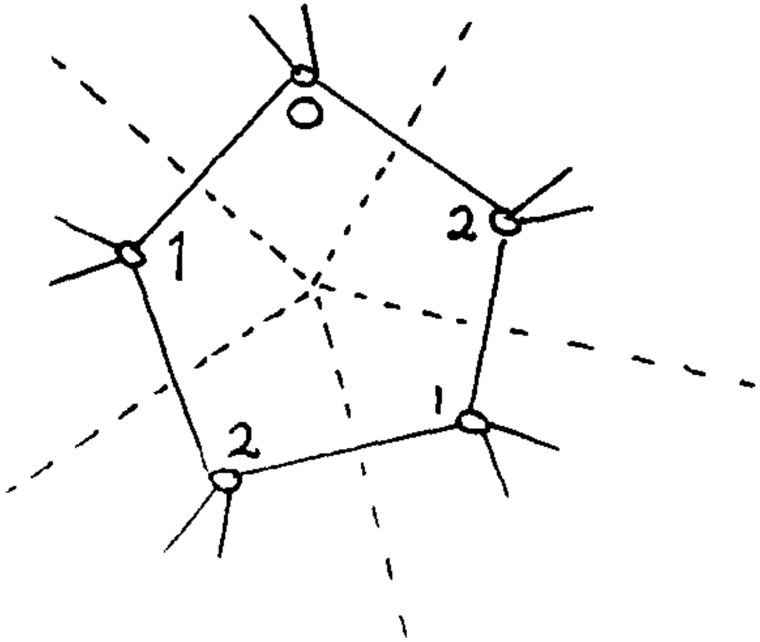
$$\chi(G) \leq 3$$

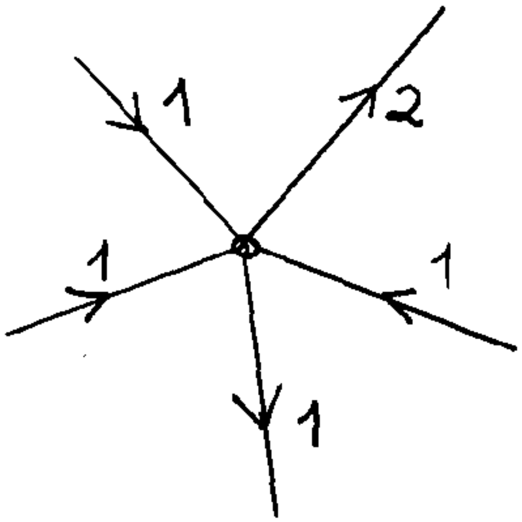


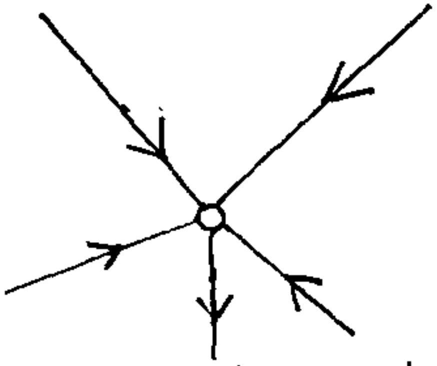
3-coloring a planar graph G $\longrightarrow$

nowhere zero 3-flow in G^* $\longrightarrow$

balanced mod 3 orientation in G^*





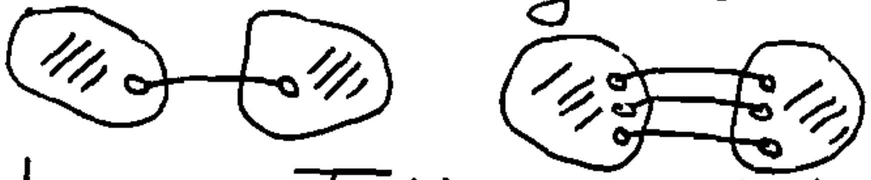


Every vertex is balanced mod 3

Tutte orientation

Grötzsch's theorem

Every planar multigraph without 1-edge-cuts and 3-edge-cuts



has a Tutte orientation.

Conjecture of Tutte (1970?):

Every 4-edge-connected multigraph has a Tutte orientation.

Weaker conjecture of Jaeger (1988):

4 → larger const. k_Γ

Conjecture (JB + CT)

There exist constants

k_c, k_g

such that

(1) every k_g-edge-conn. multigraph admits all generalized Tutte orientations

(2) every k_c-edge-conn. graph has a claw-decomposition (provided its size is $\equiv 0 \bmod 3$).

Generalized Tutte-orientations of a graph G:

For every vertex v, let $q(v) \in \{0, 1, 2\}$.

Assume also
$$\sum_{v \in V(G)} q(v) \equiv |E(G)| \pmod{3}$$

Find an orientation of G such that
$$d^+(v) \equiv q(v) \pmod{3} \quad \forall v \in V(G).$$

Theorem (JB+CT)

If one of k_t, k_c, k_g exists, then they all exist.

Moreover,

$$k_c \le k_g,$$
$$k_t \le k_c + 5$$
$$k_g \le 2k_t + 2$$

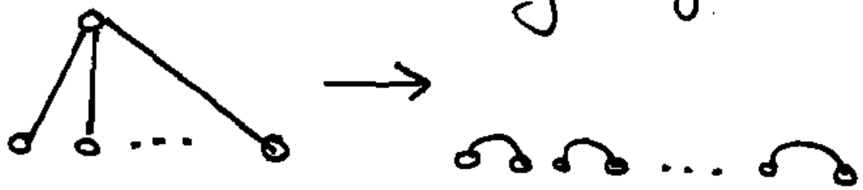
Assume every 10-edge-conn graph has a claw-decomposition (if its size is $\equiv 0 \bmod 3$)

Then every 14-edge-conn. multigraph has a Tutte-orientation.

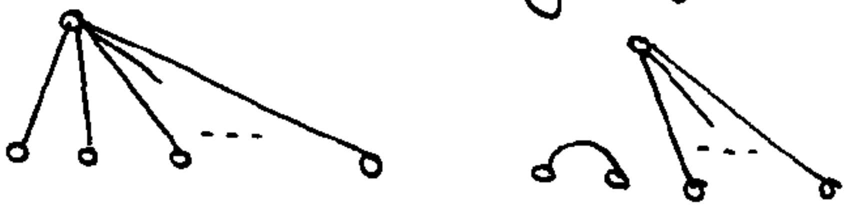
Proof (by induction)

Case 1:  Contract.

Case 2: Vertex of degree 14



Case 3: Vertex of degree ≥ 16



Case 4: 15-regular graph.

Use the claw-decomposition

Conjecture (Jaeger 1988)

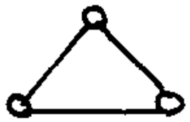
For every odd natural number k, there exists a natural number $j(k)$ such that every $j(k)$-edge-connected multigraph has an orientation which is balanced modulo k.

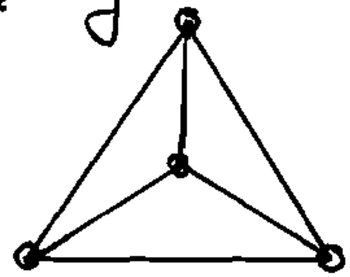
Equivalent to:

Every $j'(k)$-edge-conn. graph has a $K_{1,k}$-decomposition

Thm (Barat + CT)

Let G be a triangulation of a surface. Then G admits all generalized Tutte orientations unless G is one of

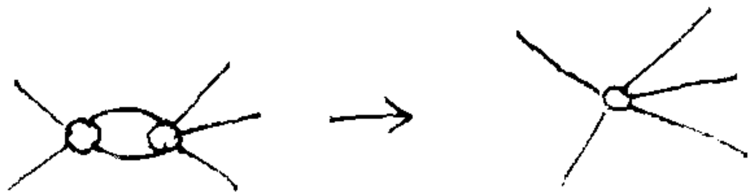




Proof in the weakening where the triangulation has at least one double edge.

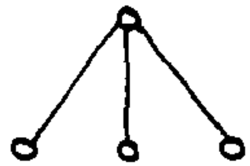
More generally, if a graph G has at least one multiple edge and the contraction of any collection of edges results in multiple edges, then G admits all generalized Tutte-orientations

Proof: Contract a multiple edge and use induction

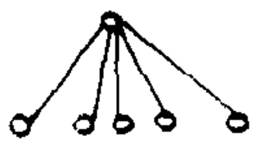


Stronger (and dangerous) Conjecture (JB + CT)

For every tree T, there exists a natural number c_T such that every c_T-edge-connected graph G has a T-decomposition



Tutte's conjecture

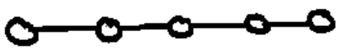


Jaeger's —

What about

?

<u>Theorem</u> G has a

 -decomposition

if $|E(G)|$ is $\equiv 0 \bmod 4$

and G is

$10^{10^{10^{14}}}$ -edge-conn.

<u>Trivial case</u>: All vertices even degree, girth ≥ 5.

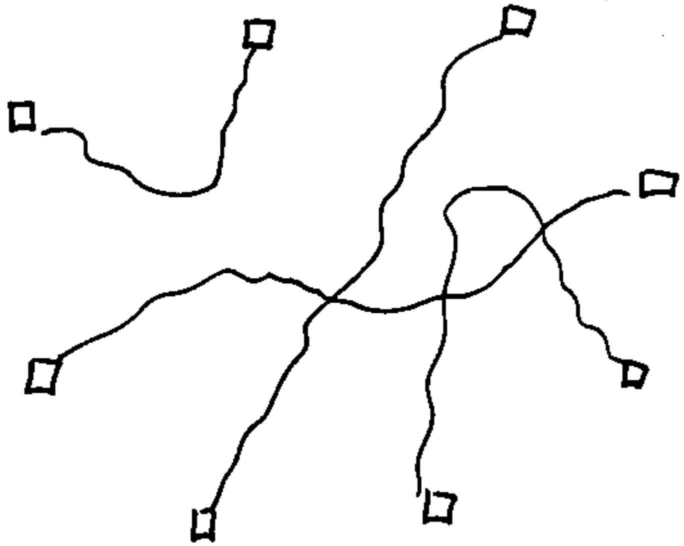
<u>Lemma</u> G connected

$S \subseteq V(G)$

$|S|$ even

Then G has a collection of edge-disjoint paths from S to S such that each vertex of S is the end of precisely one path.

S : □



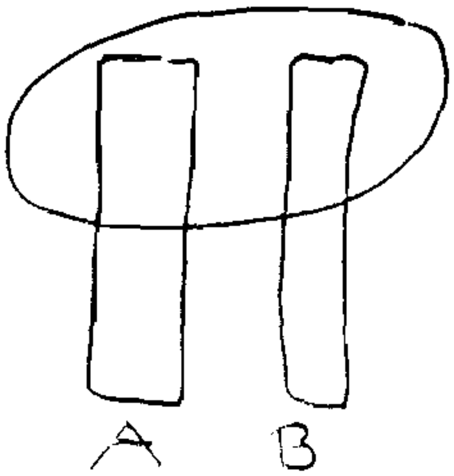
A <u>complete collection of S-paths</u>

Nearly all can be chosen to be even

Lemma

G $(2k-1)$-edge-connected.
Then $V(G)$ has a bipartition $V(G) = A \cup B$ such that $G[A,B]$ is k-edge-conn.

Proof. Select A, B such that $G[A,B]$ has as many edges as possible.



Lemma. G 7-edge-conn.

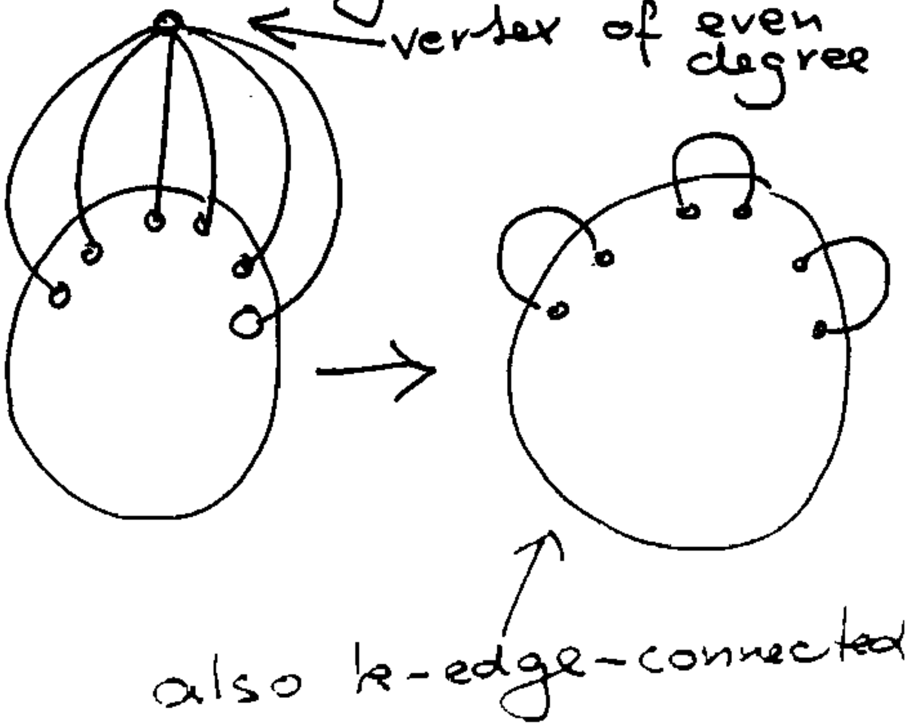
$S \subseteq V(G)$
S even.

Then G has a complete collection of S-paths such that all – except possibly one – is of even length.

Proof. Choose $V(G) = A \cup B$ such that $G[A,B]$ is 4-edge-conn.
Let T_1, T_2 be edge-disj. spanning trees.
$S_1 = S \cap A$, $S_2 = S \cap B$.
Take a complete collection of S_i-paths in T_i, $i = 1, 2$.

Special case – due to Lovász – of Mader's lifting theorem

G k-edge-connected



Thm. $m = 2^7$

G 8^m-edge-conn.
girth $> m$.

Let $S \subseteq V(G)$, $|S|$ even

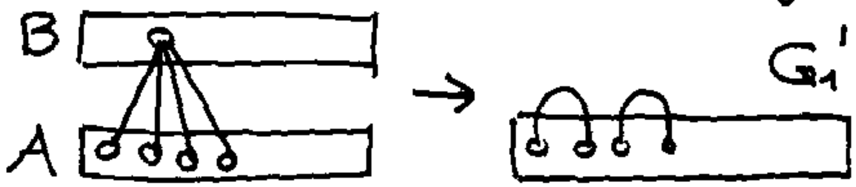
Then G has a complete collection of S-paths such that almost all have length $\equiv 0 \mod m$

Proof Let $V(G) = A \cup B$ such that $G[A, B]$ has large edge-connectivity

$G[A,B] = G_1 \cup G_2$

G_1, G_2 spanning and of large edge-conn.

Delete edges in G_1 such that all vertices in B have even degree



Mader's lifting

Apply induction of G_1' with $S \cap A$ instead of S.

Do the same for A and G_2

Thm. $m = 2^q$

G 8^{m+1}-edge-conn.
girth $> m$
size divisible by m.

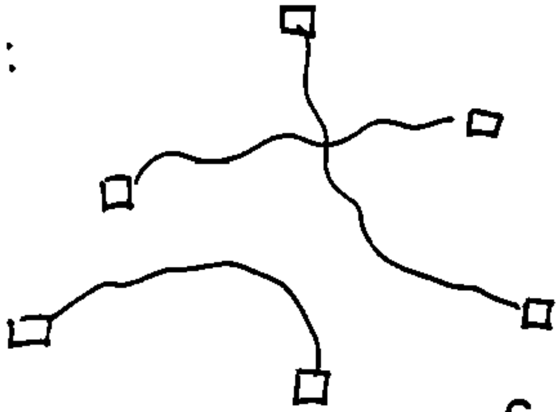
Then G has an m-path-decomposition.

Proof Write $G = G_1 \cup G_2$ where G_i is spanning and of large edge-conn.

S = the odd vertices in G.

Apply previous thm. to G_1, S.

In G_1 :



Paths connecting S of length divisible by m.

Delete the edges and decompose into m-paths.

Because of G_2 the remaining graph is highly connected and (almost) Eulerian

4-path-decomposition

o—o—o—o—o

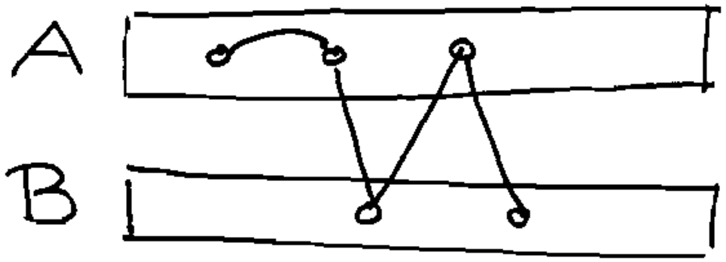
Thm If G is 207-edge-connected then G has a set E of ≤ 6 edges s.t. G−E has a 4-path-decomp.

Idea of proof:

$V(G) = A \cup B$

$G[A,B]$ is 104-edge-conn. and has 52 edge-disj. spanning trees.

Use a few spanning trees to take care of the edges in $G(A), G(B)$



So we have a bipartite graph.

Then repeat the proof in the large girth case.

The main problem is that Mader's lifting theorem creates multiple edges.

We have now decomposed a 207-edge-conn. graph into 4-paths except ≤ 6 edges.

In order to include these 6 edges we raise 207 to $10^{10^{10^{14}}}$.

<u>Conjecture</u> (CT, 1983)

$\forall\, m, g \;\exists\, f(m,g)$:

G $f(m,g)$-edge-conn.

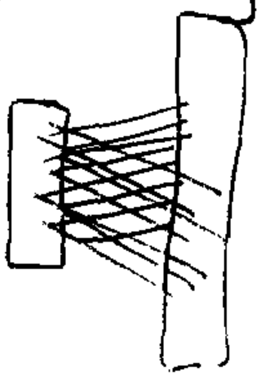
$\Downarrow$

$G \supseteq H$ m-edge-conn. girth $\geq g$

$f(m,4) \le 2m-1$

because every $(2m-1)$-edge-conn. graph contains an m-edge-connected bipartite spanning subgraph.

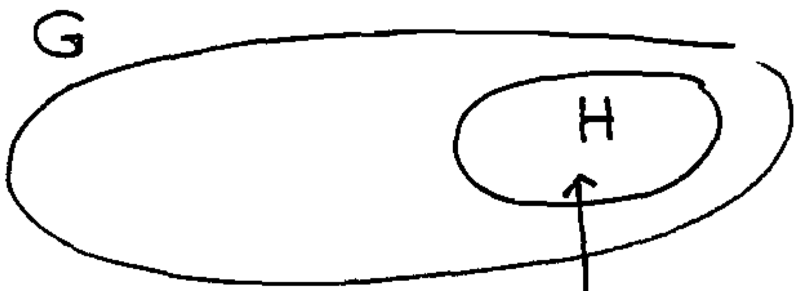
For $g \ge 5$, the subgraph H cannot be spanning.



Kühn, Osthus, 2004

$f(k,6) < 64 \cdot (8k)^{3+2\cdot 11^{64^{8k^3}}}$

Assume now G is $10^{10^{10^{14}}}$-edge-conn.



highly conn.
girth > 4

$H = H_1 \cup H_2$ each spanning highly-conn.

Decompose $G - E(H_1)$ into 4-paths except for G edges in H. Use H_2

Conjecture

$\forall\, k, g\ \exists\, h(k,g)$:

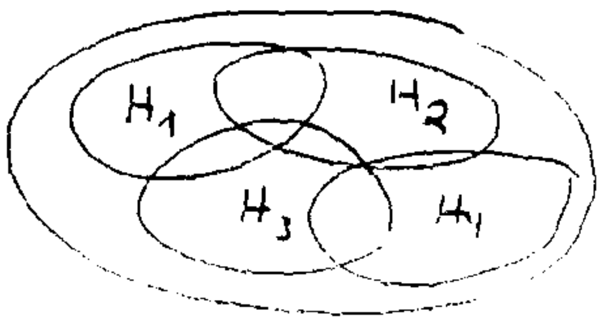
G $h(k,g)$-edge-conn.

$$\Downarrow$$

$G \supseteq H_1, H_2, \dots, H_m$ pairwise edge-disjoint subgraphs of girth $\geq g$, edge-conn. $\geq k$

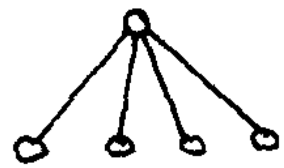
such that

$V(G) = V(H_1) \cup V(H_2) \cup \dots$

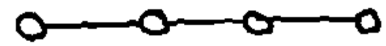


If true, then the tree-decomposition conjecture holds for every path of length a power of 2.

The next natural cases:



4-star



3-path