The De Giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics

Luis A. Caffarelli and Alexis F. Vasseur

June 30, 2010

This lecture is dedicated to the application of the De Giorgi-Nash-Moser kind of techniques to regularity issues in fluid mechanics. In a first section, we recall the original method introduced by De Giorgi to prove C^α regularity of solutions to elliptic problems with rough coefficients. In a second part, we give the main ideas to apply those techniques in the case of parabolic equations with fractional Laplacian. This allows, in particular, to show the global regularity of the Surface Quasi-Geostrophic equation in the critical case. Finally, a last section is dedicated to the application of this method to the 3D Navier-Stokes equation.

Introduction

In this lecture, we will present applications to fluid mechanics of a De Giorgi technique first introduced for the study of some elliptic equations. E. De Giorgi first used this technique in 1957 [9] to solve the 19th Hilbert problem. It consists in showing the regularity of variational solutions to nonlinear elliptic problems. To do so, he developed a geometric method to obtain boundedness and regularity of solutions to elliptic equations with discontinuous coefficients. The essence of this method has been successfully applied in several different situations, like homogenization, phase transition, inverse problems, and more recently by the authors in fluid mechanics.

In this course, we first introduce the method in the original De Giorgi setting, stressing the important aspect of the approach. Then, we will show how to adapt this method in the case of fractional diffusion which provides global regularity results for the critical Surface Quasi-Geostrophic equation.This problem was proposed earlier by several authors as a toy model for the 3D Navier-Stokes equation. The full regularity result is known to collapse for systems. However, since the De Giorgi technique is based on the physical Energy concept, some boundedness results can be obtained by this mean even in the context of systems, as long as they got a scalar Lyapunov functional (that we may call Energy or

Entropy). We will show in the last section how it can be adapted to give a new proof of the partial regularity results of the 3D Navier-Stokes equation first obtained in [3].

1 The original result of De Giorgi

1.1 The result

The 19th Hilbert problem consisted in showing that local minimizers of an energy functional

$$\mathcal{E}(w) = \int_\Omega F(\nabla w)\, dx$$

are regular provided that $p \to F(p)$ is regular.

By a local minimizer, it is meant that

$$\mathcal{E}(w) \leq \mathcal{E}(w + \varphi)$$

for any φ *compactly supported* in Ω. It is standard to show that such a minimizer w satisfies the Euler-Lagrange equation

$$\mathrm{div}(F'(\nabla w)) = 0, \qquad x \in \Omega. \tag{1}$$

De Giorgi showed that with the following assumptions

$$F \text{ is strictly convex}, \qquad \lim_{|p|\to\infty} \frac{F(p)}{|p|^2} = C > 0, \tag{2}$$

any solution to (1) is C^∞ in Ω.

We can notice that the assumption of convexity is necessary. Even in one dimension, if F reaches his minimum in two different points, $p_1 < p_2$, then we can construct Lipshitz only minimizers zig-zagging with slopes p_1 and p_2.

Note that (1) can be rewritten in the non divergence form as

$$F''(\nabla w) : D^2 w = 0.$$

Thanks to the strict convexity property of F, this provides a strictly elliptic equation on w. From the standard Calderon-Zygmund theory (which was known at the time), if ∇w is C^α, we can see this equation as a linear equation on w with elliptic C^α coefficients, by freezing the dependance on w in $F''(\nabla w)$. This provides $C^{2,\alpha}$ regularity on w. Bootstrapping the argument gives, finally, C^∞ regularity on w. However, at this point, we have only $\nabla w \in L^2(\Omega)$. And this theory does not work for weak solutions.

The idea is then, for every $1 \leq i \leq N$ to consider the derivative with respect to x_i of (1). Denote $u = \partial_i w$, this gives

$$\mathrm{div}(F''(\nabla w)\nabla u) = 0.$$

Note that, thanks to (2), there exists $\Lambda > 0$ such that for every $x \in \Omega$

$$\frac{1}{\Lambda} I \leq F''(\nabla w) \leq \Lambda I,$$

where I is the identity $N \times N$ matrix.

De Giorgi showed the following theorem.

Theorem 1.1. *Let Ω be a bounded open set of $\mathbb{R}^N$, and $\Lambda > 0$. Consider $A(x)$ a measurable matrix valued function defined on Ω such that*

$$\frac{1}{\Lambda} I \leq A(x) \leq \Lambda I, \qquad x \in \Omega. \tag{3}$$

Let $u \in H^1(\Omega)$ be a weak solution to

$$-\operatorname{div}(A(x) \cdot \nabla u) = 0, \qquad x \in \Omega. \tag{4}$$

Then $u \in C^\alpha(\tilde{\Omega})$ for any $\tilde{\Omega} \subset\subset \Omega$, with

$$\|u\|_{C^\alpha(\tilde{\Omega})} \leq C \|u\|_{L^2}(\Omega).$$

The constant α depends only on Λ and N. The constant C depends on Λ, N, Ω, and $\tilde{\Omega}$.

By applying this theorem on $\partial_i w$, this gives that $\nabla w \in C^\alpha$. Then bootstrapping the Calderon-Zygmund result gives the proof of the Hilbert problem.

From now on, we will denote L any operator $-\operatorname{div}(A(x)\nabla \cdot)$, where A is uniformly elliptic, that is, verifies (3).

1.2 proof of Theorem 1.1

1.2.1 General ideas

We show the result for $\Omega = B_1$ and $\tilde{\Omega} = B_{1/2}$. The general result, then, can be obtained by standard zooms in the following way. Let $d > 0$ be the distance from $\tilde{\Omega}$ to Ω^c. For any $x_0 \in \tilde{\Omega}$, we define

$$u_d(y) = u(x_0 + dy), \qquad y \in B_1.$$

Note that u_d verify equation (4) with diffusion matrix $A_d(y) = A(x_0 + dy)$ which verifies the same uniform elliptic estimates (3). So u_d is C^α in $B_{1/2}$ and u is C^α in $\tilde{\Omega}$. Note that α does not depend on d.

The proof of Theorem 1.1 is split into two steps. In the first step, a L^∞ bounds is obtained from the L^2 norm. The second step pushes this L^∞ bound to the the C^α regularity.

In the first step we work on a family of ball

$$B_k = \{|x| \leq 1 + 2^{-k}\}. \tag{5}$$

We consider the family of truncated functions

$$u_k = (u - C_k)_+, \qquad C_k = 1 - 2^{-k}. \tag{6}$$

We denote

$$U_k = \int_{B_k} |u_k|^2 \, dx. \tag{7}$$

We derive a nonlinear estimate of the form

$$U_k \leq C^k U_{k-1}^{\beta}, \qquad \text{for a } \beta > 0. \tag{8}$$

This shows that for U_0 small enough, U_k converges to 0 when k converges to infinity. This ensures that $u_\infty = (u-1)_+$ is equal to 0 on $B_{1/2}$, and so $u \leq 1$ on $B_{1/2}$. Note that Equation (4) is linear. Then, the game consists in "non-linearizing" the equation to obtain the non-linear estimate (8). This is based on the interplay between the Sobolev inequality (which gives a control of L^p norms of u_k from the control of the L^2 norm of ∇u_k), and the energy inequality coming from the elliptic equation which provides an "anti-Sobolev" inequality, i.e. a control on $\|\nabla u_k\|_{L^2}$ from a control on $\|u_k\|_{L^2}$. Note that those tools are linear. The nonlinear tool in this game is the Tchebychev inequality. It can be used thanks to the truncation u_k using C_k. The shrinking family of ball (B_k) is used to control the flux of energy in the energy inequality: The energy flowing through the boundary of B_k can be controlled by the energy contained in B_{k-1}.

The second step consists in obtaining a so-called oscillation lemma. We denote $\operatorname{osc}_D u = \sup_D u - \inf_D u$.

Lemma 1.2. *Let u be a solution of $Lu = 0$ in B_1 where A verifies (3). Then there exists $\lambda(\Lambda, N) < 1$ such that*

$$\operatorname{osc}_{B_{1/2}} u \leq \lambda \operatorname{osc}_{B_1} u \ .$$

This lemma implies C^α regularity of the solutions. Its strength is that it gives a definition that depends only on the L^∞ norm. The proof of C^α regularity follows that way. Take any x_0 in $B_{1/2}$. We introduce the rescaled functions

$$\bar{u}_1(y) = u(x_0 + y/2),$$
$$\bar{u}_n(y) = \bar{u}_{n-1}(y/2).$$

As before, $\bar{u}_n$ are solutions to (4) with diffusion matrices $A_n(y) = A(x_0 + y/2^n)$. Note that A_n verifies (3) for the same fixed Λ. We apply recursively Lemma 1.2 on $\bar{u}_n$. This gives

$$\sup_{|x_0 - x| \leq 2^{-n}} |u(x_0) - u(x)| \leq 2\|u\|_{L^\infty(B_1)} \lambda^n.$$

Note that this estimate does not depend on x_0. Hence u is in $C^\alpha(B_{1/2})$ with

$$\alpha = -\frac{\ln \lambda}{\ln 2}.$$

1.2.2 First step: from L^2 to L^∞

The two first ingredients are the Sobolev inequality, given by

$$\|v\|_{L^p(B_1)} \leq C\|\nabla v\|_{L^2(B_1)}$$

for $p(N) = \frac{2N}{N-2}$, whenever v is supported in B_1, and the energy inequality given by

Lemma 1.3. *(Energy inequality) If $u \geq 0$, $Lu \leq 0$ and $\varphi \in C_0^\infty(B_1)$ then*

$$\int_{B_1} (\nabla[\varphi u])^2 \, dx \leq C\|\nabla\varphi\|_{L^\infty}^2 \int_{B_1 \cap \operatorname{supp}\varphi} u^2 \, dx.$$

If A is symmetric then $C = \Lambda^2$.

Proof. We multiply $Lu = -\operatorname{div}(A(x)\nabla u)$ by $\varphi^2 u$. Since the first term is nonpositive and the second one nonnegative, we get

$$\int \nabla^T(\varphi^2 u) A \nabla u \, dx \leq 0.$$

We have to transfer a φ from the left ∇ to the right ∇.

We use, for this, the estimate

$$\int \nabla^T(\varphi u) A u (\nabla\varphi) \, dx \leq \varepsilon \int \nabla^T(\varphi u) A \nabla(\varphi u) \, dx + \frac{\Lambda}{\varepsilon} \int |\nabla\varphi|^2 u^2 \, dx.$$

We denote $u_+ = \sup(0, u)$. The main result of this section is the following.

Proposition 1. (From L^2 to L^∞) *There exists a constant $\delta = \delta(N, \Lambda)$ such that if $\|u_+\|_{L^2(B_1)}^2 \leq \delta$, then*

$$\sup_{B_{1/2}} u_+ \leq 1 \ .$$

Applying the proposition on $(\sqrt{\delta}/\|u_+\|_{L^2})u$ (which is still solution to (4) with the same A, and so the same Λ) gives the following corollary.

Corollary 1. *If u is a solution of $Lu = 0$ in B_1, then*

$$\|u_+\|_{L^\infty(B_{1/2})} \leq \frac{1}{\sqrt{\delta}}\|u_+\|_{L^2(B_1)} \ .$$

1.3 Proof

We consider a sequence of truncations

$$\varphi_k u_k$$

where φ_k is a sequence of shrinking cut-off functions converging to $\chi_{B_{1/2}}$.

More precisely:

$$\varphi_k \begin{cases} \equiv 1 & \text{in } \ B_k \\ \equiv 0 & \text{in } \ B_{k-1}^c \end{cases}$$

$$|\nabla \varphi_k| \le C\ 2^k\ .$$

Note that where $u_{k+1} > 0$, $u_k > 2^{-(k+1)}$. Therefore $\{(\varphi_{k+1} u_{k+1}) > 0\}$ is contained in $\{(\varphi_k u_k) > 2^{-(k+1)}\}$.

We have from the Sobolev inequality with $p = \frac{2N}{N-2}$:

$$\left[\int (\varphi_{k+1} u_{k+1})^p \, dx \right]^{2/p} \le C \int (\nabla [\varphi_{k+1} u_{k+1}])^2 \, dx.$$

But, from Hölder

$$\int (\varphi_{k+1} u_{k+1})^2 \, dx \le \left[\int (\varphi_{k+1} u_{k+1})^p \, dx \right]^{2/p} \cdot |\{\varphi_{k+1} u_{k+1} > 0\}|^{\frac{1}{N}},$$

so we get

$$U_{k+1} \le C \left(\int [\nabla(\varphi_{k+1} u_{k+1})]^2 \, dx \right) |\{\varphi_{k+1} u_{k+1} > 0\}|^{\frac{1}{N}}\ .$$

We now control the right hand side by U_k through the energy inequality: From energy we get

$$\int |\nabla(\varphi_{k+1} u_{k+1})|^2 \, dx \le C\ 2^{2k} \int_{\operatorname{supp} \varphi_{k+1}} u_{k+1}^2 \, dx.$$

Since $\varphi_k \equiv 1$ on $\operatorname{supp} \varphi_{k+1}$ and $u_{k+1} \le u_k$, this can be controlled by

$$\mathcal{C}\ 2^{2k} \int (\varphi_k u_k)^2 \, dx = C\ 2^{2k} U_k.$$

To control the last term, we have from the observation above:

$$|\{\varphi_{k+1} u_{k+1} > 0\}|^{\frac{1}{N}} \le |\{\varphi_k u_k > 2^{-k}\}|^{\frac{1}{N}}\ .$$

And by Chebyshev, this is controlled by

$$\le 2^{\frac{2k}{N}} \left(\int (\varphi_k u_k)^2 \right)^{\frac{1}{N}}\ .$$

So we get

$$U_{k+1} \le C\ 2^{4k} (U_k)^{1+\frac{1}{N}}\ .$$

Then, for $U_0 = \delta$ small enough $U_k \to 0$. The buildup of the exponent in U_k, forces U_k to go to zero. In fact, U_k has faster than geometric decay, i.e., for any $M > 0$, $U_k < M^{-k}$ if $U_0(M)$ is small enough.

1.3.1 Step 2. Oscillation decay.

The main result of this section is the following.

Proposition 2. *Let $v \leq 1$, $Lv = 0$ in B_2. Assume that $|B_1 \cap \{v \leq 0\}| \geq \mu$ $(\mu > 0)$. Then $\sup_{B_{1/2}} v \leq 1 - \lambda$, where λ depends only on μ, Λ, and N.*

In other words, if v is a solution of $Lv = 0$, smaller than one in B_1, and is "far from 1" in a set of non trivial measure, it cannot get too close to 1 in $B_{1/2}$.

Let us first show how this leads to Lemma 1.2. Consider the function

$$v(x) = \frac{2}{\operatorname{osc} u}\left(u(x) - \frac{\sup u + \inf u}{2}\right).$$

We have $-1 \leq v \leq 1$. Assume that v is half of the space smaller than 0 in B_1. Then we can apply Proposition 2 on v which gives that $\operatorname{osc}_{B_{1/2}} v \leq 2 - \lambda$. Hence $\operatorname{osc}_{B_{1/2}} u \leq (1 - \lambda/2) \operatorname{osc} u_{B_2}$. We get the same result if v is half of the space bigger than 0, working with $(-v)$.

To prove Proposition 2, we may first note that if the set

$$|\{v \leq 0\}| \geq |B_1| - \frac{\delta}{4} ,$$

then

$$\|v_+\|^2_{L^2(B_1)} \leq \delta/4$$

and Corollary 1 would implies that $u_+|_{B_{1/2}} \leq 1/2$.

So we must bridge the gap between knowing that $|\{v \leq 0\}| \geq \frac{1}{2}|B_1|$ and knowing that $|\{v \leq 0\}| \geq |B_1| - \frac{\delta}{2}$.

The main tool is the following De Giorgi isoperimetric inequality. It may be considered as a quantitative version of the fact that a function with a jump discontinuity cannot be in H^1.

Lemma 1.4. *Consider w such that $\int_{B_1} |\nabla w_+|^2 \, dx \leq C_0$. Set*

$$|A| = |\{w \leq 0\} \cap B_1|$$
$$|C| = |\{w \geq 1\} \cap B_1|$$
$$|D| = |\{0 < w < 1\} \cap B_1|.$$

Then we have

$$C_0|D| \geq C_1(|A|\,|C|^{1-\frac{1}{n}})^2 .$$

Proof. Consider $\bar{w} = \sup(0, \inf(w, 1))$. Note that $\nabla \bar{w} = \nabla w_+ \mathbf{1}_{\{0 \leq w \leq 1\}}$. For x_0 in C we reconstruct $\bar{w}$ by integrating along any of the rays that go from x_0 to a point in A

$$1 = \bar{w}(x_0) = \int \bar{w}_r \, dr \quad \text{or}$$

$$|A| \leq \int_D \frac{|\nabla \bar{w}(y)| \, dy}{|x_0 - y|^{n-1}}$$

$$\left(\bar{w}_r \, dr \, d\sigma \leq \frac{|\nabla w_+| r^{n-1} dr \, d\sigma}{r^{n-1}} \right)$$

Integrating x_0 on C, we find

$$|A| \; |C| \leq \int_D |\nabla w_+(y)| \left(\int_C \frac{dx_0}{|x_0 - y|^{n-1}} \right) dy.$$

Among all C with the same measure $|C|$ the integral in x_0 is maximized by the ball of radius $|C|^{1/n}$, centered at y

$$\int_C \cdots \leq |C|^{1/n} \; .$$

So

$$|A| \; |C| \leq |C|^{1/n} \left(\int_D |\nabla w_+|^2 \right)^{1/2} |D|^{1/2} \; .$$

Since $\int |\nabla w_+|^2 \, dx \leq C_0$ the proof is complete. □

Proof of Proposition 2. We consider the new sequence of truncation

$$w_k = 2^k [v - (1 - 2^{-k})].$$

Note that for any k we have $w_k \leq 1$. So from the energy inequality, we have

$$\int_{B_1} |\nabla (w_k)_+|^2 \, dx \leq C_0.$$

We have also $|\{w_k \leq 0\} \cap B_1| \geq \mu$. We apply Lemma 1.4 recursively on $2w_k$ as long as

$$\int_{B_1} (w_{k+1})_+^2 \, dx \geq \delta.$$

We get

$$|\{w_{k+1} \geq 0\} \cap B_1| = |\{2w_k \geq 1\} \cap B_1| \geq \int_{B_1} (w_{k+1})_+^2 \, dx \geq \delta.$$

So, from the lemma, there exists a constant α which does not depend on k such that

$$|\{0 < w_k < 1/2\} \cap B_1| \geq \alpha.$$

Then

$$|\{w_k \leq 0\} \cap B_1| \geq |\{w_{k-1} \leq 0\} \cap B_1| + \alpha \geq \mu + k\alpha.$$

This clearly fails after a finite number of k. At this k_0 we have for sure that

$$\int_{B_1} (w_{k_0+1})_+^2 \, dx \leq \delta.$$

Proposition 1 then implies that $w_{k_0+1} \leq 1$ in $B_{1/2}$. Rescaling back to v gives the result.

2 Global regularity result for the Surface Quasi-Geostrophic equation

2.1 Introduction

The Surface-Quasi-Geostrophic equation is the layer equation of the Quasi-Geostrophic equation. It models the evolution of the temperature at the surface of the earth. Consider the potential of temperature θ be defined on $\mathbb{R}^2$. It has been studied by several authors (see [8, 7, 6]). The equation is the following.

$$\theta_t + (v \cdot \nabla)\theta = (\Delta^{1/2})\theta, \qquad t > 0,\ x \in \mathbb{R}^2,$$
$$(-v_2, v_1) = (R_1\theta, R_2\theta),$$

where R_j are the Riesz transforms of θ defined by

$$\widehat{R_j\theta} = \frac{i\xi_j}{|\xi|}\hat{\theta}.$$

In particular we have

$$\operatorname{div} v = 0.$$

The term $(v\cdot\nabla)\theta$ models the convection transport, and the term $(\Delta^{1/2})\theta$ models the friction at the surface of the earth called Ekman pumping. Note that we are in a critical case, since the regularization term $(\Delta^{1/2}\theta)$ is of the same order as the transport term $(v.\nabla\theta)$. We are concerned here about the global regularity of the solutions. The main theorem is the following.

Theorem 2.1. *For any initial value $\theta_0 \in L^2(\mathbb{R}^2)$, there exists a weak solution to the SQG which is smooth on $(0,\infty)\times\mathbb{R}^2$.*

The main difficulty is to get the C^α regularity. Then by standard potential theory and bootstrapping argument, full regularity can be obtained. Note that the result can be obtained in higher dimension provided that $\operatorname{div} v = 0$. It is worth noting that a proof of propagation of regularity can be obtained with completely different tools [10].

We recall that the Riesz operator R is bounded from L^p to L^p for any $1 < p < \infty$, but not for L^∞. It is bounded so from BMO to BMO, the space of bounded mean variation. That is, in any cube Q the "average of u minus its average" is bounded by a constant C

$$\frac{1}{|Q|}\int_Q \left|u(x) - \frac{1}{Q}\int_Q u(y)\,dy\right| dx \le C.$$

The smallest C good for all cubes defines a semi-norm (it does not distinguishes constant that we may factor out). The space of functions u in BMO of the unit cube is smaller than any L^p $(p < \infty)$ but not included in L^∞ $((\log|x|)^-$ is a typical example).

In fact functions u in BMO have "exponential" integrability

$$\int_{Q_1} e^{C|u|} < \infty.$$

The first step is to obtain a global L^∞ estimate. The proof of the following result is fairly closed to the De Giorgi result.

Theorem 2.2. *Let θ be a (weak) solution of*

$$\theta_t + v\nabla\theta = (\Delta^{1/2})\theta \quad \text{in} \quad \mathbb{R}^N \times [0,\infty) \tag{9}$$

for some incompressible vector field v (with no apriori bounds) and initial data θ_0 in L^2.

Then

$$\|\theta(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{t}\|\theta(\cdot,0)\|_{L^2(\mathbb{R}^2)} \ .$$

Since the velocity field is the Riesz transform of θ, we have the additional control

$$\text{div } v = 0 \ , \qquad \sup_{s>t}\left(\|v(s)\|^2_{L^2(\mathbb{R}^2)} + \|v(s)\|_{BMO(\mathbb{R}^2)}\right) \leq \frac{C}{t}.$$

We may fix a $t_0 > 0$ and consider now the equation as a linear one on $\theta(t-t_0)$ where v is a given velocity field. Multiplying the equation by $1/\|\theta(t_0)\|_{L^\infty}$, we are left with the task of showing the C^α regularity of a solution (still denoted θ) to the linear equation

$$\theta_t + v\nabla\theta = (\Delta^{1/2})\theta$$

with v a given velocity verifying

$$\text{div } v = 0 \ , \qquad \sup_t\left(\|v\|^2_{L^2(\mathbb{R}^2)} + \|v\|_{BMO(\mathbb{R}^2)}\right) \leq C, \tag{10}$$

with the a priori bounds

$$\begin{cases} \sup_t \|\theta(t)\|^2_{L^2(\mathbb{R}^2)} + \|\Delta^{1/4}\theta\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}^+)} \leq C \\ \text{and also} \ \ \|\theta\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^+)} \leq 1. \end{cases} \tag{11}$$

To do that we need to reproduce the local in space De Giorgi method. The main result is the following.

Theorem 2.3. *Let θ be a weak solution of (9) for some incompressible vector field v verifying (10) and initial data θ_0 in L^2. Then $\theta \in C^\alpha((t_0,\infty)\times\mathbb{R}^N)$ for any $t_0 > 0$.*

This linear result is interesting by itself. Fractional Laplacians are commonly used to model stochastic transport involving jumps (like Levy processes). The case considered here corresponds to the critical case where the stochastic transport has the exact order as the deterministic transport. This result shows that, quite surprisingly, the regularization effect of the stochastic transport always prevails against the deterministic transport. This is due to the geometric structure of the deterministic transport velocity:

$$\text{div } v = 0$$

i.e., to the fact that the deterministic transport is incompressible.

2.2 The fractional Laplacian and harmonic extensions

The C^α regularity is a local result. The main difficulty here, is that we are dealing with a nonlocal equation due to the fractional Laplacian. The main idea is to replace this nonlocal description by a local one as the cost of adding an additional dimension. The case $\Delta^{1/2}$ is an interesting case since it coincides with the Dirichlet to Neuman map (see [4] for extension to any fractional Laplancian). More precisely, given θ defined for x in $\mathbb{R}^N$, we extend it to θ^* defined for (x, y) in $(\mathbb{R}^{N+1})^+$ by combining it with the Poisson kernel:

$$P_y(x) = \frac{C\ y}{(y^2 + |x|^2)^{\frac{N+1}{2}}} = y^{-N} P_1(x/y).$$

Then $\theta^*(x, y)$ satisfies

$$\Delta_{x,y}\theta^* = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^+.$$

It can be checked that $\Lambda^{1/2}\theta(x_0) = D_y\theta^*(x_0, 0)$. Indeed, taking the Fourier-transform in x we find that

$$\widehat{\theta^*}(\xi, y) \text{ satisfies } |\xi|^2\widehat{\theta^*} = D_{yy}\widehat{\theta^*}.$$

Thus

$$\widehat{\theta^*}(\xi, y) = \widehat{\theta}(\xi)e^{-y|\xi|}.$$

In particular

$$D_y\widehat{\theta}(\xi, 0) = -\widehat{\theta}(\xi)|\xi| = (\widehat{\Delta^{1/2}\theta})(\xi).$$

Hence, the operator $\Delta^{1/2}$ which is non local in $\mathbb{R}^N$ has a local representation in $\mathbb{R}^{N+1}$. Indeed, $\Delta^{1/2}(x_0)$ is determined by the values of θ^* in a neighborhood of $(x_0, 0)$.

As we said before, the De Giorgi is based on the notion of energy (more precisely of the evolution of level set of energy $(\theta - \lambda)_+$). Using the harmonic extension, we can make sense of the Green's and "energy" formula for the half Laplacian: Let $\sigma(x), \theta(x)$ be two "nice, decaying" functions defined in $\mathbb{R}^n$, and $\bar{\sigma}(x, y)$, $\bar{\theta}(x, y)$ decaying extensions into $(\mathbb{R}^{N+1})^+$. Then, we have

$$\int_{\mathbb{R}^n} \sigma(\bar{\theta})_\nu = \int_{(\mathbb{R}^{N+1})^+} \nabla_{x,y}\bar{\sigma}\nabla_{(x,y)}\bar{\theta} + \int_{(\mathbb{R}^{N+1})^+} \bar{\sigma}\Delta_{x,y}\bar{\theta}.$$

If we choose $\bar{\theta}(x,y)$, the harmonic extension, θ^*, the term $\bar{\theta}_\nu(x,0)$ becomes $-\Delta^{1/2}\theta$, and $\Delta\theta^* \equiv 0$, giving us

$$\int_{\mathbb{R}^n} \sigma(-\Delta^{1/2})\theta = \int_{(\mathbb{R}^{N+1})^+} \nabla\sigma\nabla\theta^*.$$

Further, if we choose

$$\sigma = (\theta-\lambda)_+ \quad \text{and} \quad \bar{\sigma} = (\theta^*-\lambda)_+$$

(i.e., the *truncation* of the *extension* of θ) we get

$$\int_{\mathbb{R}^n} (\theta-\lambda)_+(-\Lambda^{1/2}\theta) = \int_{(\mathbb{R}^{n+1})^+} [\nabla(\theta^*-\lambda)_+]^2\, dx\, dy$$

To complete our discussion, we point out that the harmonic extension θ^* of θ, is the one extension that minimizes Dirichlet energy

$$E(\theta^*) = \int_{\mathbb{R}^{n+1}_+} |\nabla\theta^*|^2$$

and that this minimum defines the $H^{1/2}$ norm of θ. In particular, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} (\theta-\lambda)_+(-\Delta^{1/2})\theta &= \iint [\nabla(\theta^*-\lambda)_+]^2\, dx\, dy \\ &\geq \iint [\nabla(\theta-\lambda)^*_+]^2\, dx\, dy = \|(\theta-\lambda)_+\|^2_{H^{1/2}}\end{aligned}$$

(since the *harmonic extension of the truncation* has *less energy* than the *truncation of the harmonic extension*).

The Energy Inequality is attained, as usual, by multiplying the equation with a truncation of θ,

$$(\theta_\lambda) = (\theta-\lambda)_+$$

and integrating in $\mathbb{R}^n \times [T_1, T_2]$.

The term corresponding to the transport vanishes, and we get:

$$\frac{1}{2}\int \left[(\theta_\lambda)^2(y,T_2) - (\theta_\lambda)^2(y,T_1)\right]\, dy + 0 = \iint_{\mathbb{R}^n\times[T_1,T_2]} \theta_\lambda \Lambda^{1/2}\theta\, dy\, dt$$

The last term corresponds, for the harmonic extension $\theta^*(x,z)$ to ($x \in \mathbb{R}^n$, $z \in \mathbb{R}^+$)

$$\begin{aligned}&\int_{T_1}^{T_2} dt \left(\int_{\mathbb{R}^n} (\theta^*)_\lambda(y,0,t) D_z(\theta^*)(y,0,t)\, dy \right) \\ &= -\int_{T_1}^{T_2} \iint_{\mathbb{R}^{n+1}_+} \nabla(\theta^*)_\lambda(y,z,t)\nabla\theta^*(y,z,t)\, dy\, dz \\ &= -\int_{T_1}^{T_2} dt \iint_{\mathbb{R}^{n+1}_+} \left[\nabla\theta^*_\lambda\right]^2\, dy\, dz\ .\end{aligned}$$

Note that $(\theta^*)_\lambda$ is *not* the harmonic extension of θ_λ, but the truncation of the extension of θ, i.e., $(\theta^* - \lambda)_+$.

Nevertheless, it is an admissible (going to zero at infinity) extension of θ_λ and as such,

$$\|\theta^*_\lambda\|_{H^1(\mathbb{R}^{n+1}_+)} \geq \|\theta_\lambda\|_{H^{1/2}(\mathbb{R}^n)} \ .$$

Therefore we end up with the following energy inequality

$$\|\theta_\lambda(\cdot, T_2)\|^2_{L^2} + \int_{T_1}^{T_2} \|\theta_\lambda\|^2_{H^{1/2}}\, dt \leq \|\theta_\lambda(T_1)\|^2_{L^2} \ . \tag{12}$$

2.3 From L^2 to L^∞

The proof is similar to the classical De Giorgi case. We have a layer at time $t = 0$, but no boundary in x. We do not need yet to consider decreasing balls. However, to get the boundedness, we need to "escape" from the layer $t = 0$ in a dyadic way. So we consider cut-offs in time $T_k = -1 - 2^{-k}$. As before we consider also a sequence of increasing cut-offs $\lambda_k = 1 - 2^{-k}$ of θ (i.e., $\theta_k = \theta_{\lambda_k}$).

Using the energy inequality (12), we get, thanks to Sobolev inequality, for any $s \leq T_k$:

$$\sup_{t \geq T_k} \|\theta_k(t)\|^2_{L^2} + \int_{T_k}^{\infty} \|\theta_k(t)\|^2_{L^p}\, dt \leq \|\theta_k(s)\|^2_{L^2}.$$

This is a control of θ_k in terms of norms $L^\infty(L^2)$ and $L^2(L^p)$ for a $p > 2$. By interpolation, there exists $q > 2$, such that

$$\|\theta_k\|^2_{L^q(\mathbb{R}^N \times [T_k, \infty))} \leq \|\theta_k(s)\|^2_{L^2}$$

for any $s < T_k$. So, taking the mean value in s between T_{k-1} and T_k, we find

$$\|\theta_k\|^2_{L^q(\mathbb{R}^N \times [T_k, \infty))} \leq 2^k \int_{T_{k-1}}^{T_k} |\theta_k(s)\|^2_{L^2}\, ds.$$

We now invert the relation. For $I_k = [T_k, \infty) \times \mathbb{R}^N$ we denote

$$U_k = \iint_{I_k} (\theta_k)^2\, dx\, dt.$$

By Hölder with θ^2 and $\chi_{\theta_k > 0}$ we get (with $\bar{q}$ the conjugate exponent to $q/2$):

$$U_k \leq \left[\iint_{I_k} \theta_k^q\right]^{2/q} |\{\theta_k > 0\} \cap I_k|^{1/\bar{q}} = \alpha \cdot \beta.$$

In turn $\alpha \leq 2^k U_{k-1}$. By going from k to $k-1$, we can estimate:

$$\beta = \left|\{\theta_{k-1} > 2^{-k}\} \cap I\right|^{1/\bar{q}} \leq \left[2^{2k} \iint_{I_k} (\theta_{k-1})^2\right]^{1/\bar{q}} \quad \text{(by Chebichev)}$$

(since $\theta_k \leq \theta_{k-1}$ and further $\theta_k > 0$ implies $\theta_{k-1} > 2^{-k}$). This gives the recurrence relation

$$U_k \leq 2^{\bar{C}k} U_k^{1+1/(2\bar{q})}.$$

Due to the $1 + 1/(2\bar{q})$ nonlinearity, $U_k \to 0$ if U_0 was small enough, i.e., if $\|\theta_0\|_{L^2} \leq \delta_0$ then $\|\theta(\cdot,t)\|_{L^\infty} \leq 1$, for $t \geq 1$. Since the equation is linear in θ, we can apply this result to $\frac{\delta_0}{\|\theta_0\|_{L^2}}\theta(t_0 t, t_0 x)$ which gives

$$\|\theta(\cdot,t)\|_{L^\infty} \leq \frac{\|\theta_0\|_{L^2}}{t_0^{N/2}\delta_0} \quad \text{for } t \geq t_0.$$

2.4 Regularity C^α

We now pass to the issue of regularity, i.e., the "oscillation lemma". We need to reproduce the local in space De Giorgi method to get Theorem 2.3. This is where the harmonic extension is particularly important: at the cost of adding one variable, we are able to localize the energy inequality.

The most difficult part is to get the oscillation lemma under the smallness condition on energy:

Proposition 3. *Assume condition* (10) *on the velocity. Then there exists* $\varepsilon_0 > 0$ *and* $0 < \lambda < 1$*, such that for any* θ *solution to* (9) *the following holds true. We denote* θ^* *its harmonic extension. Assume that*

- $\theta^*(t,x,z) \leq 1, \qquad t \in [-4,0], x \in B_4, z \in [0,4],$
- $\displaystyle\int_{-4}^0 \int_{B_4} |\theta_+(t,x)|^2 \, dx \, dt + \int_{-4}^0 \int_{B_4} \int_0^4 |\theta_+^*(t,x)|^2 \, dz \, dx \, dt \leq \varepsilon_0,$

Then

$$\theta(t,x) \leq 1 - \lambda, \qquad (t,x) \in (-1,0) \times B_1. \tag{13}$$

This calls for a couple of remarks:

- Note that (13) implies the existence of λ^* such that (13) is valid for θ^* on $((-1,0) \times B_1 \times (0,1))$ with λ^*. Indeed, θ^* is harmonic in x, z. So, for any fix t, by standard comparison principle, it is smaller that the harmonic solution (independent on time) with boundary conditions 1 for $z = 4$ and for $|x| = 4$, and λ for $z = 0$. Note that, thanks to the strong maximum principle, this fixed harmonic solution is strictly smaller than one on $B_1 \times [0,1]$. The constant $1 - \lambda^*$ is its maximum on this set.
- With the above remark, we get the oscillation decay if we can get rid of the energy smallness condition. This can be done using the De Giorgi isoperimetric lemma as before. The non local feature of the equation for this part of the proof does not induce much difficulty. The main thing is to incorporate the time dependency. We refer to the paper [5] for this part.

- With this oscillation decay at hand, we can obtain the oscillation lemma as before by rescaling the function θ. A last difficulty occurs, so, since the condition (10) is not entirely invariant by the blow-up $(t,x) \to (\varepsilon t, \varepsilon x)$. Obviously, the BMO norm is invariant. But not the mean value of v on a ball. Note that we would not have this problem is v would have been bounded in L^∞. This difficulty comes from the lack of boundedness of the Riesz operator from L^∞ to L^∞. This problem can be solved by using the transport feature of the equation, especially the Galilean invariance. while rescaling, we follow the mean flow $(t,x) \to (\varepsilon t, \varepsilon(x - t\int v))$ (see [5] for more details).

Let us concentrate here on the most technical result Proposition 3. It is based on the following localized energy inequality.

Lemma 2.4. *Let t_1, t_2 be such that $t_1 < t_2$. There exists a constant $\Phi > 0$ depending on the constant in (10) such that the following holds true. For any solution to (9), and cut-off function η such that for every $t_1 < t < t_2$: $\eta[\theta^*]_+(t) = 0$ on $z = 4$, and on $|x| = 4$, we have*

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{B_4}\int_0^4 |\nabla(\eta[\theta^*]_+)|^2\,dz\,dx\,dt + \int_{B_4}(\eta[\theta]_+)^2(t_2,x)\,dx \\ \leq &\int_{B_4}(\eta[\theta]_+)^2(t_1,x)\,dx + \Phi\int_{t_1}^{t_2}\int_{B_4}([\nabla\eta][\theta]_+)^2\,dx\,dt \\ &+2\int_{t_1}^{t_2}\int_{B_4}\int_0^4([\nabla\eta][\theta^*]_+)^2\,dz\,dx\,dt. \end{aligned} \tag{14}$$

This lemma can be obtained using the same ingredients as before. It gives a good control in $L^2_t(H^1_{x,z})$ of θ^*. However, it gives a control in $L^\infty_t(L^2)$ only on the trace $z = 0$. To perform the De Giorgi nonlinearization procedure, we need to extract more uniform in time control of θ^* for $z > 0$. Note that this degeneracy of the energy inequality is not only technical. Using the extension map, we did not solve the non-local feature of the problem. We have merely encoded it in $z > 0$. The hard work remains to control it.

We have fluxes of energy in time, position x, and z. Therefore we consider shrinking sets of the form:

$$T_k = -(1+2^{-k}), \qquad B_k = \{|x| \leq 1 + 2^{-k}\}, \qquad B_k^* = B_k \times (0, \delta^k).$$

Note that we add more flexibility by shrinking in z at a faster pace $\delta << 1$.

As usual, we want to consider a family of cut-off θ_{λ_k}. The fundamental idea, is to get such a family which verifies

$$\theta^*_{\lambda_k}(t,x,\delta^k) = 0, \qquad \text{uniformly in } (t,x). \tag{15}$$

This can be performed for $k = 1$ in the following way. Using a cut-off function η in x and z in the energy inequality, we get a uniform in time bound on the energy on the trace $z = 0$: $\int \theta_+^2 \, dx$. This quantity is very small if ε_0 is small. Then we use again a comparison principle. For every fixed t, θ^* is an harmonic function which is (by comparison principle) bounded by above by the sum of two harmonic functions. The boundary conditions are for the first one, 1 on the surfaces $|x| = 4$ and $z = 4$, and 0 on the surface $z = 0$. The second harmonic function has 0 as boundary values on the surfaces $|x| = 4$ and $z = 4$, and $\theta(t, x)$ for $z = 0$. The strong maximum principle assures that the first harmonic function is smaller than, let say, $1 - \lambda_1$ for $z = 2, |x| \leq 2$. And the Poisson formula shows that the second harmonic function is as small as we want (for instance $\lambda_1/2$) on the same surface, provided that ε_0 is very small. We can consider now (using the linearity of the equation):

$$\theta_k = (\bar{\theta} - C_k)_+,$$

with

$$\bar{\theta} = \frac{2}{\lambda_1}\left(\theta - 1 + \frac{\lambda_1}{2}\right),$$
$$C_k = 1 - 2^{-k}.$$

We have already $\theta_0^* = 0$ for $z = 2, t > -2, |x| < 2$.

We need now to propagate this property for all k, by induction. Since $\theta_k^* = 0$ for $z = \delta^k$, by comparison principle, for $z < \delta^k$, θ_k^* is smaller that the sum of two harmonic functions. The first one is the contribution of the side of the cylinder: It is equal to 1 for $|x| = 1 + 2^{-k}$ and 0 on the two surfaces $z = 0$ and $z = \delta^{-k}$. The second one is the contribution of the trace at $z = 0$ and can be computed using the Poisson kernel. The first remark is that, the contribution of the side is exponentially decreasing in x with a rate proportional to δ^k. Hence its value in B_{k+1} is smaller that $Ce^{-(2/\delta)^k}$. This is very tiny if δ is reasonably small. For δ fixed, the value of the contribution of the trace $z = 0$ can be bounded using the Poisson kernel by the energy at $z = 0$ multiplying by $1/(\delta^k)^N$. Assume now that up to this k, we were able to perform the De Giorgi non linearization. Then, the energy at the level k is smaller that M^{-k} if the initial energy for $k = 0$ is smaller than M^{-1} (which is very small if ε_0 is very small). So, if we choose this M^{-1} very small compared to δ, we can get the contribution of the side and of the trace $z = 0$ smaller than, let say, 2^{-k}. Since $\theta_{k+1} = (\theta_k - 2^{-k})_+$, this gives the property that $\theta_{k+1} = 0$ for $z = \delta^{k+1}, |x| \leq 1 + 2^{-(k+1)}$.

In turn, this property shows that the values of θ_{k+1} for $z > 0$ depends only on the side (which is very tiny as we have seen) and on the trace which is controlled through the energy inequality. We can, then, perform the De Giorgi non linearization at the level $k + 1$.

An important thing to check in this procedure, is that there is no cost in the energy inequality while taking δ very small. This is indeed the case: since $\theta_k^* = 0$ for $z = \delta^k$, we can use the energy inequality with the cut-off function in x only. Then $\nabla\eta$ does not depend on δ.

3 An application to the Navier-stokes equation

The De Giorgi method does not work in general for systems. However, it has been noticed before that Stampacchia truncations methods can provide some interested bounds on the solutions. This method has been successfully applied by Alikakos for system of reaction-diffusion in [1] and Beirao Da Veiga for the Navier-Stokes system of equations in [2].

Indeed, the first part of the De Giorgi method can be used to obtained L^∞ bounds on systems for which a natural scalar quantity is at play. Note that in a lot of cases (like the Navier-Stokes equation) L^∞ bounds can lead to full regularity using bootstrapping arguments. For the Navier-Stokes equation the natural scalar quantity is the kinetic energy $|u|^2$.

We consider the incompressible Navier Stokes equation in dimension 3, namely:

$$\begin{aligned} &\partial_t u + \operatorname{div}(u \otimes u) + \nabla P - \Delta u = 0 \qquad t \in]0,\infty[,\ x \in \mathbb{R}^3, \\ &\operatorname{div} u = 0, \end{aligned} \tag{16}$$

with initial value $u^0 \in L^2(\mathbb{R}^3)$. We consider only suitable solutions, that is, which verify

$$\partial_t \frac{|u|^2}{2} + \operatorname{div}(u\frac{|u|^2}{2}) + \operatorname{div}(uP) + |\nabla u|^2 - \Delta \frac{|u|^2}{2} \leq 0 \qquad t \in]0,\infty[,\ x \in \mathbb{R}^3. \tag{17}$$

In this section, we will show how the De Giorgi techniques can be applied to give an alternative proof of the partial regularity result first shown by Caffarelli, Kohn and Nirenberg [3]. More precisely we will show the first part of the proof. In our case it corresponds to the following theorem.

Theorem 3.1. *For every $p > 1$, there exists a universal constant C^*, such that any solution u of (16) (17) in $[-1,1] \times B(1)$ verifying:*

$$\sup_{t\in[-1,1]} \left(\int_{B(1)} |u|^2\, dx \right) + \int_{-1}^1 \int_{B(1)} |\nabla u|^2\, dx\, dt + \left[\int_{-1}^1 \left(\int_{B(1)} |P|\, dx \right)^p dt \right]^{\frac{2}{p}} \leq C^*, \tag{18}$$

is bounded by 1 on $[-1/2, 1] \times B(1/2)$.

Let us first show that this result leads to the partial regularity result of Scheffer, that is the Hausdorff dimension of the set of points (t, x) where u is not smooth is smaller than $5/3$. The result in [3] (see also [11]) pushes to 1 with some extra work.

From Sobolev and the energy inequality we have $u \in L^\infty(L^2) \cap L^2(L^6)$. By interpolation, this gives $u \in L^{10/3}_{t,x}$. The pressure is given by

$$-\Delta P = \sum_{i,j} \partial_i \partial_j (u_i u_j).$$

By the boundedness of the Riesz transform in L^p, this gives that P is bounded in $L^{5/3}_{t,x}$. For any fixed (t_0, x_0) we consider the rescaled function

$$u_\varepsilon(t,x) = \varepsilon u(t_0 + \varepsilon^2 t, x_0 + \varepsilon x), \qquad P_\varepsilon(t,x) = \varepsilon^2 P(t_0 + \varepsilon^2 t, x_0 + \varepsilon x),$$

which is still solution to (16), (17). Applying Theorem 3.1, we see that if u is not bounded in a neighborhood of radius ε of (t_0, x_0) then

$$\int_{(-1,1)\times B_1} (|u_\varepsilon|^{10/3} + |P_\varepsilon|^{5/3}) dx\, dt = \frac{1}{\varepsilon^{5/3}} \int_{t_0 + (-\varepsilon^2, \varepsilon^2)\times(x_0 + B_\varepsilon)} (|u|^{10/3} + |P|^{5/3}) dx\, dt \geq C^*.$$

Using Tchebychev, this gives that

$$\begin{aligned} &\#\{B_\varepsilon \text{ covering the bad points}\} \\ \leq\ & \frac{1}{C^*} \int \frac{1}{\varepsilon^{5/3}} \int_{t_0 + (-\varepsilon^2, \varepsilon^2)\times(x_0 + B_\varepsilon)} (|u|^{10/3} + |P|^{5/3}) dx\, dt\, dx_0\, dt_0 \\ \leq\ & \varepsilon^{5-5/3} \int (|u|^{10/3} + |P|^{5/3}) dx\, dt. \end{aligned}$$

3.1 the level set energy functions

To apply the De Giorgi method, we introduce the truncations:

$$\begin{aligned} &B_k = B(1/2(1 + 2^{-3k})) \qquad T_k = 1/2(-1 - 2^{-k}), \\ &Q_k = [T_k, 1] \times B_k, \\ &B_{k-1/3} = B_{1/2(1 + 2 * 2^{-3k})}. \end{aligned}$$

To deal with the non locality of the pressure we will also introduce:

$$B_{k-2/3} = B_{1/2(1 + 4 * 2^{-3k})}.$$

Then we introduce a new function:

$$v_k = [|u| - (1 - 2^{-k})]_+.$$

Notice that v_k^2 can be seen as a level set of energy since $v_k^2 = 0$ for $|u| < 1 - 2^{-k}$ and is of the order of $|u|^2$ for $|u| \gg 1 - 2^{-k}$.

Let us define:

$$U_k = \sup_{t \in [T_k, 1]} \left(\int_{B_k} |v_k(t,x)|^2\, dx \right) + \int_{Q_k} |d_k(t,x)|^2\, dx\, dt,$$

where:

$$d_k^2 = \frac{(1 - 2^{-k}) \mathbf{1}_{\{|u| \geq (1 - 2^{-k})\}}}{|u|} |\nabla |u||^2 + \frac{v_k}{|u|} |\nabla u|^2.$$

Notice that:

$$U_0 = \sup_{t \in [-1,1]} \left(\int_{B(1)} |u(t,x)|^2\, dx \right) + \int_{-1}^{1} \int_{B(1)} |\nabla u(t,x)|^2\, dx\, dt.$$

We want to study the limit when k goes to infinity of U_k. More precisely, we want to obtain the following Proposition.

Proposition 4. *let $p > 1$. There exists universal constants C_p, $\beta_p > 1$ depending only on p such that for any solution to* (16), (17) *in* $[-1,1] \times B(1)$, *if* $U_0 \leq 1$ *then we have for every $k > 0$:*

$$U_k \leq C_p^k (1 + \|P\|_{L^p(0,1;L^1(B_0))}) U_{k-1}^{\beta_p}. \tag{19}$$

Theorem 3.1 follows from this theorem exactly as for the elliptic case. Indeed, if $\|P\|_{L^p(0,1;L^1(B_0))})$ is bounded by 1 and U_0 is small enough, then U_k converges to 0 when k goes ∞. At the limit this gives that $|u| \leq 1$ on $Q_{1/2}$.

Multiplying (16) by $uv_k/|u|$, we find

$$\begin{aligned} \partial_t \frac{v_k^2}{2} + \mathrm{div}\left(u \frac{v_k^2}{2}\right) + d_k^2 - \Delta \frac{v_k^2}{2} \\ + \mathrm{div}(uP) + (v_k/|u| - 1) u \cdot \nabla_x P \leq 0. \end{aligned} \tag{20}$$

3.2 Bound on U_k

Let us introduce functions $\eta_k \in C^\infty(\mathbb{R}^3)$ verifying:

$$\begin{aligned} &\eta_k(x) = 1 \qquad \text{in } B_k \\ &\eta_k(x) = 0 \qquad \text{in } B_{k-1/3}^C \\ &0 \leq \eta_k(x) \leq 1 \\ &|\nabla \eta_k| \leq C 2^{3k} \\ &|\nabla^2 \eta_k| \leq C 2^{6k}. \end{aligned}$$

We multiply (20) by $\eta_k(x)$ and integrate on $[\sigma, t] \times \mathbb{R}^3$ for $T_{k-1} \leq \sigma \leq T_k \leq t \leq 1$ to find:

$$\begin{aligned} \int \eta_k(x) \frac{|v_k(t,x)|^2}{2}\, dx + \int_\sigma^t \int \eta_k(x) d_k^2(s,x)\, dx\, ds \\ \leq \int \eta_k(x) \frac{|v_k(\sigma,x)|^2}{2}\, dx \\ + \int_\sigma^t \int \nabla \eta_k(x) u \frac{|v_k(s,x)|^2}{2}\, dx\, ds \\ + \int_\sigma^t \int \Delta \eta_k(x) \frac{|v_k(s,x)|^2}{2}\, dx\, ds \\ - \int_\sigma^t \int \eta_k(x) \left\{ \mathrm{div}(uP) + \left(\frac{v_k}{|u|} - 1\right) u \nabla P \right\}\, dx\, dt. \end{aligned}$$

Integrating in σ between T_{k-1} and T_k and divided by $T_{k-1} - T_k = 2^{-(k+1)}$, we find:

$$\begin{aligned}
&\sup_{t\in[T_k,1]}\left(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,dx+\int_{T_k}^t\int \eta_k(x)d_k^2(s,x)\,dx\,ds\right)\\
&\quad\le 2^{k+1}\int_{T_{k-1}}^{T_k}\int \eta_k(x)\frac{|v_k(\sigma,x)|^2}{2}\,dx\,d\sigma\\
&\qquad+\int_{T_{k-1}}^1\left|\int \nabla\eta_k(x)u\frac{|v_k(s,x)|^2}{2}\,dx\right|\,ds\\
&\qquad+\int_{T_{k-1}}^1\left|\int \Delta\eta_k(x)\frac{|v_k(s,x)|^2}{2}\,dx\right|\,ds\\
&\qquad+\int_{T_{k-1}}^1\left|\int \eta_k(x)\left\{\operatorname{div}(uP)+\left(\frac{v_k}{|u|}-1\right)u\nabla P\right\}\,dx\right|\,dt.
\end{aligned}$$

Since $\eta_k\equiv 1$ on B_k,

$$\begin{aligned}
U_k &\le \sup_{t\in[T_k,1]}\left(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,dx\right)+\int_{T_k}^1\int \eta_k(x)d_k^2(s,x)\,dx\,ds\\
&\le 2\sup_{t\in[T_k,1]}\left(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,dx+\int_{T_k}^t\int \eta_k(x)d_k^2(s,x)\,dx\,ds\right).
\end{aligned}$$

We claim that:

$$\begin{aligned}
U_k \le{}& C2^{6k}\int_{Q_{k-1}}|v_k(s,x)|^2\,dx\,ds\\
&+C2^{3k}\int_{Q_{k-1}}|v_k(s,x)|^3\,dx\,ds\\
&+2\int_{T_{k-1}}^1\left|\int \eta_k(x)\left\{\operatorname{div}(uP)+\left(\frac{v_k}{|u|}-1\right)u\nabla P\right\}\,dx\right|\,dt.
\end{aligned}\tag{21}$$

We use the bound on $\nabla\eta_k$ and $\Delta\eta_k$, the fact that η_k is supported in Q_{k-1}, and the decomposition:

$$u\frac{v_k^2}{2}=\left\{u\left(1-\frac{v_k}{|u|}\right)+\frac{uv_k}{|u|}\right\}\frac{v_k^2}{2}.$$

Noticing that $|u(1-v_k/|u|)|\le 1$:

$$\begin{aligned}
\left|u\left(1-\frac{v_k}{|u|}\right)\frac{v_k^2}{2}\right| &\le \frac{v_k^2}{2}\\
\left|\frac{u}{|u|}v_k\frac{v_k^2}{2}\right| &\le \frac{v_k^3}{2}.
\end{aligned}$$

The aim is now to non linearize the right hand side terms.

3.3 Raise of the power exponents

We want to bound the right-hand side term of (21) with nonlinear power of U_{k-1} bigger than 1. Let us first treat the first two terms. of (21):

$$\begin{aligned} C2^{6k}\int_{Q_{k-1}} |v_k(s,x)|^2\,dx\,ds + C2^{3k}\int_{Q_{k-1}} |v_k(s,x)|^3\,dx\,ds \\ \leq C2^{6k}\|v_k^2\|_{L^{5/3}(Q_{k-1})}\|1_{\{v_k>0\}}\|_{L^{5/2}(Q_{k-1})} \\ +C2^{3k}\|v_k^3\|_{L^{10/9}(Q_{k-1})}\|1_{\{v_k>0\}}\|_{L^{10}(Q_{k-1})} \end{aligned}$$

From the definition of v_k we have that $v_k \leq v_{k-1}$, and so:

$$\|v_k^2\|_{L^{5/3}(Q_{k-1})} = \|v_k\|^2_{L^{10/3}(Q_{k-1})} \leq \|v_{k-1}\|^2_{L^{10/3}(Q_{k-1})}.$$

This quantity is bounded by CU_{k-1} by Sobolev and interpolation. In the same way we have:

$$\||v_k|^3\|_{L^{10/9}(Q_{k-1})} = \|v_k\|^3_{L^{10/3}(Q_{k-1})} \leq U_{k-1}^{3/2}.$$

Therefore, thanks to a Tchebychev argument:

$$\begin{aligned} C2^{6k}\int_{Q_{k-1}} |v_k(s,x)|^2\,dx\,ds + C2^{3k}\int_{Q_{k-1}} |v_k(s,x)|^3\,dx\,ds \\ \leq C2^{6k+4k/3}U_k^{5/3}. \end{aligned} \tag{22}$$

We now need to treat the pressure terms.

3.4 The pressure terms

First we consider the classical decomposition between short range and long range terms. The idea is that the local terms can be treated as the terms depending on v_k while the long range term are smooth in x.

More precisely, we can decompose $P|_{B_{k-2/3}}$ into two parts:

$$P|_{B_{k-2/3}} = P_{k1}|_{B_{k-2/3}} + P_{k2}|_{B_{k-2/3}},$$

where:

$$\begin{aligned} \|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))} + \|P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))} \\ \leq C2^{12k}\left(\|P\|_{L^p(T_{k-1},1;L^1(B_{k-1}))} + \|u\|^2_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\right). \end{aligned}$$

and P_{k2} is solution on $[T_{k-1},1]\times\mathbb{R}^3$ to:

$$-\Delta P_{k2} = \sum_{i,j}\partial_i\partial_j\left[\phi_k u_j u_i\right].$$

The term P_{k2} is treated as above by decomposing in its definition

$$u_i = v_k \frac{u_i}{|u|} + u_i(1 - v_k/|u|).$$

The main point is that the second term is bounded by 1.

For the term P_{k1} we may use its boundedness in x. Indeed we can bound the associated term by:

$$C\|v_k\|_{L^\infty(L^2)} \|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))} \|\mathbf{1}_{\{v_k>0\}}\|_{L^{p'}(T_{k-1},1;L^2(B_{k-1}))}. \quad (23)$$

We can then use Tchebychev as before.

References

[1] (0537465) N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations, **4** (1979), 827–868.

[2] (1608053) H. Beirão da Veiga, *Concerning the regularity of the solutions to the Navier-Stokes equations via the truncation method. I*, Differential Integral Equations, **10** (1997), 1149–1156.

[3] (0673830) L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., **35** (1982), 771–831.

[4] (2354493) L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, **32** (2007), 1245–1260.

[5] L. Caffarelli and A. Vasseur, *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*, Annals of Math., to appear.

[6] (2215601) D. Chae, *On the regularity conditions for the dissipative quasi-geostrophic equations*, SIAM J. Math. Anal., **37** (2006), 1649–1656 (electronic).

[7] (1962043) D. Chae and J. Lee, *Global well-posedness in the super-critical dissipative quasi-geostrophic equations*, Comm. Math. Phys., **233** (2003), 297–311.

[8] (2196360) P. Constantin, *Euler equations, Navier-Stokes equations and turbulence*, in “Mathematical foundation of turbulent viscous flows”, Lecture Notes in Math. **1871**, pp. 1–43, Springer, Berlin, 2006.

[9] (0093649) E. De Giorgi, *Sulla differenziabilità e l’analiticità delle estremaili degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Math. Nat., **3** (1957), 25–43.

[10] (2276260) A. Kiselev, F. Nazarov and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation*,Invent. Math., **167** (2007), 445–453.

[11] (2374209) A. Vasseur, *A new proof of partial regularity of solutions to Navier-Stokes equations*, Nonlinear Differential Equations Appl., **14** (2007), 753–785.