Survey on Polynomial Automorphism Groups

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Group of polynomial automorphisms

Notation

We write $R^{[n]}$ for the polynomial ring $R[X_1, \ldots, X_n]$ over R.

For R a commutative ring the symbol $\mathrm{GA}_n(R)$ denotes the *general automorphism group*, by which we mean the automorphism group of $\mathbb{A}^n_R = \operatorname{Spec} R^{[n]}$ over $\operatorname{Spec} R$. An element of $\mathrm{GA}_n(R)$ is represented by a vector $\varphi = (F_1, \ldots, F_n) \in (R^{[n]})^n$.

When $R = k$ a field, $\mathrm{GA}_n(k)$ is also called the *affine Cremona group.*

Subgroups of GA_n

Subgroups that play a role: GL_n, Af_n, EA_n, TA_n, J_n, $\mathrm{H}_{i,n}$

EA_n is the subgroup generated by the elementary automorphisms. An *elementary* automorphism is one of the form

$$e_i(f) = (X_1, \ldots, X_{i-1}, X_i + f, X_{i+1}, \ldots, X_n)$$

for some $i \in \{1, \ldots, n\}$, $f \in R[X, \hat{i}]$. (fact: $\mathrm{E}_n = \mathrm{EA}_n \cap \mathrm{GL}_n$.)

$\mathrm{TA}_n = \langle \mathrm{Af}_n, \mathrm{EA}_n \rangle$ is the *tame* subgroup.

J_n is the triangular, or Jonquière, group consisting of $(F_1, \ldots, F_n)$ where $F_i \in R[X_1, \ldots, X_i]$.

We also have the subgroups $\mathrm{H}_{1,n}, \mathrm{H}_{2,n}, \ldots, \mathrm{H}_{n,n}$, where $\mathrm{H}_{i,n}$ is the stabilizer of the $R \oplus RX_1 \oplus \cdots \oplus RX_i$ in $R[X_1, \ldots, X_n]$. We have

$$\mathrm{H}_{i,n} = \mathrm{Af}_i(R) \ltimes \mathrm{GA}_{n-i}(R[X_1, \ldots, X_i])$$

Note $\mathrm{H}_{n,n} = \mathrm{Af}_n(R)$.

Structure of $\mathrm{GA}_2(k)$

We begin with the following classical theorem in polynomial automorphisms.

Theorem (Jung, van der Kulk)

The group of polynomial automorphisms of $\mathbb{A}^2_k$, k a field, is generated by the linear and the elementary automorphisms. More strongly,

$$GA_2(k) = Af_2(k) *_{Bf_2(k)} J_2(k)\,.$$

The generation statement was proved by Jung in 1942 for k of characteristic 0, and generalized to arbitrary characteristic by van der Kulk in 1953, who also (essentially) proved the structure statement.

Structure of $\mathrm{GA}_2(k)$, continued

Theorem (Jung, van der Kulk)

The group of polynomial automorphisms of $\mathbb{A}^2_k$, k a field, is generated by the linear and the elementary automorphisms. More strongly,

$$GA_2(k) = Af_2(k) *_{Bf_2(k)} J_2(k)\,.$$

The Jung-van der Kulk Theorem gives $\mathrm{TA}_2(k) = \mathrm{GA}_2(k)$.

Note that $\mathrm{J}_2(k)$ coincides with $\mathrm{H}_{1,2}$, the stabilizer of $k \oplus kX \subset k[X, Y]$. Also $\mathrm{Af}_2(k)$ is $\mathrm{H}_{2,2}$, the stabilizer of $k \oplus kX \oplus kY$. So we have

$$\mathrm{GA}_2(k) = \mathrm{TA}_2(k) = H_{2,2} *_{H_{2,2} \cap H_{1,2}} H_{1,2}$$

Why study rings other than fields?

In trying to understand automorphisms over a field we are quickly led to considering other rings, especially polynomial rings over fields. Examples of mysterious polynomials and potential counterexamples to cancellation can also be generated by considering such.

Note that

$$\mathrm{GA}_2(k[T]) \subset \mathrm{GA}_3(k)\,.$$

More generally,

$$\mathrm{GA}_n(R^{[m]}) \subset \mathrm{GA}_{n+m}(R)\,.$$

We call this inclusion "restriction of scalars".

Non-tameness

The Jung-van der Kulk Theorem is false for R a domain, not a field. A standard example of a non-tame automorphism is

$$\Big(X + a(aY - X^2),\, Y + 2X(aY - X^2) + a(aY - X^2)^2\Big)$$

where a is any non-zero non-unit in a domain R. This automorphism can be realized as $\exp(aY - X^2)D$, where

$$D(X) = a\,, \quad D(Y) = 2X \qquad \text{(i.e., } D = a\partial_X + 2X\partial_Y) \;.$$

It has the following tame factorization over $R[1/a]$:

$$\left(X,\, Y + \frac{1}{a}X^2\right) \circ \left(X + a^2 Y,\, Y\right) \circ \left(X,\, Y - \frac{1}{a}X^2\right) \;.$$

For $R = k[T]$, $a = T$, this is the example given by Nagata in 1972.

Nagata's example

Nagata's example can be viewed as an element of $\mathrm{GA}_3(k)$ by restriction of scalars:

$$\left(T, X + T(TY - X^2), Y + 2X(TY - X^2) + T(TY - X^2)^2\right)$$

It is not tame as over $k[T]$. Nagata (1972) conjectured it is not tame over k, i.e., does not lie in $\mathrm{TA}_3(k)$, a conjecture that remained open for 30 years.

A remarkable breakthrough came in 2002:

Theorem (Shestakov, Umirbaev)

For $\operatorname{char} k = 0$, the Nagata automorphism is <u>not</u> tame.

However it had long been known (Smith, Wright) that this automorphism is stably tame, with one more variable needed to achieve tameness.

Actions of G_a on $\mathbb{A}_k^n$

In characteristic zero, one can run a parameter through any locally nilpotent derivation D of $k^{[n]}$ by writing $\exp(tD)$, $t \in k$. This defines an action of the additive group G_a on k^n, hence a homomorphism $(k, +) \hookrightarrow \mathrm{GA}_n(k)$. Much effort has been devoted to understanding such subgroups, up to conjugacy.

Theorem (Rentschler, 1968)

For $\operatorname{char} k = 0$, any G_a-action on $\mathbb{A}_k^2$ is conjugate to one of the form $(X, Y + tf(X))$.

The analogue in characteristic $p > 0$, where a G_a-action is defined by a "locally finite iterative higher derivation". Here:

Theorem (Miyanishi, 1971)

For $\operatorname{char} k = p$, any G_a-action on $\mathbb{A}_k^2$ is conjugate to one of the form $\left(X, Y + tf_0(X) + t^p f_1(X) + +t^{p^2} f_2(X) + \cdots + t^{p^r} f_r(X)\right)$.

We get a G_a on $\mathbb{A}_k^3$ by running the parameter t through the Nagata automorphism:

$$\Big(T, X + tT(TY - X^2), Y + 2tX(TY - X^2) + t^2 T(TY - X^2)^2\Big) \\ = \exp(t(TY - X^2)D)$$

where $D(T) = 0$, $D(X) = T$, $D(Y) = 2X$, so $D = T\partial_X + 2X\partial_Y$.

In 1984 Bass observed that this action is non-triangularizable by virtue of the fact that its fixed locus has an isolated singularity, whereas the fixed locus of a triangular action is cylindrical.

In 1987 Popov, using s similar strategy, showed there exist non-triangular actions of G_a on $\mathbb{A}_k^n$ for all $n \geq 3$.

Tame G_a-actions

It is not known how to classify all G_a-actions on $\mathbb{A}_k^3$. We restrict the question.

A G_a-action is called tame if it induces a homormorphism $(k, +) \hookrightarrow \mathrm{TA}_n(k)$.

Question

Are all tame G_a-actions on $\mathbb{A}_k^n$ triangularizable?

The answer is yes for $n = 2$, no for $n \geq 4$.

For $n = 3$ the question is open. Later we will present a recent result that might help resolve this question.

Possible generators for $\mathrm{GA}_n(k)$?

Question

Is $GA_n(k)$ generated by $Af_n(k)$ together with automorphisms of the form $\exp D$, where D is an locally nilpotent derivation on $k^{[n]}$.

The following example in $\mathrm{GA}_3(k)$, a modification of Nagata's example using the technique of "pseudo-conjugation", suggests that the above question may not be true.

$$\left(T, X, Y + \frac{1}{T^2}X^2 + \frac{2}{T}X^3\right) \circ \left(T, X + T^3 Y, Y\right) \circ \left(T, X, Y - \frac{1}{T^2}X^2\right)$$

There is no known factorization of this by linear and exponential automorphisms.

Conjecture

Are all G_m on $\mathbb{A}_k^n$ linearizable, i.e., conjugate to an action of the form

$$(t^{a_1}X_1, \dots, t^{a_n}X_n)$$

with $a_1, \dots, a_n \in \mathbb{Z}$?

- $n = 1$: not difficult
- $n = 2$: Bialynicki-Birula (1967)
- $n = 3$, $k = \mathbb{C}$: Koras, Russell + Kaliman, Makar-Limanov (1997)
- $n \geq 4$, char $k = p > 0$: false, Asanuma (1994)

Linearization of G_m-actions on $\mathbb{A}^3_k$

Theorem (Koras, Russell / Kaliman, Makar-Limanov, 1997)

All G_m-actions on $\mathbb{A}^3_{\mathbb{C}}$ are linearizable.

The proof was a long saga. The "weights" a_1, a_2, a_3 can be determined from the action by looking at the action on the tangent space of a fixed point. The "hard case" was $a_1 < 0$, $a_2, a_3 > 0$. The quest to solve the "hard case" led to trying to prove certain k-algebras A were not isomorphic to $k^{[3]}$, the simplest case being $\mathbb{C}[T, X, Y, Z]/(X + X^2Y + Z^3 + T^2)$.

The linearization theorem was generalized to:

Theorem (Popov, 1998)

Every action of a connected reductive algebraic group on $\mathbb{A}^3_{\mathbb{C}}$ is linearizable.

Makar-Liminov invariant

The tool used to distinguish the Koras-Russell threefolds from $\mathbb{A}^3_{\mathbb{C}}$ was the Makar-Limanov invariant subring

$$\mathrm{ML}(A) = \bigcap_D \mathrm{Ker}(D)$$

where D runs through all locally nilpotent derivations of A. This turned out to be a quite useful tool for many purposes.

Later Derksen defined the invariant subring

$$\mathrm{DK}(A) = \text{subring generated by} \bigcup_D \mathrm{Ker}(D)$$

which was used in Neena Gupta's recent proof that the cancellation property does not hold for $k^{[3]}$ for k of characteristic $p > 0$.

Structure of $\mathrm{TA}_3(k)$

Recall the subgroups $H_{i,n} = \mathrm{stab}\,(k \oplus kX_1 \oplus \cdots \oplus kX_i)$ in $\mathrm{GA}_n(k)$. Note $H_{n,n} = \mathrm{Af}_n$. Also $H_{n-1,n} \subset \mathrm{TA}_n$.

$H_{1,3}$ contains the Nagata automorphism, so is not contained in $\mathrm{TA}_3(k)$. So let

$$\widetilde{H}_{1,3} = H_{1,3} \cap \mathrm{TA}_3(k) = Af_1(k) \ltimes \mathrm{TA}_2(k[X_1])$$

The second equality follows from the very deep results of Shestakov-Umirbaev, which say that in $\mathrm{GA}_3(k)$ we have

$$\mathrm{GA}_2(k[X_1]) \cap \mathrm{TA}_3(k) = \mathrm{TA}_2(k[X_1])\,.$$

The following has been proved using Umirbaev's theorem on generators and relations.

Theorem (Wright)

For k a field of characteristic zero, $TA_3(k)$ is the amalgamated product of the three groups $\widetilde{H}_{1,3}, H_{2,3}, H_{3,3}$ along their pairwise intersections.

Structure of $\mathrm{TA}_3(k)$, continued

Restating:

Theorem (Wright)

For k a field of characteristic zero, $TA_3(k)$ is the amalgamated product of the three groups $\widetilde{H}_{1,3}, H_{2,3}, H_{3,3}$ along their pairwise intersections.

This invites these questions:

Question

Is the associated 2-dimensional simplicial complex 2-connected?

If yes, this might be a tool to address tame G_a-actions on $\mathbb{A}^3_k$.

Question

Is the subgroup $\langle H_{1,3}, H_{2,3}, H_{3,3}\rangle \subset GA_3(k)$ the amalgamated product of $H_{1,3}, H_{2,3}, H_{3,3}$ along their pairwise intersections?

Generation of $GA_3(k)$?

As to whether $GA_3(k) = \langle H_{1,3}, H_{2,3}, H_{3,3} \rangle$, we point to this example:

Example (Freudenburg, 1996)

In $k[X, Y, Z]$, define

$$F = XZ - Y^2, \quad G = ZF^2 + 2X^2YF + X^5, \quad R = X^3 + YF$$

Then $\Delta_{F,G} = |J(F, G, *)|$ is a locally nilpotent derivation with kernel $k[F, G]$ and local slice R (so (F,G,R) are birational variables). Note that $k[F, G]$ contains no variables.

Letting $\gamma = exp\,\Delta_{(F,G)}$, we have $\gamma = (A, B, C)$ with $A, B, C \in k[X, Y, Z]$ having degrees 9, 25, 41, respectively.

The algorithm of Shestakov-Umirbaev shows $\gamma \notin TA_3(k)$. We do not know whether $\gamma \in \langle H_{1,3}, H_{2,3}, H_{3,3} \rangle$, or if it is stably tame.

Stabilization

The study of polynomial automorphisms adopts the following concept from K-theory.

Definition

Stabilization refers to the embedding of $\mathrm{GA}_n(R)$ into $\mathrm{GA}_{n+m}(R)$ (the "stabilization homomorphism"). If $\varphi = (F_1, \ldots, F_n) \in \mathrm{GA}_n(R)$, we write $\varphi^{[m]}$ for its image

$$(F_1, \ldots, F_n, X_{m+1}, \ldots, X_{n+m})$$

in $\mathrm{GA}_{n+m}(R)$. We say, for example, an automorphism φ is *stably tame* if it becomes tame in some higher dimension.

We write $\mathrm{GA}_\infty(R)$ for the direct limit $\lim_{n\to\infty} \mathrm{GA}_n(R)$, and similarly for the other automorphism groups (TA_∞, EA_∞, etc.).

Stable tameness of the Nagata automorphism

Let D be a locally nilpotent derivation on $R^{[n]}$, $a \in \mathrm{Ker} D$.
Extend D to $R^{[n+1]}$ by setting $D(X_{n+1}) = 0$.

Define $\tau \in \mathrm{GA}_{n+1}(R)$ by $\tau = (X_1, \ldots, X_n, X_{n+1} + a)$.

Theorem (Smith's commutator formula, 1989)

$$exp(aD)^{[1]} = \tau^{-1}\, exp(-X_{n+1}D)\, \tau\, exp(X_{n+1}D)$$

Nagata example: On $k[T, X, Y]$, $\eta = \exp\left((TY - X^2)D\right)$ where $D = T\partial_X + 2X\partial_Y$.

Smith's formula shows that $\eta^{[1]}$ is tame. Smith's formula cannot be used to show the altered Nagata automorphism

$$\left(T, X, Y + \frac{1}{T^2}X^2 + \frac{2}{T}X^3\right) \circ \left(T, X + T^3Y, Y\right) \circ \left(T, X, Y - \frac{1}{T^2}X^2\right)$$

is stably tame. We will see, however, that it is.

Recent result (Adv. Math.)

Theorem (Berson, van den Essen, Wright, 2010)

Let R be a regular ring, $\varphi \in GA_2(R)$. Then φ is stably tame.

Stronger result for characteristic zero, R one-dimensional:

Theorem (One-dimensional $\mathbb{Q}$-algebra case)

Let R be a Dedekind $\mathbb{Q}$-algebra, and let $\varphi \in GA_2(R)$. Then, φ becomes tame with the addition of three more dimensions. In other words, $GA_2(R) \subset TA_5(R)$.

Hence all automorphisms in $\mathrm{GA}_3(k)$ that fix one coordinate lie in $\mathrm{TA}_6(k)$.

Question

Are three new dimensions actually needed, say for $R = k[Z]$?

More general versions

The Main Theorem is an immediate consequence of the following, thanks to the Jung-van der Kulk Theorem.

Theorem (Main Theorem, First General Form)

For a fixed integer $n \geq 2$ assume it is true that for all fields k, all elements of $GA_n(k)$ are stably tame. Then the same is true replacing "field" by "regular ring".

Which, in turn, follows from:

Theorem (Main Theorem, Second General Form)

Let R be a regular ring, $\varphi \in GA_n(R)$. Assume $\overline{\varphi}_{\mathcal{P}}$ is stably tame in $GA_n(k(\mathcal{P}))$ for all $\mathcal{P} \in Spec(R)$. Then φ is stably tame.

Characterizing polynomial rings

The latter statement hearkens to:

Theorem (Asanuma, 1987)

Let R be a regular local ring, A a finitely generated, flat R-algebra for which $A \otimes k(\mathcal{P}) \cong_{k(\mathcal{P})} k(\mathcal{P})^{[n]}$ for all $\mathcal{P} \in Spec\,(R)$. Then A is stably a polynomial ring over R, i.e., $A^{[m]} \cong_R R^{[n+m]}$ for some $m \geq 0$.

Theorem (Sathaye, 1983)

Let R be a $\mathbb{Q}$-algebra which is a DVR with maximal ideal πR, A a finitely generated R-algebra for which $A \otimes k(\mathcal{P}) \cong_{k(\mathcal{P})} k(\mathcal{P})^{[2]}$ for $\mathcal{P} = (0), \pi R$. Then $A \cong R^{[2]}$.

Theorem (Bass, Connell, Wright / Suslin, 1977)

Let R be a Noetherian ring, A a finitely generated R-algebra for which $A_{\mathcal{P}} \cong R_{\mathcal{P}}^{[n]}$ for all $\mathcal{P} \in Spec\,(R)$. Then $A \cong S(P)$ for some projective R-module P. (If P is free then $A \cong R^{[n]}$.)

How to recognize a coordinate

We turn to the question of when a single polynomial F is a *coordinate*, i.e., can be completed into an automorphism.

Definition

$F \in R^{[n]}$ is called a hyperplane if $R^{[n]}/(F) \cong R^{[n-1]}$.

Theorem (Abhyankar, Moh / Suzuki, 1975)

For k a field, $\operatorname{char} k = 0$, hyperplanes in $k^{[2]}$ are coordinates.

- Russell and Sataye showed this holds replacing k by $k[T]$.
- False for char $k = p > 0$: $X + X^{sp} + Y^{p^e}$, $p^e \nmid sp$, $sp \nmid p^e$

Conjecture (Abhyankar, Sathaye)

For k a field, $\operatorname{char} k = 0$, hyperplanes in $k^{[n]}$ are coordinates.

Open for $n \geq 3$.

Polynomials with coordinate-like behaviour

The "non-rectifiable line" in $\mathbb{A}^2_k$ given by $X + X^{sp} + Y^{p^e}$, for $\operatorname{char} k = p > 0$, $p^e \nmid sp$, $sp \nmid p^e$, inspired the following:

Example (ala Weisfeiler)

Let R be Noetherian domain, $\operatorname{char} R = p > 0$, $a \in R, \neq 0$, a not a unit. Let $F = aU + X + X^{sp} + Y^{p^e} \in R[U, X, Y]$ and let $A = R[U, X, Y]/(F)$.

Note that A satisfies the hypothesis of Asanuma's theorem, if A is regular, hence is stably a polynomial ring.

Asanuma shows that in fact $A^{[1]} \cong R^{[3]}$ and $A \ncong R^{[2]}$.

Counterexample to cancellation

Applying this with $R = k[T]$, where $\operatorname{char} k = p > 0$, and $a = T^m$, we get

$$A = k[T, U, X, Y]/(T^m U + X + X^{sp} + Y^{p^e}) = k[t, u, x, y]$$

having the property that $A^{[1]} \cong k[T]^{[3]}$, hence $A^{[1]} \cong k^{[4]}$.

A very recent breakthrough is:

Theorem (Gupta, 2012)

$A \ncong k^{[3]}$ for $m \geq 2$.

Thus we have a counterexample to the cancellation in characteristic $p > 0$.

This was accomplished by showing that the Derksen invariant $\mathrm{DK}\, A \subseteq k[t, x, y] \subsetneq A$.

Vénéreau polynomials

Vénéreau (2001): Over $k[T, T^{-1}, U]$, consider the following variation of the Nagata example:

$$\varphi = \left(X, Y + \frac{1}{U}X^2\right) \circ \left(X + \frac{U^2}{T}Y, Y\right) \circ \left(X, Y - \frac{1}{U}X^2\right)$$
$$= \left(X + \frac{U}{T}(UY - X^2),\ \ *\ \right)$$

Now restrict scalars to $k[T, T^{-1}]$ and compose on the left with $\tau = (U + T^{m+1}X, X, Y)$, $m \geq 1$, to get

$$\tau\varphi = \left(U + T^m(TX + U(UY - X^2)), X + \frac{U}{T}(UY - X^2),\ \ *\ \right)$$

The first coordinate, $B_m = U + T^m(TX + U(UY - X^2))$, is the m^{th} *Vénéreau polynomial.*

Vénéreau polynomials, continued

Vénéreau polynomial: $B_m = U + T^m(TX + U(UY - X^2))$

Vénéreau noted:

- B_m is a coordinate over the residue fields of all prime ideals in $k[T]$. (This implies using Asanuma's theorem that it is a stable coordinate.)
- B_m is a hyperplane over $k[T]$. (This follows from Sathaye's theorem and the Bass-Connell-Wright/Suslin theorem.)

Vénéreau showed B_m a coordinate over $k[T]$ for $m \geq 3$ and asked about $m = 1, 2$.

Freudenburg (2009) showed B_1, B_2 are 1-stable coordinates.

Lewis (2011) showed B_2 is a coordinate.

Question

Is $B_1 = U + T(TX + U(UY - X^2))$ a coordinate?

Related question

Let $\varphi = (F_1, \ldots, F_n) \in \mathrm{EA}_n(k[T, T^{-1}, U]$ and let r be the smallest non-negative integer such that $T^r F_1 \in k[T, U, X_1, \ldots, X_n]$. Restrict scalars to $k[T, T^{-1}]$ and compose on the left with $\tau = (U + T^{r+m} X_1, X_1, \ldots, X_n)$, $m \geq 1$, to get

$$\tau\varphi = (U + T^{r+m} F_1, F_1, \ldots, F_n)$$

Again we ask:

Question

Is the first coordinate $U + T^{r+m} F_1$ a coordinate?

Lewis showed the answer is yes in many cases. He uses the technique of "pseudo-conjugation".

For $n = 1$ the answer is yes (and all such coordinates are stably tame, by B-vdE-W). For $n = 2$ this question is unsolved.

Problems for the next generation

- Solve the Jacobian Conjecture (Problem #16 on Smale's list).
- Gain a greater understanding of $\mathrm{GA}_3(k)$ (e.g., generators, G_a-actions, characteristic $p > 0$). Determine whether "very wild" automorphisms such as Freudenburg's example lie in $\langle H_{1,3}, H_{2,3}, H_{3,3} \rangle$.
- Develop the structure of $\mathrm{GA}_n(k)$ as an infinite-dimensional algebraic group.
- Solve the Abhyankar-Sathaye Conjecture for $n = 3$: Hyperplanes in $k^{[3]}$ are coordinates (char $k = 0$).
- Determine whether the first Vénéreau plynomial B_1, and the other related polynomials we discussed, are coordinates,
- Understand elements of $\mathrm{GA}_n(k)$ in terms of an appropriate birational factorization of the induced map $\mathbb{P}^n_k \dashrightarrow \mathbb{P}^n_k$.

THANK YOU

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