GEOMETRIES OF CERTAIN $\mathbb{Q}$-FANO THREEFOLDS AND THE RATIONALITY OF SOME MODULI SPACES

FRANCESCO ZUCCONI

1. Introduction

The idea to use even spin curves for studies of 3-folds or higher dimensional varieties goes back to Tjurin [Tj]. Mukai is the first who extended this idea. One of his famous results concerns the geometry of lines on a general smooth prime Fano threefold X of genus twelve. He showed that the Hilbert scheme of lines on X is a smooth curve $\mathcal{H}_1$ of genus three, there exists a theta characteristic θ on $\mathcal{H}_1$ without global sections, and X is recovered from the even spin curve $(\mathcal{H}_1, \theta)$ as a certain variety of power sums; see: [Muk].

In the previous works [TZ1] and [TZ2], we interpreted Mukai's work from the standpoint of quintic normal rational curves on the smooth quintic del Pezzo threefold B, and succeeded in generalizing his results by considering smooth rational curves on B of any degree. In these works, we rather proceeded in the opposite direction to Tjurin and Mukai, namely, we gave applications of geometries of 3-folds to studies of even spin curves. One of the main results is the existence of the Scorza quartics for general even spin curves of arbitrary genus ([TZ2, Theorem 1.4.1]).

We step further in this direction and prove the following result:

Theorem 1.1. *The moduli space of even spin curves of genus four is rational.*

In the course of the proof of Theorem 1.1, the interplay of sextic normal rational curves on B, even spin curves of genus four, and sets of six points on the projective plane modulo PGL_2 action is important. An interesting feature of this interplay is the correspondence of the following:

- A birational selfmap $B \dashrightarrow B$, where the indeterminacy of the map in each direction is a general sextic normal rational curve;
- The interchange of two g^1_3's of a general curve of genus four;
- The association map between two sets of six points on the projective plane modulo PGL_2 action.

Our second result concerns the rationality of the moduli space $\mathcal{F}_{8,2\times\frac{1}{2}(1,1,1)}$ of genus-8 Fanos with two singular points of type $\frac{1}{2}(1,1,1)$. In [Ta06] it was classified the class of primary $\mathbb{Q}$-Fano 3-folds with non-Gorenstein singularities, with only cyclic quotient terminal singularities and with a Du Val $K3$ surface in the anticanonical linear system; see: [Ta06, Theorem 1.5]. These Fanos have at most genus 8 and those of genus 8 have at most two singular points, which are $\frac{1}{2}(1,1,1)$-singularities. For this class of Fanos $\mathcal{F}_{8,2\times\frac{1}{2}(1,1,1)}$ is an analogue of $\mathcal{F}_{12}$ for smooth Fanos.

Theorem $\mathcal{F}_{8,2\times\frac{1}{2}(1,1,1)}$ *is a rational variety.*

$\mathcal{F}_{8,2\times\frac{1}{2}(1,1,1)}$ can be interpretated as a moduli space of curves too.

In fact let $\mathcal{S}_g^+$ be the moduli space of even spin curves of genus g. Let $\mathcal{S}_4^{+g_3^1-\mathrm{Sym}} \subset \mathcal{S}_4^+$ be the loci of points $[(C,\theta)]$ such that C has two distinct trigonal series, denote them by δ and respectively δ', $h^0(C,\mathcal{O}_C(\theta)) = 0$, and θ is a theta-characteristic such that $h^0(C,\mathcal{O}_C(\theta-\delta+\delta) = 1$. We call $\mathcal{S}_4^{+g_3^1-\mathrm{Sym}}$ the moduli space of *g_3^1-symmetric spin curves* since if $[(C,\theta)] \in \mathcal{S}_4^{+g_3^1-\mathrm{Sym}}$ then $h^0(C,\mathcal{O}_C(\theta-\delta+\delta') = 1$ iff $h^0(C,\mathcal{O}_C(\theta-\delta'+\delta) = 1$. We can show:

Theorem *$\mathcal{S}_4^{+g_3^1-\mathrm{Sym}}$ is birational to $\mathcal{F}_{8,2\times\frac{1}{2}(1,1,1)}$. In particular $\mathcal{S}_4^{+g_3^1-\mathrm{Sym}}$ is a rational variety.*

We see the above result as an analogue of the Mukai result that $\mathcal{F}_{12}$ is birational to S_3^+.

References

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D.I.M.I. Università degli studi di Udine Via delle scienze 208-206, 33100 Udine Italia
E-mail address: zucconi@dimi.uniud.it