Lefschetz Properties and Jordan Type
in
Algebra, Geometry and Combinatorics
Centro Internazionale per la Ricerca Matematica (CIRM)
Levico, Italy, 25–29 June 2018

1. Karim Adiprasito: *Lefschetz theorems beyond the projective case*

   I will give a new method to prove the Lefschetz theorem beyond projectivity, and describe as a corollary classes of varieties that satisfy the Hard Lefschetz theorem.

2. Nasrin Altafi Razlighi: *The weak Lefschetz Property and action of the cyclic*

   Consider Artinian ideals in the polynomial ring with $n \geq 3$ variables generated by homogeneous polynomials of degree $d \geq 2$ invariant under an action of the cyclic group $\mathbb{Z}/d\mathbb{Z}$. For $n = 3$ a result by Mezzetti and Miró-Roig classifies the actions where the ideals are monomial Togliatti systems, meaning that they fail the WLP by failing injectivity in degree $d - 1$. We generalize their result by classifying the actions where the ideals fail the WLP by failing surjectivity in degree $d - 1$. In general, for every $n \geq 3$ we give a complete classification of the actions of the cyclic group where the corresponding ideals fail the WLP. This is a joint work with Mats Boij.

3. Luca Chiantini: *On the determinantal rank*

   I will discuss some properties on the representation of forms as a sum of determinants or pfaffians of matrices with fixed degrees.The problem is strictly related with the weak Lefschetz properties of some quotients of polynomial rings.

4. Graham Denham: *The conormal variety of a matroid*

   I will describe work in progress with Federico Ardila and June Huh in which we build some new toric varieties from matroids, with the goal of answering combinatorial questions about the latter. Our construction hinges on showing that the Artinian Gorenstein algebras arising along the way have a priori unexpected Lefschetz properties. This sheds some new light on the tropical characteristic classes of López de Medrano, Rincón, and Shaw, as well as on the hvector of the broken circuit complex of an arbitrary matroid.

5. Roberta Di Gennaro: *Generalized logarithmic bundle and Lefschetz properties*

In (Faenzi-Valles, JLMS, 2014) the authors have deepened the study of the logarithmic bundle associated to the finite set of points Z obtained by duality from a line arrangements in $\mathbb{P}^2$. In particular they proved the isomorphism between the logarithmic bundle T_Z and $p_*q^*(I_Z(1))$ (where p and q are the left and right projections from the incidence variety $\{(x,l) \in \mathbb{P}^2 \times \mathbb{P}^2 | x \in l\}$. So, there, they define and begun to study the generalized logarithmic bundle $T_Z^{(n)} = p_*q^*(I_Z(n))$ for $n \in \mathbb{N}$. In (DiGennaro Ilardi Valles, JLMS, 2014) and (DiGennaro-Ilardi, JPAA, 2018), the relationship between arrangements such that the logarithmic bundle has not balanced splitting on a general line and suitable ideals of power of linear forms failing the Strong Lefschetz Property at range 2 is highlighted. Here, by continuing the study of $T_Z^{(n)}$, we would like to generalize this relation: ideals failing SLP at any range k are related to $T_Z^{(k-1)}$ with not balanced splitting on a general line. Moreover, we are also interested in the relation with the unexpected curves (Cook-Harbourne-Migliore-Nagel, to appear in CM) and in suitable examples.

6. Alex Dimca: *Lefschetz properties of the Jacobian module and splitting types of associated two bundles*

   Let S be the polynomial ring in x, y, z with complex coefficients. For a homogeneous polynomial f in S, one can consider a graded S-module $N(f)$, measuring how far the projective curve $C : f = 0$ is from being free. The graded S-module $N(f)$ has a Lefschetz property, which I have established in a joint paper with D. Popescu. This allows us to define a numerical invariant $\nu(C)$, which plays a key role in the study of the splitting type of the rank two vector bundle of logarithmic vector fields along C.

7. Rodrigo Jose Gondim Neves: *Hessians and the Lefschetz properties*

   We want to highlight the importance of Hessians (classical, higher order and mixed ones) to control WLP, SLP and the Jordan type of standard graded Artinian Gorenstein algebras. We present criteria based in the rank of Hessian to an AG algebra have the WLP and the SLP. We show that the Jordan type of an AG algebra is determined by the rank of mixed Hessians. We also present new classes of forms with vanishing Hessian with unexpected geometry.

8. Brian Harbourne: *Failures of the Lefschetz Property and unexpected hypersurfaces in projective space*

   I will discuss joint work with J. Migliore, U. Nagel and D. Cook regarding unexpected curves and failures of SLP. I also plan to discuss work in progress, joint with J. Migliore, U. Nagel and Z. Teitler, on related results in higher dimensions.

9. Tony Iarrobino: *Problems on Jordan type of multiplication maps*

   We give several problems concerning the Jordan type of multiplication maps for elements ℓ in the maximal ideal of an Artin algebra. Recall that the Jordan type P_ℓ of the multiplication m_ℓ is the partition giving the sizes in the Jordan block decomposition for m_ℓ, which is nilpotent. It in general gives more information than whether ℓ is weak or strong Lefschetz. The problems concern the set of Jordan types possible for a given Artin algebra A, and the loci of elements ℓ for which P_ℓ is fixed, as invariants of A. In particular they concern the behavior of these invariants for constructions such as tensor product, connected sums, flat extensions, and associated graded algebra of A when A is Artinian Gorenstein.

These problems are proposed with Hai Long Dao, Leila Khatami, Pedro Marques, Chris McDaniel, Alexandra Seceleanu, and Yong-Su Shin (see "Jordan type problems for 'Lefschetz Properties' at Levico, June, 2018", 6p, April 15, 2018.)

10. Chris McDaniel: *Some Problems Related to Strong Lefschetz Properties*

    I will present some results and problems concerning Jordan types of multiplication maps, free extensions of graded Artinian Gorenstein algebras, and their connections to invariant theory. This is based on joint work with A. Iarrobino, and P. Macias Marques.

11. Mateusz Michalek: *Monomial Togliatti systems*

    Togliatti systems join in a beautiful way two seemingly unrelated topics:

    1) the weak Lefschetz property (WLP) and

    2) varieties with degenerate osculating spaces.

    The first examples of such systems were presented already in 1929 by Togliatti who studied monomial maps from $\mathbb{P}^2$ to $\mathbb{P}^5$. However, only recently Mezzetti, Miró-Roig and Ottaviani, using apolarity, proved general results relating projections of Veronese embedding with degenerate general osculating spaces and Artinian ideals that fail the WLP. In our talk we will present the above mentioned algebraic and geometric properties, providing further examples. We will define Togliatti systems and show new results, obtained jointly with Miró-Roig, on their classification, answering a conjecture of Ilardi (corrected by Mezzetti, Miró-Roig and Ottaviani).

12. Gleb Nenashev: *On ideals generated by two generic quadratic forms in the exterior algebra*

    We consider two algebras: the first algebra is the exterior algebra modulo an ideal generated by two generic quadratic forms and the second one is an square-free polynomial ring modulo the ideal generated by the squares of two generic linear forms. We conjecture that their Hilbert series are equal to each other and we present a theoretical combinatorial interpretation of their coefficients. We have proved that the conjecture for the second case follows from that for the first case. Furthermore, we show that the conjectured series is an upper bound in the coefficient- wise sense for both cases. (This is joint work with Veronica Crispin Quinonez and Samuel Lundqvist).

13. Yong-Su Shin: *The Strong Lefschetz Property and Representation Theory*

    This is a joint work with Seok-Jin Kang and Young-Rock Kim. It is well-known that an Artinian monomial complete intersection quotient of n-variable polynomial ring over a field of characteristic 0 has the strong Lefschetz property. We find an explicit basis compatible with S_3-module structure of an Artinian monomial complete intersection quotient of codimension 3 based on the Schur-Weyl duality.

14. Larry Smith: *Minimal Prime Ideals in Equivariant Coinvariant Algebras*

    Let $\rho : G \to GL(n, \mathbb{F})$ be a representation of the finite group G over the field $\mathbb{F}$ and $R = \mathbb{F}[V]$ be the algebra of polynomial functions on the vector space $V = \mathbb{F}^n$. The **equivariant coinvariant** algebra of ρ is the enveloping algebra $R \otimes_{R^G} R$ associated to the inclusion $R^G \hookrightarrow R$. In an old paper of J. Watanabe he showed for $\mathbb{F} = \mathbb{C}$ that $R \otimes_{R^G} R$ is Cohen-Macaulay if and only if ρ is a reflection representation and in such a case it is also reduced, meaning it has no nilpotent elements. In joint work with C. McDaniel we have extended this to a large class of reflection representations

over arbitrary fields. What we do not know how to deal with is the case that ρ is a reflection representation, not even in the case $\mathbb{F} = \mathbb{C}$. I will try to explain some concrete cases and present some concrete problems that arise out of them.

15. Justyna Szpond: *Fermat-type arrangements in geometry and algebra*

A Fermat arrangement of lines in $\mathbb{P}^2$ is given by linear factors of the polynomials

$$(x^n - y^n)(y^n - z^n)(z^n - x^n).$$

For the ideals I of singular points of these arrangements (i.e. points where at least two arrangement lines meet) and $n \geq 3$ there is an interesting non-containment relation

$$I^{(3)} \not\subset I^2$$

between the third symbolic power of I and its second ordinary power. Ideals of points with this behavior were not known until 2013.

It is natural to generalize the above setting and study polynomials

$$F_{N,n} = \prod_{0 \leq i < j \leq N} (x_i^n - x_j^n)$$

and arrangements of hyperplanes they define in $\mathbb{P}^N$. We call these arrangements Fermat-type.

In two recent articles with Grzegorz Malara we studied the containment problem for ideals of codimension 2 flats defined by these arrangements.

On the other hand, recently Harbourne, Migliore, Nagel with Cook II and separately with Teitler initiated the study of hypersurfaces with unexpected interpolation behavior. Such hypersurfaces appear in situations related to Fermat-type arrangements.

In my talk I will report on these two stories.