# Secant Loci and Syzygies
## (joint work with Edoardo Sernesi)

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# The goal

Find sufficient conditions in terms of secant loci for the vanishing of syzygies of curves.

# Origins

## Previous results

- Green's conjecture for tetragonal curves - F.-O. Schreyer, C. Voisin
- A result of E. Arbarello and E. Sernesi

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## Setup

$C \overset{|L|}{\hookrightarrow} \mathbb{P}^r$, $L$ special over a field $k = \overline{k}$ of characteristic zero.

The multiplication map:

$$u_L : \bigoplus_q S^q H^0(L) \otimes_k H^0(K_C) \xrightarrow{u_L} \bigoplus_q H^0(L^q \otimes K_C)$$

morphism of graded modules over $S := \mathrm{Sym}\, H^0(L)$.

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**Definition (Arbarello–Sernesi)**

$I_{K,L} := \ker(u_L)$ graded module over $S$ s.t.

$$I_{K,L,2} := \ker\{H^0(L) \otimes H^0(K_C) \to H^0(L \otimes K_C)\}.$$

It is called the semi–canonical ideal.

The definition makes sense for any $L$.

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# Origins

**Theorem (Arbarello–Sernesi, 1978)**

*Assume* $r \geq 4$. *The module* $I_{K,L}$ *is generated in degree two unless* $C$ *lies on a surface of minimal degree in* $\mathbb{P}^r$.

# Origins

**Arbarello–Sernesi's Theorem**

The proof is a fine analysis of the generators of the ideal (Petri).
It relies on the existence of an effective divisor $D = x_1 + \cdots + x_r$ s.t.

(1) $h^0(L(-D)) = 2$,

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## Translation

In terms of projective geometry,

(1) $\langle D\rangle = \mathbb{P}^{r-2}$,

(2) $\langle D\rangle \cap C = \mathrm{supp}(D)$,

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**For $L = K_C$**

The three conditions give a primitive $\mathfrak{g}^1_{g-1}$.

Brill-Noether theory: there exists always a primitive $\mathfrak{g}^1_{g-1}$ except for trigonal curves and plane quintics.

The homogeneous ideal of a non–hyperelliptic canonical curve is generated by quadrics if and only if the curve is neither trigonal nor plane quintic (K. Petri, 1922).

# The goal

Go one step further and analyse the module of syzygies of $I_{K,L}$.

# The result

## Theorem (A.–Sernesi, 2015)

*Assume $r \geq 5$. Suppose that the curve $C$ is non-tetragonal and $I_{K,L}$ is generated in degree two. The module of syzygies of $I_{K,L}$ is generated in degree one if the dimension of the secant locus $V_{r-1}^{r-2}(L)$ equals the expected dimension and in any component of $V_{r-1}^{r-2}(L)$ there exists an effective divisor $D = x_1 + \cdots + x_{r-1}$ s.t.*

(1) $h^0(L(-D)) = 3$,

(2) $L(-D)$ *is base–point–free,*

(3) $h^0(L(-D + x_i)) = 3$ *for any $i$.*

## Remark

If $C$ carries a $\mathfrak{g}_4^1$, say $\mathcal{O}_C(\eta)$, and $\eta$ imposes independent conditions on $|L|$ then the module of syzygies of $I_{K,L}$ cannot be generated in degree one. (Green–Lazarsfeld, 1984.)

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# Secant Loci

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$\Xi_n \subset C \times C_n$ the universal divisor on the $n$–th symmetric product $C_n$ of $C$, $\pi : C \times C_n \to C$, $\pi_n : C \times C_n \to C_n$ the projections.

The secant bundle of $L$ is the rank–$n$ vector bundle on $C_n$ defined by:

$$E_{L,n} := \pi_{n*}(\pi^* L \otimes \mathcal{O}_{\Xi_n}).$$

For any $\xi \in C_n$, the fibre of $E_{L,n}$ over $\xi$ is isomorphic to $L|_\xi$.

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$\pi_{n*}\pi^*L \cong H^0(L) \otimes \mathcal{O}_{C_n}$ and hence we have a sheaf morphism

$$e_{L,n} : H^0(L) \otimes \mathcal{O}_{C_n} \to E_{L,n}.$$

$e_{L,n}$ is generically surjective for $n \leq r$.

For any $k \leq n-1$, the secant locus $V_n^k(L)$ is the closed subscheme

$$V_n^k(L) := D_k(e_{L,n}) \subset C_n.$$

$V_n^k(L) \setminus V_n^{k-1}(L)$ parametrizes the $n$–secant $(k-1)$–planes in the induced embedding.

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The expected dimension of $V_n^k(L)$ is

$$\text{expdim } V_n^k(L) = n - (r+1-k)(n-k)$$

If non–empty, then $V_n^k(L)$ has dimension $\geq n - (r+1-k)(n-k)$.

For $k = n-1$:

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In terms of the geometry of secant loci,

(1) $D \in V^{r-2}_{r-1}(L) \setminus V^{r-3}_{r-1}(L)$,

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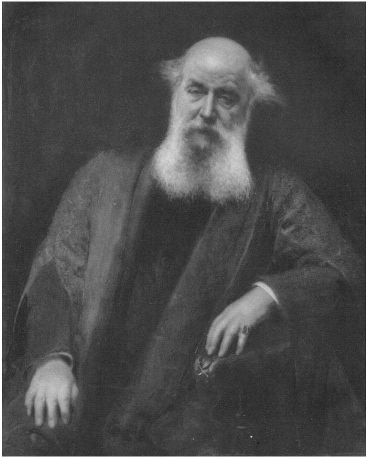

James Joseph Sylvester (1814 – 1897)

# Syzygies

It will be recollected that we have assigned as the condition of contact in three consecutive points, that a certain cubic equation shall have all its roots real. Now, as well remarked by Mr. Cayley, we cannot express this fact by less than three equations in integral terms of the coefficients. Thus if the cubic be written

$$a\lambda^3 + 3b\lambda^2 + 3c\lambda + d = 0,$$

we have as one of such ternary systems,

$$U = ac - b^2 = 0, \quad V = bd - c^2 = 0, \quad W = bc - ad = 0.$$

The significant parts of these equations are of course, however capable of being connected by integral multipliers $U'$, $V'$, $W'$, such that

$$U'U + V'V + W'W = 0.$$

Any number of functions $U, V, W$ so related, I call *syzygetic functions*, and $U'$, $V'$, $V''$ I term the *syzygetic multipliers*.* These in the case supposed are $c$, $a$, $b$, respectively.

In like manner it is evident that the members of any group of functions, more than two in number, whose nullity is implied in the relation of double contact, whether such group form a complete system or not, must be in syzygy.

# Syzygies

$P_1, \ldots, P_m$ homogeneous polynomials in $z_0, \ldots, z_r$ over $k$

A syzygy between $P_1, \ldots, P_m$ is a relation

$$Q_1 P_1 + \cdots + Q_m P_m = 0$$

with $Q_1, \ldots, Q_m \in k[z_0, \ldots, z_r]$ homogeneous.

**Example**

$P_2 P_1 - P_1 P_2 = 0$ is a syzygy between $P_1$ and $P_2$.

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# Syzygies

David Hilbert (1862 – 1943)

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Ueber die Theorie der algebraischen Formen*).

Von

David Hilbert in Königsberg.

Inhalt:



I.

**Die Endlichkeit der Formen in einem beliebigen Formensysteme.**

Unter einer algebraischen Form verstehen wir in üblicher Weise eine ganze rationale homogene Function von gewissen Veränderlichen und die Coefficienten der Form denken wir uns als Zahlen eines bestimmten Rationalitätsbereiches. Ist dann durch irgend ein Gesetz ein System von unbegrenzt vielen Formen von beliebigen Ordnungen in den Veränderlichen vorgelegt, so entsteht die Frage, ob es stets möglich ist, aus diesem Formensysteme eine endliche Zahl von Formen derart auszuwählen, dass jede andere Form des Systems durch lineare Combination jener ausgewählten Formen erhalten werden kann, d. h. ob eine jede Form des Systems sich in die Gestalt

$$F = A_1F_1 + A_2F_2 + \cdots + A_mF_m$$

bringen lässt, wo $F_1, F_2, \ldots, F_m$ bestimmt ausgewählte Formen des gegebenen Systems und $A_1, A_2, \ldots, A_m$ irgendwelche, dem nämlichen Rationalitätsbereiche angehörige Formen der Veränderlichen sind. Um diese Frage zu entscheiden, beweisen wir zunächst das folgende für unsere weiteren Untersuchungen grundlegende Theorem:

*) Vgl. die vorläufigen Mittheilungen des Verfassers: „Zur Theorie der algebraischen Gebilde", Nachrichten v. d. kgl. Ges. d. Wiss. zu Göttingen, 1888 (erste Note) und 1889 (zweite und dritte Note).





$$X_t^{(1)} = A_1^{(2)} F_{t1}^{(2)} + A_2^{(2)} F_{t2}^{(2)} + \cdots + A_{m^{(3)}}^{(2)} F_{tm^{(3)}}^{(2)} \qquad (t = 1, 2, \ldots, m^{(2)})$$

annimmt, wo $A_1^{(2)}, A_2^{(2)}, \ldots, A_{m^{(3)}}^{(2)}$ irgend welche Formen sind. Der letztere Ansatz führt auf das Gleichungssystem

$$(12)\quad F_{t1}^{(2)} X_1^{(2)} + F_{t2}^{(2)} X_2^{(2)} + \cdots + F_{tm^{(3)}}^{(2)} X_{m^{(3)}}^{(2)} = 0, \qquad (t = 1, 2, \ldots, m^{(2)})$$

wo $F_{t1}^{(2)}, F_{t2}^{(2)}, \ldots, F_{tm^{(3)}}^{(2)}$ die gegebenen Coefficienten und $X_1^{(2)}, X_2^{(2)}, \ldots, X_{m^{(3)}}^{(2)}$ die zu bestimmenden Formen sind. Dieses dritte Gleichungssystem (12) ist aus dem zweiten Gleichungssysteme (11) in der nämlichen Weise abgeleitet, wie das zweite Gleichungssystem aus der ursprünglichen Gleichung (9). Durch Fortsetzung des eben eingeschlagenen Verfahrens erhalten wir eine Kette von abgeleiteten Gleichungssystemen, in welcher stets die Zahl der zu bestimmenden Formen irgend eines Gleichungssystemes übereinstimmt mit der Zahl der Gleichungen des darauf folgenden Gleichungssystems.

Zur einheitlicheren Darstellung der weiteren Untersuchungen ist es nöthig, an Stelle der einen ursprünglichen Gleichung (9) ein beliebiges Gleichungssystem von der Gestalt

$$(13)\quad F_{t1}X_1 + F_{t2}X_2 + \cdots + F_{tm^{(1)}}X_{m^{(1)}} = 0 \qquad (t = 1, 2, \ldots, m)$$

zu setzen. Die Anwendung des oben angegebenen Verfahrens gestaltet sich dann zu einer allgemeinen Theorie solcher Gleichungssysteme, deren Kern in dem folgenden Satze liegt:

Theorem III. *Ist ein Gleichungssystem von der Gestalt* (13) *vorgelegt, so führt die Aufstellung der Relationen zwischen den Lösungen desselben zu einem zweiten Gleichungssysteme von der nämlichen Gestalt; aus diesem zweiten abgeleiteten Gleichungssysteme entspringt in gleicher Weise ein drittes abgeleitetes Gleichungssystem. Das so begonnene Verfahren erreicht bei weiterer Fortsetzung stets ein Ende und zwar ist spätestens das $n^{te}$ Gleichungssystem jener Kette ein solches, welches keine Lösung mehr besitzt.*

Der Beweis dieses Theorems ist nicht mühelos; er ergiebt sich aus den folgenden Schlüssen.

Unter den Gleichungen des vorgelegten Systems könnten einige eine Folge der übrigen sein, indem sie von jedem Formensysteme befriedigt werden, welches diesen letzteren Gleichungen genügt. Nehmen wir an, dass solche Gleichungen bereits ausgeschaltet sind, so ist, wenn überhaupt Lösungen vorhanden sein sollen, nothwendig, die Zahl $m$ der Gleichungen des Systems (13) kleiner als die Zahl $m^{(1)}$ der zu bestimmenden Formen und ausserdem sind die $m$-reihigen Determinanten

# Minimal resolutions

## Setup

$V$ is a $k$–vector space of dimension $r+1$

$z_0, \ldots, z_r$ a basis in $V$

$S := \mathrm{Sym}\ V = k[z_0, \ldots, z_r] = \bigoplus_d S_d$ the symmetric algebra of $S$

$\mathfrak{m} = (z_0, \ldots, z_r) \subset S$ the irrelevant ideal

$M = \bigoplus_j M_j$ a finitely generated graded $S$-module.

# Minimal resolutions

**Theorem (Hilbert, 1890)**

*There exists a free resolution of graded $S$–modules:*

$$0 \leftarrow M \leftarrow F_0 \leftarrow \cdots \leftarrow F_{i-1} \stackrel{d_i}{\leftarrow} F_i \leftarrow \cdots \leftarrow F_{r+1} \leftarrow 0$$

*with $F_i = \oplus_j S(-i-j)^{b_{ij}}$ such that $\mathrm{Im}(d_i) \subset \mathfrak{m} \cdot F_{i-1}$. This is called the minimal resolution of $M$ and is unique up to automorphisms of its factors.*

# Minimal resolutions

## Explanation

Minimality:

- the matrix associated to $d_i$ does not contain any non-zero constant.
- when reduced modulo $\mathfrak{m}$, all the differentials become zero.

# Minimal resolutions

The elements of $F_i$ are the syzygies of $M$, the numbers $b_{ij} = b_{ij}(M)$ are the graded Betti numbers of $M$.

If we organise $b_{ij}$ in a table, we obtain the Betti table of $M$.

| | $i \quad \rightarrow$ |
|---|---|
| | $\cdots$ |
| $j$ | $b_{ij}$ |
| $\downarrow$ | $\cdots$ |

$$0 \leftarrow M \leftarrow \oplus_j S(-j)^{b_{0j}} \leftarrow \cdots \leftarrow \oplus_j S(-i-j)^{b_{ij}} \leftarrow \cdots \leftarrow \oplus_j S(-r-1-j)^{b_{r+1,j}} \leftarrow 0.$$

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## Example (Ulrich bundles)

$X \subset \mathbb{P}^r$ a smooth irreducible $n$–dimensional variety, $E$ an Ulrich bundle on $X$.

Betti table of the section module $H^0_*(E) = \bigoplus H^0(E(i))$:

| | 0 | $\cdots$ | $i$ | $\cdots$ | $r-n$ |
|---|---|---|---|---|---|
| 0 | $\deg(E)$ | $\cdots$ | $\deg(E)\binom{r-n}{i}$ | $\cdots$ | $\deg(E)$ |

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# Minimal resolutions

## Example (Twisted cubic)

Equations: $(2 \times 2)$–minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

Relations:

$$\begin{vmatrix} z_0 & z_1 & z_2 \\ z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} z_1 & z_2 & z_3 \\ z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

Betti table of the coordinate ring:

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | – | – |
| 1 | – | 3 | 2 |

# Minimal resolutions

## Example (Koszul resolution)

$M = S/\mathfrak{m}$ the residual field.

$$0 \leftarrow S/\mathfrak{m} \leftarrow S \leftarrow V \otimes_k S(-1) \leftarrow \ldots \leftarrow \wedge^{r+1} V \otimes_k S(-r-1) \leftarrow 0$$

the map $\wedge^p V \otimes S(-p) \to \wedge^{p-1} V \otimes S(-p+1)$ is given by

$$z_{i_1} \wedge \ldots \wedge z_{i_p} \otimes P \mapsto \sum_{\ell} (-1)^{\ell} z_{i_1} \wedge \ldots \widehat{\ell} \ldots \wedge z_{i_p} \otimes z_{i_\ell} P.$$

# Minimal resolutions

## Remark

$$0 \leftarrow M \leftarrow \oplus_j S(-j)^{b_{0j}} \leftarrow \cdots \leftarrow \oplus_j S(-i-j)^{b_{ij}} \leftarrow \cdots \leftarrow \oplus_j S(-r-1-j)^{b_{r+1,j}} \leftarrow 0.$$

$$b_{ij} = \dim \operatorname{Tor}_i(M, S/\mathfrak{m})_{i+j}.$$

# Minimal resolutions

## Moral

The Betti number $b_{pq}$ coincides with the dimension of the space $K_{p,q}(M)$, called Koszul cohomology space of $M$, and defined as the cohomology at the middle of the induced complex (called the Koszul complex)

$$\wedge^{p+1} V \otimes M_{q-1} \to \wedge^p V \otimes M_q \to \wedge^{p-1} V \otimes M_{q+1}$$

$$z_{i_1} \wedge \ldots \wedge z_{i_p} \otimes x \mapsto \sum_{\ell} (-1)^{\ell} z_{i_1} \wedge \ldots \widehat{\ell} \ldots \wedge z_{i_p} \otimes z_{i_\ell} x.$$

# Minimal resolutions

## Geometric cases

- $X \subset \mathbb{P}^r$ a non-degenerate variety, $M = S_X$.
- $X \subset \mathbb{P}^r$ a non-degenerate variety, $M = I_X$.
- $X \subset \mathbb{P}^r$ a non-degenerate variety, $L = \mathcal{O}_X(1)$, $M = R(X, L) := \oplus_n H^0(X, L^n)$.
- $X$ a projective variety, $L \in \mathrm{Pic}(X)$, $V \subset H^0(L)$, $\mathcal{F}$ a coherent sheaf, $M = R(X, \mathcal{F}, L) := \oplus_n H^0(X, \mathcal{F} \otimes L^n)$.

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# Minimal resolutions

**Notation:**

For $M = R(X, \mathcal{F}, L)$, we use the notation $K_{i,j}(X, \mathcal{F}; L, V)$.
Further notation:
$K_{i,j}(X, \mathcal{F}; L)$ if $V = H^0(X, L)$,
$K_{i,j}(X; L, V)$ if $\mathcal{F} = \mathcal{O}_X$,
$K_{i,j}(X; L)$ if $V = H^0(X, L)$ and $\mathcal{F} = \mathcal{O}_X$.

# Minimal resolutions

## Important notice

$X \subset \mathbb{P}^r$, $V = H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$, and $L = \mathcal{O}_X(1)$.

Then $X$ is projectively normal if and only if $K_{0,j}(X; L) = 0$ for all $j \geq 1$.

If $X$ is projectively normal, then to homogeneous ideal is generated by quadrics if and only if $K_{1,j}(X; L) = 0$ for all $j \geq 2$.

Further, the module of relations between the quadrics is generated by linear forms if and only if $K_{2,j}(X; L) = 0$ for all $j \geq 2$.

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# Minimal resolutions

**Definition (Green, 1984)**

The property $K_{i,j}(X;L) = 0$ for all $i \leq p$ and $j \geq 2$ is called the property $(N_p)$.

*Meaning.* Purity of the minimal resolution up to the $p$th step.

| | 0 | 1 | ... | $p$ | $p+1$ | ... |
|---|---|---|---|---|---|---|
| 0 | 1 | – | ... | – | – | ... |
| 1 | – | $b_{11}$ | ... | $b_{p1}$ | $b_{p+1,1}$ | ... |
| 2 | – | – | ... | – | $b_{p+1,2}$ | ... |
| 3 | – | – | ... | – | $b_{p+1,3}$ | ... |
| $\vdots$ | – | – | ... | – | $\vdots$ | ... |

# Minimal resolutions

**Conjecture (Green, 1984)**

*If a canonical curve* $C$ *fails property* $(N_p)$ *then* $\mathrm{Cliff}(C) \leq p$*.*

The case $p = 2$ was solved by Voisin and Schreyer.

# Minimal resolutions

$X \subset \mathbb{P}^r$ curve, $L = \mathcal{O}_X(1)$, can define the analogue of property $(N_p)$ for the (shift of the) Arbarello-Sernesi module

$$R(X, K_X, L)(-1) = \bigoplus_q H^0(L^{q-1} \otimes K_X).$$

By duality, it amounts to the vanishing of $K_{p,1}(X; L)$ for $p$ large.

**Theorem (Green–Lazarsfeld, 1984, Ein–Lazarsfeld, 2015)**

*If* $\deg(L) \gg 0$ *then* $K_{p,1}(X; L) = 0$ *if and only if* $p \geq h^0(L) - \mathrm{gon}(X)$.

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**Example (Seven points in $\mathbb{P}^3$)**

$X = \{p_1, \ldots, p_7\} \subset \mathbb{P}^3$ set of distinct points in linearly general position.

$I_{X,2}$ generated by three quadrics.

The three quadrics have a linear syzygy if and only if $X$ lies on a twisted cubic. This twisted cubic is one example of a syzygy variety.

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Moreover, the Betti tables in the two cases are the following

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## Definition

$X \subset \mathbb{P}^r = \mathbb{P}V^*$ non–degenerate, $p \geq 2$, $0 \neq \alpha \in K_{p,1}(S_X)$.

$$K_{p,1}(S_X) \cong K_{p-1,2}(I_X) = \ker\{\wedge^{p-1}V \otimes I_{X,2} \to \wedge^{p-2}V \otimes I_{X,3}\}$$

$$\alpha = \sum_{|I|=p-1} z_I \otimes Q_I, \; Q_I \in I_{X,2}.$$

Define the syzygy variety of $\alpha$

$$\mathrm{Syz}(\alpha) = \mathbb{V}\left((Q_I)_{|I|=p-1}\right).$$

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# Syzygy varieties

Classification: large $p$.

**Theorem (M. Green's $K_{p,1}$ Theorem)**

*Assume $X$ is of pure dimension $n$. Then*

- $K_{p,1}(S_X) = 0$ *for all* $p > r - n$,
- $K_{r-n,1}(S_X) \neq 0$ *if and only if* $X$ *is a variety of minimal degree* $(r - n + 1)$ *and in this case* $X = \mathrm{Syz}_{r-n}(X)$,
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$X$ curve embedded by a complete linear system of degree $\geq 2g + 2$, third case: $K_{r-2,1}(S_X) \neq 0$ then $X$ is a normal elliptic curve or $X$ lies on a surface of minimal degree $(r - 1)$ which induces a double cover of $\mathbb{P}^1$.

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- $K_{p,1}(S_X) = 0$ *for all* $p > r - n$,
- $K_{r-n,1}(S_X) \neq 0$ *if and only if* $X$ *is a variety of minimal degree* $(r - n + 1)$ *and in this case* $X = \mathrm{Syz}_{r-n}(X)$,
- *If* $K_{r-n,1}(S_X) = 0$ *and* $K_{r-n-1,1}(S_X) \neq 0$ *then either* $X = \mathrm{Syz}_{r-n-1}(X)$ *and* $\deg(X) = r - n + 2$ *or* $\mathrm{Syz}_{r-n-1}(X)$ *is an* $(n+1)$*–fold of minimal degree* $(r - n)$.

$X$ curve embedded by a complete linear system of degree $\geq 2g + 2$, third case: $K_{r-2,1}(S_X) \neq 0$ then $X$ is a normal elliptic curve or $X$ lies on a surface of minimal degree $(r - 1)$ which induces a double cover of $\mathbb{P}^1$.

# Syzygy varieties

Classification: large $p$.

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**Theorem (S. Ehbauer)**

*Assume $X$ is a curve. If $K_{r-3,1}(S_X) \neq 0$, then $\mathrm{Syz}_{r-3}(X)$ is either*

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# Syzygy varieties

## Example

If $X$ is a generic canonical curve of genus $6$ then it is a quadratic section of a del Pezzo surface in $\mathbb{P}^5$. This del Pezzo surface is a syzygy variety $\mathrm{Syz}_2(K_X)$. The Betti table:

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | – | – | – | – |
| 1 | – | 6 | 5 | – | – |
| 2 | – | – | 5 | 6 | – |
| 3 | – | – | – | – | 1 |

# Syzygy varieties

**Theorem (A.–Bruno–Sernesi, work in progress)**

*If $C$ is a canonical 4–gonal curve of genus $\geq 6$ then $\mathrm{Syz}_2(C) = C$ unless $C$ is either bielliptic or a quadric section of a del Pezzo surface. In the bielliptic case, the second syzygy variety is a cone over the elliptic curve. For quadric sections of del Pezzo's the second syzygy variety is the del Pezzo surface itself.*

Syzygy varieties tend to have small degree.

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# The result

**Theorem**

*Suppose that the curve is non-tetragonal and $I_{K,L}$ is generated in degree two. The module of syzygies of $I_{K,L}$ is generated in degree one if the dimension of the secant locus $V^{r-2}_{r-1}(L)$ equals the expected dimension and in any component of $V^{r-2}_{r-1}(L)$ there exists an effective divisor $D = x_1 + \cdots + x_{r-1}$ s.t.*

(1) $h^0(L(-D)) = 3$,

(2) $L(-D)$ *is base–point–free,*

(3) $h^0(L(-D + x_i)) = 3$ *for any* $i$.

# Proof idea

Arbarello-Sernesi: the map $u_L$ is surjective.

Koszul cohomology: translate into

$$K_{1,j}(I_{K,L}) \cong K_{2,j}(C, K_C \otimes L^{-1}; L) = 0 \text{ for } j \geq 2;$$

this is the analogue of the property $(N_2)$ for the graded module

$$\bigoplus_q H^0(C, L^{q-1} \otimes K_C).$$

Duality for Koszul cohomology

$$K_{r-3,1}(C; L) = 0,$$

equivalently

$$K_{r-4,2}(I_C) = \ker\{\wedge^{r-4} V \otimes I_{C,2} \to \wedge^{r-5} V \otimes I_{C,3}\} = 0$$

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# Proof idea

Use the syzygy variety

$$\mathrm{Syz}_{r-3}(C) := \bigcap_{0 \neq \alpha \in K_{r-4,2}(I_C)} \mathrm{Syz}(\alpha).$$

# Proof idea

**Theorem (Ehbauer)**

*If* $K_{r-3,1}(S_C) \neq 0$*, then* $\mathrm{Syz}_{r-3}(C)$ *is either*

- *a surface of minimal degree* $(r-1)$*, or*
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- *a threefold of minimal degree* $(r-2)$*.*

# Proof idea

*1st Step.* Prove that the hypotheses of the theorem are preserved under generic inner projections; can assume $r = 5$ and hence

$$\text{expdim } V_4^3(L) = 2 \times 4 - 5 - 2 = 1.$$

*2nd Step.* Prove that our hypotheses prevent the curve from lying on a surface of minimal degree 4 in $\mathbb{P}^5$ or a smooth del Pezzo surface in $\mathbb{P}^5$ or a singular surface of degree 5 in $\mathbb{P}^5$ or on a threefold of degree 3 in $\mathbb{P}^5$.

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# Many Happy Returns, Edoardo!