Families of Rational Curves
which determine the structure of the
(projective) Space

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The Tangent Map

Let $X$ be a smooth projective variety and $\mathcal{V} \subset \text{RatCurves}^n(X)$, a closed irreducible component; fix a point $x \in X$ and consider $\mathcal{V}_x$. The tangent map $\Phi_x$ is defined by

$\Phi_x([f]) = [(Tf)_0(\partial/\partial t)]$, where $f$ is smooth at 0.

Notation. By $P$ we denote the "natural projectivisation". With $t$ we denote a local coordinate around $0 \in P^1$. 
Let $X$ be a smooth projective variety and $\mathcal{V} \subset RatCurves^n(X)$, a closed irreducible component; fix a point $x \in X$ and consider $\mathcal{V}_x$.

**Definition**

The rational map $\Phi_x : \mathcal{V}_x \rightarrow P(T_xX)$, defined, at $[f] \in \mathcal{V}_x$ which is smooth at 0, by

$$\Phi_x([f]) = [(Tf)_0(\partial/\partial t)]$$

is called the tangent map (c.f. [Mori79, pp.602-603]). It sends a member of $\mathcal{V}_x$ which is smooth at 0 to its tangent direction.

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**Proposition**

*If* \( f : \mathbb{P}^1 \to C \subset X \) *is an unbending member of* \( \mathcal{V}_x \), *the tangent map can be extended to* \([f]\), *even when* \( C \) *is singular at* \( x \), *because the differential* 

\[ Tf : T(\mathbb{P}^1) \to f^*T(X) \]

*is injective.*

*Moreover* \( \Phi_x \) *is immersive at* \([f]\) *\( \in \mathcal{V}_x \).*
The Tangent Map

**Proposition**

If $f : \mathbb{P}^1 \to C \subset X$ is an unbending member of $\mathcal{V}_x$, the tangent map can be extended to $[f]$, even when $C$ is singular at $x$, because the differential $Tf : T(\mathbb{P}^1) \to f^*T(X)$ is injective. Moreover $\Phi_x$ is immersive at $[f] \in \mathcal{V}_x$.

In particular for an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, the tangent map $\Phi_x$ is generically finite over its image.
**Proof** The proof that \( \Phi_x \) is immersive is taken from Hwang.
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$$T[f](\mathcal{V}_x) = H^0(\oplus \mathcal{O}^p \oplus \mathcal{O}(-1)^{n-1-p}) \subset T[f](\mathcal{V}) = H^0(\oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}).$$
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**Proof** The proof that $\Phi_x$ is immersive is taken from Hwang. Let $V = u^{-1}V$ the Hilbert family corresponding to $V$, $B = \emptyset$ or $x$:  
\[ T_{[f]} V_B = H^0(\mathbb{P}^1, f^* T_X(−B)) = H^0(\mathbb{P}^1, \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}(-B)). \]
Passing to the quotient by $Aut(\mathbb{P}^1)$, i.e. passing to $V$, we delete the part corresponding to $T(\mathbb{P}^1)$:
\[ T_{[f]}(V_x) = H^0(\oplus \mathcal{O}^{p} \oplus \mathcal{O}(-1)^{n-1-p}) \subset T_{[f]}(V) = H^0(\oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}). \]
Take $v \in T_{[f]}(V_x) \subset T_{[f]}(V)$; we can find a deformation $f_t$ of $f_0 := f$ such that $\frac{df}{dt} |_{t=0} = v$. Let $z$ be a local coordinate in $\mathbb{P}^1$ centered at 0.
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**Proof** The proof that $\Phi_x$ is immersive is taken from Hwang. Let $V = u^{-1} V$ the Hilbert family corresponding to $V$, $B = \emptyset$ or $x$:

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Then the differential $d\Phi_x : T[f] (V_x) \to T_{\Phi_x (f)} P(T_x X)$ send $v$ to

$$d\Phi_x (v) = \frac{d}{dt} \bigg|_{t=0} \frac{df_t}{dz} \bigg|_{z=0} = \frac{d}{dz} \bigg|_{z=0} \frac{df_t}{dt} \bigg|_{z=0} = \frac{dv}{dz} \bigg|_{z=0}.$$

To derive $v$ with respect to $z$ we think it in $T[f] (V) = H^0(\oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p})$; a non zero section here has non vanishing differential.
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Take $v \in T_{[f]}(\mathcal{V}_x) \subset T_{[f]}(\mathcal{V})$; we can find a deformation $f_t$ of $f_0 := f$ such that $\frac{df}{dt}|_{t=0} = v$. Let $z$ be a local coordinate in $\mathbb{P}^1$ centered at 0. Then the differential $d\Phi_x : T_{[f]}(\mathcal{V}_x) \rightarrow T_{\Phi_x([f])} P(T_xX)$ send $v$ to

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To derive $v$ with respect to $z$ we think it in $T_{[f]}(\mathcal{V}) = H^0(\oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p})$; a non zero section here has non vanishing differential.
Using the above mentioned result of Kebekus one can prove the following.

**Theorem**

For an unbreakable uniruling \( \mathcal{V} \) and a general point \( x \in X \), the tangent morphism \( \Phi_x : \mathcal{V}_x \rightarrow P(T_xX) \) can be defined by assigning to each member \( C \) of \( \mathcal{V}_x \) its tangent direction. This morphism \( \Phi_x \) is finite over its image.
**Proof.** Let $i_x : U_x \to X$ be the evaluation map; by Kebekus the preimage $i_x^{-1}(x)$ contains a section, which we call $\sigma_\infty \cong \mathcal{V}_x$, and at most a finite number of further points. Let $U_x$ be the inverse image of $\mathcal{V}_x$ in the universal family.
Proof. Let \( i_x : U_x \rightarrow X \) be the evaluation map; by Kebekus the preimage \( i_x^{-1}(x) \) contains a section, which we call \( \sigma_\infty \cong \mathcal{V}_x \), and at most a finite number of further points. Let \( U_x \) be the inverse image of \( \mathcal{V}_x \) in the universal family.

Since all curves are immersed at \( x \), the tangent morphism of \( i_x \) gives a nowhere vanishing morphism of vector bundles,

\[
T i_x : T_{U_x} \mathcal{V}_x |_{\sigma_\infty} \rightarrow i_x^* (T_{X|_x}).
\]

The tangent map \( \Phi_x \) is given by the projectivization of this map.
**Proof.** Let $i_x : U_x \to X$ be the evaluation map; by Kebekus the preimage $i_x^{-1}(x)$ contains a section, which we call $\sigma_\infty \cong \mathcal{V}_x$, and at most a finite number of further points. Let $U_x$ be the inverse image of $\mathcal{V}_x$ in the universal family. Since all curves are immersed at $x$, the tangent morphism of $i_x$ gives a nowhere vanishing morphism of vector bundles,

$$T i_x : T_{U_x}|_{\mathcal{V}_x|_{\sigma_\infty}} \to i_x^*(T_X|_x).$$

The tangent map $\Phi_x$ is given by the projectivization of this map. Assume, by contradiction, that $\Phi_x$ is not finite: by the above morphism, we can find a curve $C \subset \mathcal{V}_x$ such that $N_{\sigma_\infty, U_x}$ is trivial along $C$. But $\sigma_\infty$ is contracted and the normal bundle must be negative.
The next result was proved in general by Hwang and Mok.

**Theorem**

*For an unbreakable uniruling $\mathcal{V}$ and a general point $x \in X$, the tangent morphism $\Phi_x : \mathcal{V}_x \to P(T_xX)$ is birational (i.e. generically injective) over its image.*
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Therefore $\Phi_x$ is the normalization of its image in $P(T_xX)$. 
The Tangent Map if tangent bundle is ample

Note (c.f Mori ’79 Corollary 7.ii) that if $TX$ is ample (in particular $-K_X$ is ample and $X$ is uniruled) and we take a locally unsplit (unbreakable) family of rational curves, $\mathcal{V}$, then for every element $[f] \in \mathcal{V}$ we have

$$f^*TX = \mathcal{O}_\mathbb{P}^1(2) \oplus \mathcal{O}_\mathbb{P}^1(1) \oplus \ldots \oplus \mathcal{O}_\mathbb{P}^1(1).$$
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Thus the tangent map $\Phi_x : \mathcal{V}_x \rightarrow P(T_x X)$ is defined at every point, it is finite and at every point it is immersive. Thus it is an etale cover of $P(T_x X) = \mathbb{P}^{n-1}$. But $\mathbb{P}^{n-1}$ is simply connected and therefore $\Phi_x$ is birational and thus an isomorphism.
The Variety of Minimal Rational Tangents

**Definition**

We define $S_x \subseteq P(T_xX)$ as the closure of the image of the map $\Phi_x$ and we call it tangent cone of curves from $\mathcal{V}$ at the point $x$.

J.-M. Hwang and N. Mok call this Variety of Minimal Rational Tangents. The name tangent cone follows from the fact that $S_x$ is (at least around $[f]$) the tangent cone to Locus($V_x$).
Variety of Minimal Rational Tangents

For our purposes we need the following observation which follows from the above discussion.

Lemma

*The projectivised tangent space of the tangent cone $S_x$ at $\Phi_x([f])$ is equal to $P((f^*TX_0^+) \subset P((f^*TX_0) = P(T_xX).$*
Variety of Minimal Rational Tangents

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**Lemma**  
The projectivised tangent space of the tangent cone $S_x$ at $\Phi_x([f])$ is equal to $P((f^*TX)_0^+ \subset P((f^*TX)_0) = P(T_xX)$.

**Proof**  The tangent space to Locus($V_x$) at $f(p)$, for $p \neq 0$, is the image of the evaluation of sections of the twisted pull-back of $TX$ which is

$$\text{Im}(T\hat{F})_p = (f^*TX)_p^+ \subset (f^*TX)_p = T_{f(p)}X.$$  

Thus passing with $p$ to 0 we get the result.
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Let $X$ be a complex projective manifold of dimension $n \geq 3$. Assume that $TX$ is ample. Then $X$ is isomorphic to the projective space.
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The next Theorem was first proved by Cho-Miyaoka-Shepherd Barron; subsequently Kebekus gave a shorter proof.

**Theorem**

Let $X$ be a complex projective manifold of dimension $n \geq 3$. Assume that for every curve $C \subset X$ we have $-K_X \cdot C \geq n + 1$. Then $X$ is isomorphic to the projective space.

Note that Mori’s Theorem follows immediately from it.
Proof. Take an unbreakable uniruling $\mathcal{V}$. By our assumption and the above results, for a general point $x \in X$ we have that $\mathcal{V}_x$ is smooth and $\dim(\mathcal{V}_x) = -K_X \cdot C - 2 = (n - 1)$. 
Proof. Take an unbreakable uniruling $\mathcal{V}$. By our assumption and the above results, for a general point $x \in X$ we have that $\mathcal{V}_x$ is smooth and $\dim(\mathcal{V}_x) = -K_X \cdot C - 2 = (n - 1)$. By the above results we have that $\mathcal{V}_x \cong \sigma_\infty \cong \mathbb{P}^{n-1}$. 
Proof. Take an unbreakable uniruling $\mathcal{V}$. By our assumption and the above results, for a general point $x \in X$ we have that $\mathcal{V}_x$ is smooth and $\text{dim}(\mathcal{V}_x) = -K_X \cdot C - 2 = (n - 1)$.

By the above results we have that $\mathcal{V}_x \cong \sigma_\infty \cong \mathbb{P}^{n-1}$.

Let $\tilde{i}_x : \mathcal{V}_x \to \tilde{X} = Bl_x X$ be the lift up of $i_x$; since $T\tilde{i}_x$ has rank one along $\sigma_\infty$, then $T\tilde{i}_x$ has maximal rank along $\sigma_\infty$, in particular $N_{\sigma_\infty, U_x} \cong N_{E/\tilde{X}} = O_{\mathbb{P}^{n-1}}(-1)$. 
Proof

Consider the Stein factorization of the universal map
\[ i_x : U_x \to X : U_x \to Y \to X, \]
where the first map \( \alpha : U_x \to Y \) contracts the divisor \( \sigma_\infty \) and the second \( \beta : Y \to X \) is a finite map.
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where the first map \( \alpha : U_x \to Y \) contracts the divisor \( \sigma_\infty \) and the second \( \beta : Y \to X \) is a finite map.

Since \( R^1 \pi_* (\mathcal{O}_{U_x}) = 0 \) and \( \mathcal{O}_{U_x}(\sigma_\infty)|_{\sigma_\infty} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \), the push forward of the twisted ideal sheaf sequence

\[
0 \to \mathcal{O}_{U_x} \to \mathcal{O}_{U_x}(\sigma_\infty) \to \mathcal{O}_{U_x}(\sigma_\infty)|_{\sigma_\infty} \to 0
\]

gives on \( V_x \cong \mathbb{P}^{n-1} \) a sequence,

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0 \to \mathcal{O}_{\mathbb{P}^{n-1}} \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to 0,
\]

where \( U_x \cong \mathbb{P}(\mathcal{E}^*) \). Since \( Ext^1_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1), \mathcal{O}_{\mathbb{P}^{n-1}}) = 0 \), then \( U_x \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}) \).
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where \( U_x \cong \mathbb{P}(\mathcal{E}^*) \). Since \( \text{Ext}^1_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1), \mathcal{O}_{\mathbb{P}^{n-1}}) = 0 \), then \( U_x \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}) \).

An application of Zariski’s main theorem shows that \( \alpha \) is the standard contraction of \( \sigma_\infty \), that is \( Y = \mathbb{P}^n \).
Proof

We have that adjunction formula for a finite, surjective morphism:

$$-K_{\mathbb{P}^n} = \beta^*(-K_X) + \text{branch divisor}. $$

Let $l$ be a line through $\alpha(x)$ and $t = \beta(l)$; $t$ is a curve associated with $\mathcal{V}_x$. 
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Let $l$ be a line through $\alpha(x)$ and $t = \beta(l)$; $t$ is a curve associated with $V_x$. Thus we have

$n + 1 = -K_X \cdot t = (\beta^*(-K_X)) \cdot l = (-K_{\mathbb{P}^n} - (\text{branch divisor})) \cdot l = n + 1 - (\text{branch divisor}) \cdot l$

Then the branch divisor is empty and $\beta$ is birational, thus an isomorphism.
Another generalization

The following generalization of Mori’s is due to A. and Wisniewski.

**Theorem**

*Let $X$ be a complex projective manifold of dimension $n \geq 3$. Assume that there exist a subsheaf $E \subset TX$ which is an ample vector bundle. Then $X$ is isomorphic to the projective space.*
Another generalization

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**Theorem**

Let $X$ be a complex projective manifold of dimension $n \geq 3$. Assume that there exist a subsheaf $E \subset TX$ which is an ample vector bundle. Then $X$ is isomorphic to the projective space.

**Proof.** By the assumption we can apply the Theorem of Miyaoka, therefore $X$ is uniruled.

Take an unbreakable uniruling $\mathcal{V}$: for a general $f \in \mathcal{V}$ we have $f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^d \oplus \mathcal{O}^{(n-d-1)}$, where $d = \text{deg}(f^*(-K_X)) - 2$. 
Lemma

For any \( f \in \mathcal{V} \) the pull-back \( f^* E \) is isomorphic either to \( \mathcal{O}(1)^{\oplus r} \) or to \( \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus (r-1)} \). In particular the family of curves parametrized by \( \mathcal{V} \) is unsplit.

Proof. For a general \( f \in \mathcal{V} \) the pull-back \( f^* E \) is an ample subbundle of \( f^* \mathcal{T}_X = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus (d)} \oplus \mathcal{O}^{\oplus (n-d-1)} \) and thus it is as in the lemma.
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For any $f \in V$ the pull-back $f^* E$ is isomorphic either to $O(1)^{\oplus r}$ or to $O(2) \oplus O(1)^{\oplus (r-1)}$. In particular the family of curves parametrized by $V$ is unsplit.

Proof. For a general $f \in V$ the pull-back $f^* E$ is an ample subbundle of $f^* TX = O(2) \oplus O(1)^{\oplus (d)} \oplus O^{\oplus (n-d-1)}$ and thus it is as in the lemma. Since $E$ is ample this is true also for all $f \in V$. 
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Lemma

For any \( f \in \mathcal{V} \) the pull-back \( f^*E \) is isomorphic either to \( \mathcal{O}(1)^{r} \) or to \( \mathcal{O}(2) \oplus \mathcal{O}(1)^{(r-1)} \). In particular the family of curves parametrized by \( V \) is unsplit.

Proof. For a general \( f \in \mathcal{V} \) the pull-back \( f^*E \) is an ample subbundle of \( f^*TX = \mathcal{O}(2) \oplus \mathcal{O}(1)^{(d)} \oplus \mathcal{O}^{(n-d-1)} \) and thus it is as in the lemma. Since \( E \) is ample this is true also for all \( f \in V \).

Since \( \text{deg}(f^*E) = r \) or \( \text{deg}(f^*E) = r + 1 \) and \( r > 1 \), and for any ample bundle \( \mathcal{E} \) over a rational curve we have \( \text{deg}(\mathcal{E}) \geq \text{rank}(\mathcal{E}) \), it follows that no curve from \( V \) can be split into a sum of two or more rational curves, hence \( V \) is unsplit.
Proof

We shall analyze $X$ using the notions of rc$\mathcal{V}$ relation and rc$\mathcal{V}$ fibration.
Proof

We shall analyze $X$ using the notions of rc$V$ relation and rc$V$ fibration. The following is a key observation.

**Lemma**

Let $X$, $E$ and $V$ be as above and moreover assume that $\varphi^0 : X^0 \to Z^0$ is an rc$V$ fibration. Then $E$ is tangent to a general fiber of $\varphi^0$. That is, if $X_g$ is a general fiber of $\varphi^0$, then the injection $E|_{X_g} \to TX|_{X_g}$ factors via $E|_{X_g} \hookrightarrow TX_g$. 
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**Proof** Choose a general $X_g$ (in particular smooth) and let $x \in X_g$ and $f \in \mathcal{V}_x$ be general as well. By construction $Locus(\mathcal{V}_x) \subset X_g$. 

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**Proof** Choose a general $X_g$ (in particular smooth) and let $x \in X_g$ and $f \in \mathcal{V}_x$ be general as well. By construction $\text{Locus}(\mathcal{V}_x) \subset X_g$. The tangent space to $\text{Locus}(\mathcal{V}_x)$ at $f(p)$ is the image of the evaluation of sections of the twisted pull-back of $TX$, which is $= (f^*TX)_p^+$, therefore $(f^*TX)_p^+ \subset (f^*TX_g)_p$ for every $p \in \mathbb{P}^1 \setminus \{0\}$. 
Proof

We shall analyze $X$ using the notions of rc-$\mathcal{V}$ relation and rc-$\mathcal{V}$ fibration. The following is a key observation.

**Lemma**

Let $X$, $E$ and $\mathcal{V}$ be as above and moreover assume that $\varphi^0 : X^0 \to Z^0$ is an rc-$\mathcal{V}$ fibration. Then $E$ is tangent to a general fiber of $\varphi^0$. That is, if $X_g$ is a general fiber of $\varphi^0$, then the injection $E|_{X_g} \to TX|_{X_g}$ factors via $E|_{X_g} \hookrightarrow TX_g$.

**Proof** Choose a general $X_g$ (in particular smooth) and let $x \in X_g$ and $f \in \mathcal{V}_x$ be general as well. By construction $\text{Locus}(\mathcal{V}_x) \subset X_g$.

The tangent space to $\text{Locus}(\mathcal{V}_x)$ at $f(p)$ is the image of the evaluation of sections of the twisted pull-back of $TX$, which is $= (f^*TX)_p^+$, therefore $(f^*TX)_p^+ \subset (f^*TX_g)_p$ for every $p \in \mathbb{P}^1 \setminus \{0\}$.

This implies that $E|_{X_g} \to TX|_{X_g}$ factors to $E|_{X_g} \to TX_g$ generically and since the map $TX_g \to TX|_{X_g}$ has cokernel which is torsion free (it is the normal sheaf which is locally free) this yields $E|_{X_g} \hookrightarrow TX_g$, a sheaf injection.
Proof

**Proposition**

The general fiber of $\varphi^0$, $X_g$, is $\mathbb{P}^k$ and $E|_{X_g} = \mathcal{O}(1)^{r}$ or $E|_{X_g} = TX_g$.

**Proof** By abuse we denote the general fiber with $X := X_g$. We consider here only the case when for $f \in V$ the pull-back $f^*E$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{(r-1)}$. In particular $f^*E \subset (f^*TX)^+$. 

Proof

Theorem

The general fiber of $\varphi^0, X_g$, is $\mathbb{P}^k$ and $E|_{X_g} = \mathcal{O}(1)^{+} r$ or $E|_{X_g} = TX_g$.

Proof

By abuse we denote the general fiber with $X := X_g$. We consider here only the case when for $f \in \mathcal{V}$ the pull-back $f^*E$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(1)^{+}(r-1)$. In particular $f^*E \subset (f^*TX)^{+}$.

Comparing the splitting type of $f^*E$ and $f^*TX$ we see that the tangent map $Tf : T\mathbb{P}^1 \rightarrow f^*TX$ factors to a vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \rightarrow f^*E$. (In other words, we have surjective morphism $(f^*E)^* \rightarrow \Omega_{\mathbb{P}^1} \cong \mathcal{O}(-2))$. 

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Proposition

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The vector bundle (nowhere degenerate) injection $T\mathbb{P}^1 \to f^*E$ implies $(f^*TX)^+ \hookrightarrow f^*E$. In fact, choose a general $f$ which is an immersion at $0 \rightarrow x$. Then $\Phi_x([f]) \in P(E_x) = P((f^*E)_0) \subset P(T_xX) = P((f^*TX)_0)$ and the same holds for morphisms in a neighborhood of $[f]$ in $V_x$. Thus around $\Phi_x([f])$ the tangent cone $S_x$ is contained in $P(E_x) = P((f^*E)_0)$, so is its tangent space $P((f^*TX)_0^+)$. 

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Proof

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Therefore \( f^*E = (f^*TX)^+ \) and thus \( \deg(f^*E) = \deg(f^*(-K_X)) \).
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The embedding \( E \hookrightarrow TX \) gives rise to a non-trivial morphism \( \det(E) \to \Lambda^rTX \) and thus to a non-zero section of \( \Lambda^rTX \otimes K_X \). We use dualities to have the equalities:

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\begin{align*}
    h^0(X, \Lambda^rTX \otimes K_X) &= h^n(X, \Omega^n_X) = h^r(X, \Omega^n_X) = h^r(X, K_X) = h^{n-r}(X, \mathcal{O}_X) \\
\end{align*}
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and, since \( X \) is Fano, the latter number is non-zero only if \( r = n \).
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Thus $\Lambda^rTX \otimes (\det E)^{-1} \cong O_X$ so $E \hookrightarrow TX$ is nowhere degenerate, hence an isomorphism.
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Thus \( \Lambda^r TX \otimes (\det E)^{-1} \cong \mathcal{O}_X \) so \( E \hookrightarrow TX \) is nowhere degenerate, hence an isomorphism.

We conclude by the Theorem of Mori.
Finally we prove that $\dim Z_0$ is zero, i.e. $X$ is rationally connected. By contradiction if $\dim Z_0 \geq 1$ one can prove that:

**Lemma**

*Outside a subset of codimension $\geq 2$ the morphism $\varphi_0$ is a $\mathbb{P}^k$-bundle (in the analytic topology).*

Then we take a complete curve $B \subset Z_0$ and we consider the $\mathbb{P}^k$-bundle $\varphi_0 : X_B := \varphi_0^{-1}(B) \to B$ with the ample vector bundle $E|_{X_B}$.
Proof

We get a contradiction applying the following straightforward result, due to Campana and Peternell.

**Lemma**

Let $X$ be a $n$-dimensional projective manifold, $\varphi : X \to Y$ a $\mathbb{P}^k$ bundle ($k < n$) of the form $X = \mathbb{P}(V)$ with a vector bundle $V$ on $Y$. Then the relative tangent sheaf $T_{X/Y}$ does not contain an ample locally free subsheaf.