Attainability in Repeated Games with Vector Payoffs

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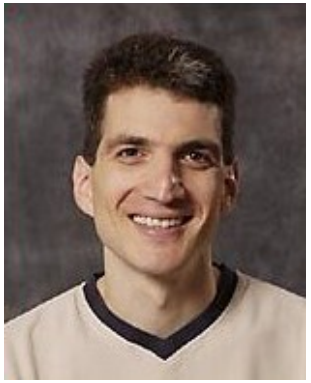


Outline

Repeated game with vector payoffs

- Two players play repeated game
- vector payoffs are **d-dimensional**
- the total payoff up to stage t is $x(t)$
- example for $d = 2$ (in discrete-time)

$$\begin{pmatrix} (6,7) & (1,7) & (6,2) & (1,2) \\ (6,-4) & (1,-4) & (6,-9) & (1,-9) \\ (-3,-1) & (-8,-1) & (-3,-6) & (-8,-6) \\ (-3,10) & (-8,10) & (-3,5) & (-8,5) \end{pmatrix}$$

Repeated game with vector payoffs: stage 1

- Two players play repeated game
- vector payoffs are **d-dimensional**
- the total payoff up to stage t is $x(t)$
- example for $d = 2$ (in discrete-time)

$$\begin{pmatrix} \boxed{(6,7)} & (1,7) & (6,2) & (1,2) \\ (6,-4) & (1,-4) & (6,-9) & (1,-9) \\ (-3,-1) & (-8,-1) & (-3,-6) & (-8,-6) \\ (-3,10) & (-8,10) & (-3,5) & (-8,5) \end{pmatrix}$$

- $t = 1$: (Top,Left) $\rightarrow x(1) = (6,7)$

Repeated game with vector payoffs: stage 2

- Two players play repeated game
- vector payoffs are **d-dimensional**
- the total payoff up to stage t is $x(t)$
- example for $d = 2$ (in discrete-time)

$$\begin{pmatrix} (6,7) & (1,7) & (6,2) & (1,2) \\ (6,-4) & (1,-4) & (6,-9) & (1,-9) \\ (-3,-1) & (-8,-1) & (-3,-6) & (-8,-6) \\ (-3,10) & (-8,10) & (-3,5) & \boxed{(-8,5)} \end{pmatrix}$$

- $t = 1$: (Top,Left) $\rightarrow x(1) = (6,7)$
- $t = 2$: (T,L)-(Bottom,Right) $\rightarrow x(2) = (-2,12)$

Repeated game with vector payoffs: stage 3

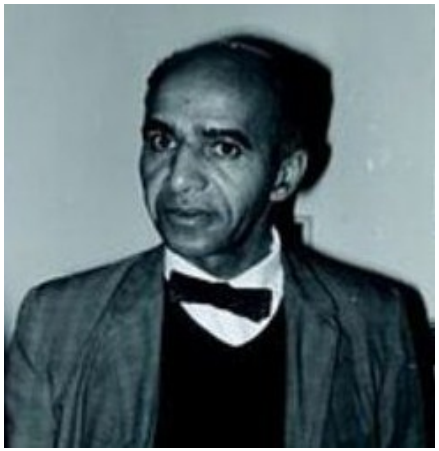
- Two players play repeated game
- vector payoffs are **d-dimensional**
- the total payoff up to stage t is $x(t)$
- example for $d = 2$ (in discrete-time)

$$\begin{pmatrix} \boxed{(6,7)} & (1,7) & (6,2) & (1,2) \\ (6,-4) & (1,-4) & (6,-9) & (1,-9) \\ (-3,-1) & (-8,-1) & (-3,-6) & (-8,-6) \\ (-3,10) & (-8,10) & (-3,5) & (-8,5) \end{pmatrix}$$

- $t = 1$: (Top,Left) $\rightarrow x(1) = (6,7)$
- $t = 2$: (T,L)-(Bottom,Right) $\rightarrow x(2) = (-2,12)$
- $t = 3$: (T,L)-(B,R)-(Top,Left) $\rightarrow x(3) = (4,19)$

Approachability [Blackwell, '56]

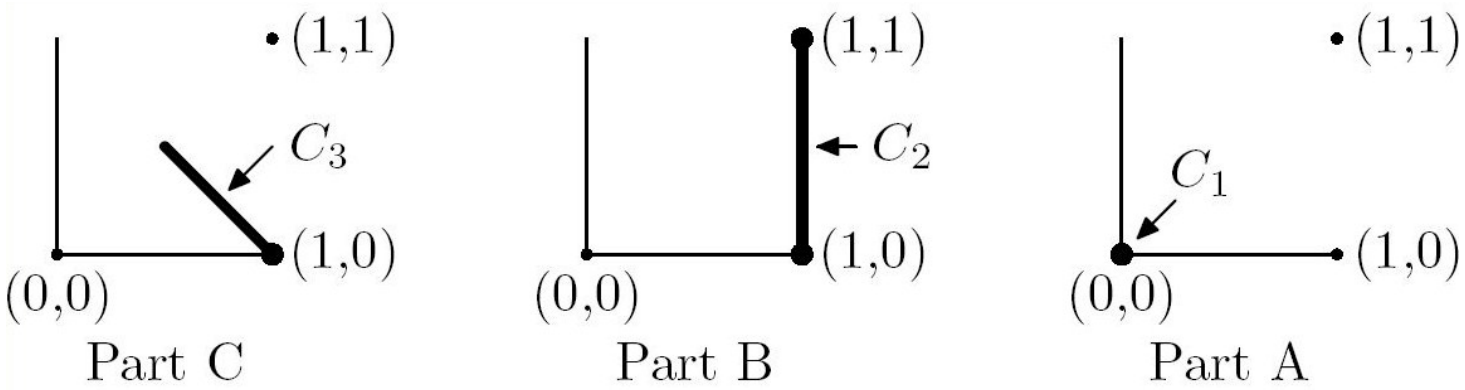
A set of payoff vectors A is **approachable** by P_1 if she has a strategy such that the **average payoff** up to stage t, $\bar{x}(t) := \frac{x(t)}{t}$, converges to A, regardless of the strategy of P_2.



Blackwell, D. (1956b) "An Analog of the MinMax Theorem For Vector Payoffs," Pacific J. of Math., 6, 1-8.

Example [Solan, Maschler, and Zamir, 2010]

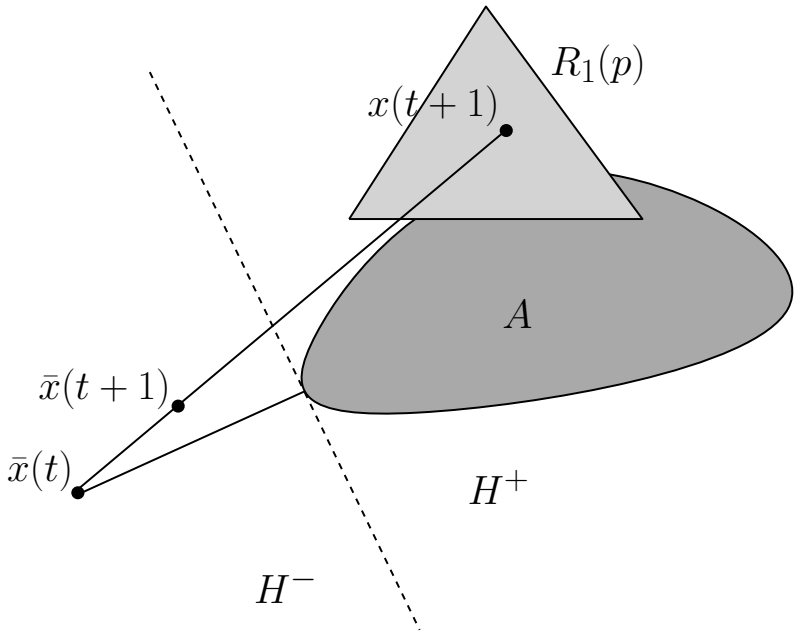
	L	R
T	$(0,0)$	$(0,0)$
B	$(1,1)$	$(1,0)$



- C_1 approachable when P_1 plays T in every stage
- C_2 approachable when P_1 plays B in every stage
- C_3 approachable when P_1 plays

$$\begin{cases} B & \text{if } \bar{x}_1(t-1) + \bar{x}_2(t-1) < 1 \\ T & \text{otherwise} \end{cases}$$

Geometric insight



A approachable if for any $\bar{x}(t) \in H^-$, $\exists\, p$ s.t. $R_1(p) \subset H^+$.

- $y(t)$ is projection of $\bar{x}(t)$ onto A
- take supporting hyperplane (dashed line) for A in $y(t)$

$$H = \{z \in \mathbb{R}^d | \langle z - y(t), \bar{x}(t) - y(t) \rangle = 0\}$$

- $R_1(p)$ set of payoff vectors when P_1 plays mixed strategy p.

just a few references

- Extension to infinite dimensional spaces

  Lehrer, "Approachability in infinite dimensional spaces," IJGT, 31(2) 2002

- Connections to differential games

  Soulaimani, Quincampoix, Sorin "Repeated Games and Qualitative Differential Games..." SICON 48, 2009

- continuous-time approachability

  Hart, Mas-Colell, "Regret-based continuous-time dynamics", Games and Economic Behavior 45, 2003

The notion of attainability

A set of payoff vectors A is **attainable** by P_1 if she has a strategy such that the **total payoff** up to stage t, $x(t)$, "converges" to A, regardless of the strategy of P_2.

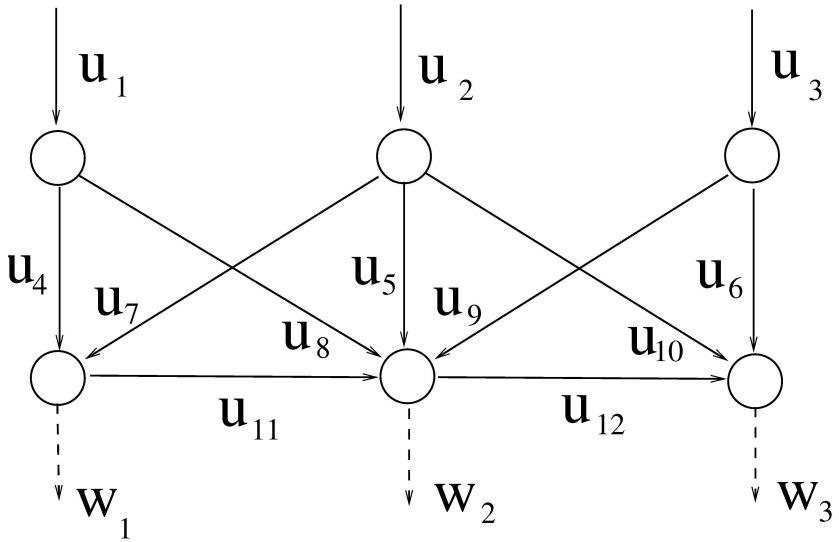
Lehrer, Solan, Bauso, "Repeated games over networks with vector payoffs: the notion of attainability" NetGCooP 2011 (journal version at arXiv:1201.6054v1).

Background and historical notes

INTERNATIONAL SCHOOL OF MATHEMATICS "G. STAMPACCHIA"

Workshop on "Stochastic Methods in Game Theory"
Ettore Majorana Centre, Erice, 8-16 September 2010.

Motivation: Control Theory



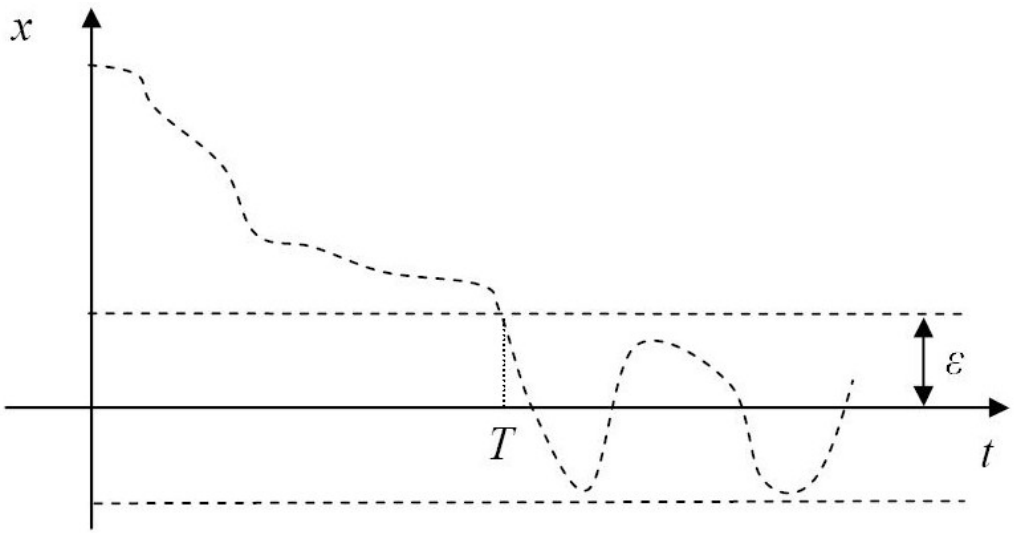
- *Uncontrolled* flows/demand $w(t) \in \mathcal{W}, \quad \forall t,$
- *Controlled* flows/supply $u(t) \in \mathcal{U}, \quad \forall t.$
- buffer (excess supply) dynamics:

$$\dot{x}(t) = Bu(t) - w(t),\ x(0) = \zeta$$

- B incidence matrix.

[Bauso, Blanchini, Pesenti, *Automatica*, 2006], [Bauso, Blanchini, Pesenti, *TAC*, 2010]

Robust stabilizability

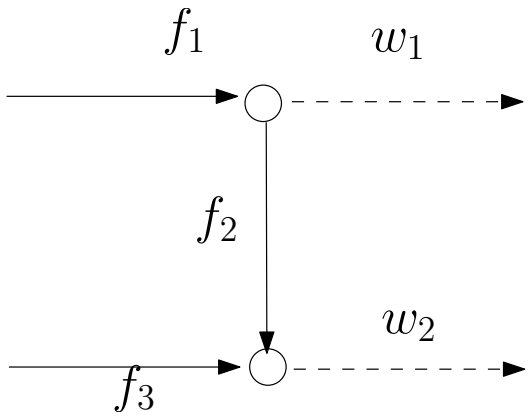


- We need to bound total excess supply $x(t) = \int_0^t \dot{x}(\tau)d\tau + \zeta$

reinterpret as continuous time repeated game where

- P_1 plays $u(t)$ and P_2 plays w
- $\dot{x}(t)$ is instantaneous payoff
- $x(t)$ is total payoff up to time t

back to first example



$$u(t) \in \left\{ \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 6 \end{bmatrix} \right\}$$

$$w(t) \in \left\{ \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$$

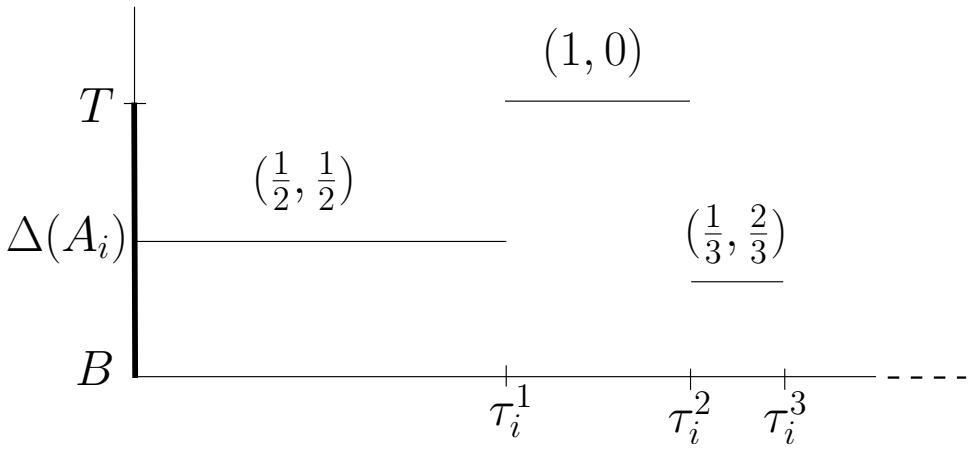
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} - \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

$$\begin{pmatrix} (6,7) & (1,7) & (6,2) & (1,2) \\ (6,-4) & (1,-4) & (6,-9) & (1,-9) \\ (-3,-1) & (-8,-1) & (-3,-6) & (-8,-6) \\ (-3,10) & (-8,10) & (-3,5) & (-8,5) \end{pmatrix}$$

simulation

The Model

- A repeated game $(A_1, A_2, \mathbf{g})$
- A_i action space of P_i
- $\mathbf{g} : A_1 \times A_2 \to [-1, 1]^d$ is d-dimensional payoff
- $(a_i^t)_{t \in \mathbb{R}_+}$ is *non-anticipating behavior strategy* for player i
  - $(a_i^t)_{t \in \mathbb{R}_+}$ takes values in $\Delta(A_i)$
  - $\exists$ increasing sequence of times $\tau_i^1 < \tau_i^2 < \tau_i^3 < \ldots$ s.t. a_i^t is measurable w.r.t. the information τ_i^k, $\tau_i^k \leq t < \tau_i^{k+1}$.

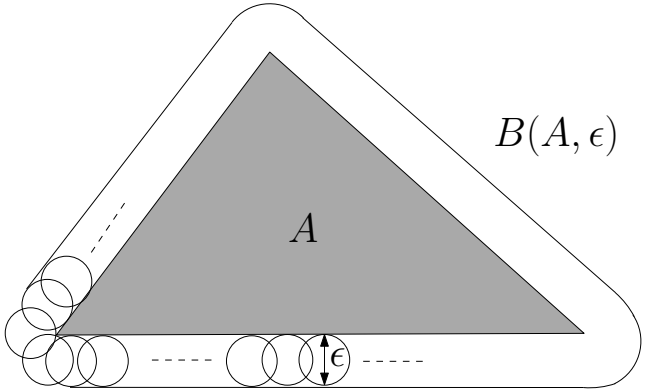


Attainability: definition

- g_t =payoff at time t given the mixed actions of the players
- $x(t) = \int_{\tau=0}^{t} g_\tau(\text{mixed action pairs at time } \tau) d\tau$

Def.: A set A in $\mathbb{R}^d$ is attainable by P_1 if there is $T > 0$ such that for every $\epsilon > 0$ there is a strategy σ_1 of $P1$ s.t.

$$dist(x(t)[\sigma_1, \sigma_2], A) \leq \epsilon, \quad \forall t \geq T, \forall \sigma_2.$$



$$B(A, \epsilon) := \{z : dist(z, A) \leq \epsilon\}$$

Main results in a nutshell

Thm. 1: The following conditions are equivalent.

B1 **Vector** $\vec{0} \in \mathbb{R}^d$ is attainable by P1;

B2 $v_\lambda \geq 0$ for every $\lambda \in \mathbb{R}^d$.

Thm. 2: **Vector** $z \in \mathbb{R}^d$ $(\neq \vec{0})$ is attainable by P1 $\Leftrightarrow$

B1 The vector $\vec{0} \in \mathbb{R}^d$ is attainable by $P1$

and either one between ***B3*** and ***B4*** holds (yet to be introduced)

Thm. 3: The following statements are equivalent:

C1 $v_\lambda > 0$ for every $\lambda \in \mathbb{R}^d$;

C2 **Every vector** $z \in \mathbb{R}^d$ is attainable by player 1.

Main results in a nutshell

Thm. 1: The following conditions are equivalent.

B1 **Vector** $\vec{0} \in \mathbb{R}^d$ is at

$$v_\lambda \geq 0$$

B2 $v_\lambda \geq 0$ for every $\lambda \in \mathbb{R}^d$.

Thm. 2: **Vector** $z \in \mathbb{R}^d$ $(\neq \vec{0})$ is attainable by P1 $\Leftrightarrow$

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and either one between ***B3*** and ***B4*** holds (yet to be introduced)

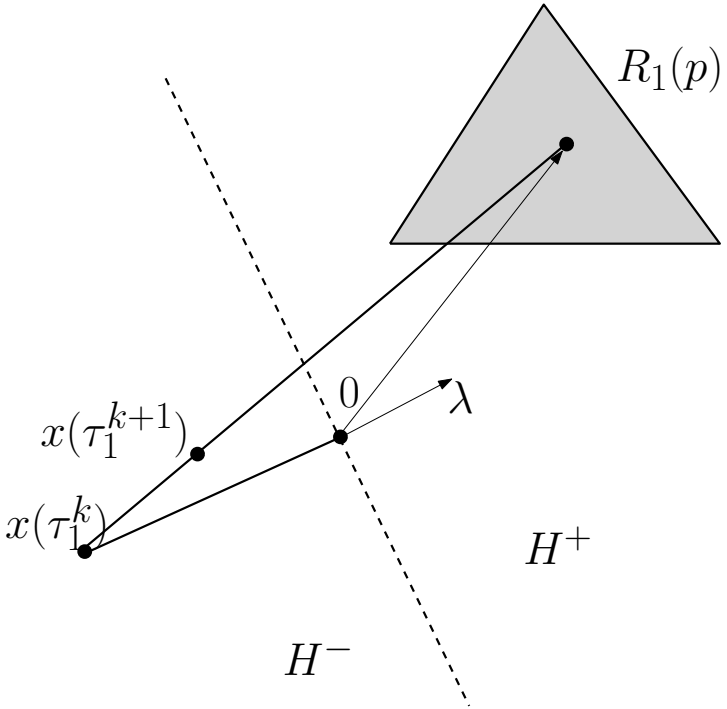
Thm. 3: The following statements are equivalent:

C1 $v_\lambda > 0$ for every $\lambda \in$

$$v_\lambda > 0$$

C2 **Every vector** $z \in \mathbb{R}$ is attainable by player 1.

Value of projected game



- $v_\lambda > 0$ for every $\lambda \in \mathbb{R}^d$.

$$\begin{pmatrix} (\#,\#) & (\#,\#) \\ (\#,\#) & (\#,\#) \end{pmatrix} \Rightarrow \begin{pmatrix} \langle \lambda, (\#,\#) \rangle & \langle \lambda, (\#,\#) \rangle \\ \langle \lambda, (\#,\#) \rangle & \langle \lambda, (\#,\#) \rangle \end{pmatrix}$$

- $v_\lambda > 0$ is value of projected game

Theorem 2

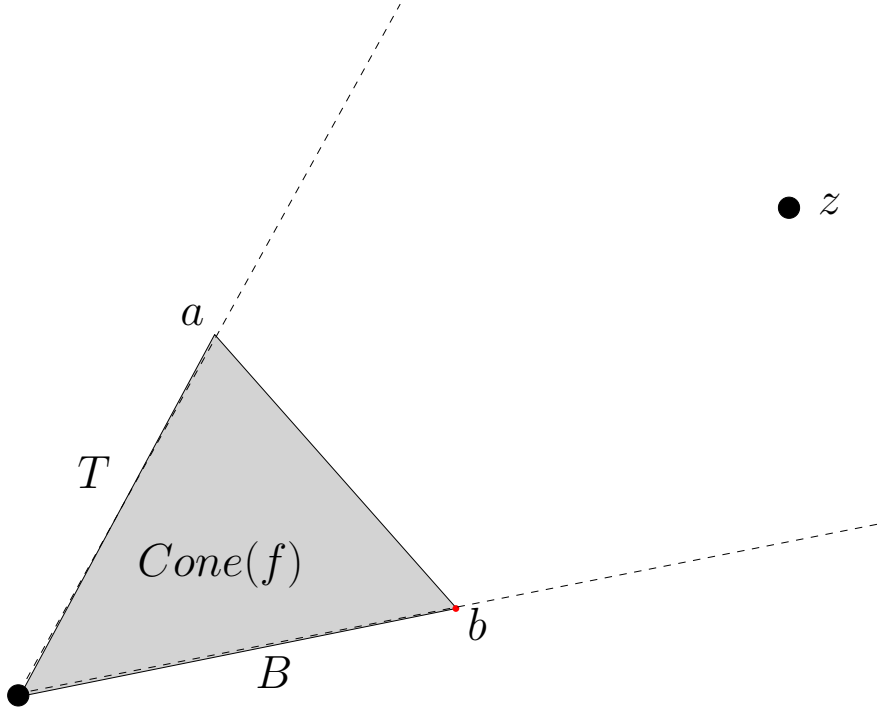
Vector $z \in \mathbb{R}^d$ $(\neq \vec{0})$ is attainable by P1 $\Leftrightarrow$

B1 The vector $\vec{0} \in \mathbb{R}^d$ is attainable by $P1$

B3 for every function $f : \Delta(A_1) \to \Delta(A_2)$, vector z is in

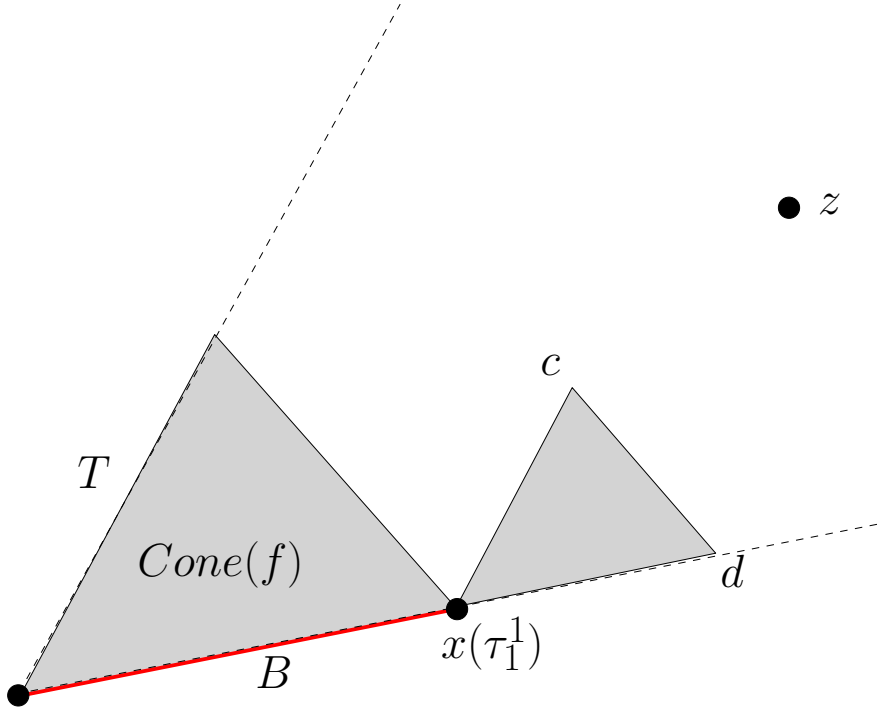
$$Cone(f) := \left\{ y \in \mathbb{R}^d | \, y = \sum_{p \in A_1} \alpha_p g(p, f(p)) \colon \alpha_p \geq 0 \, \forall p \right\}$$

Sketch of proof



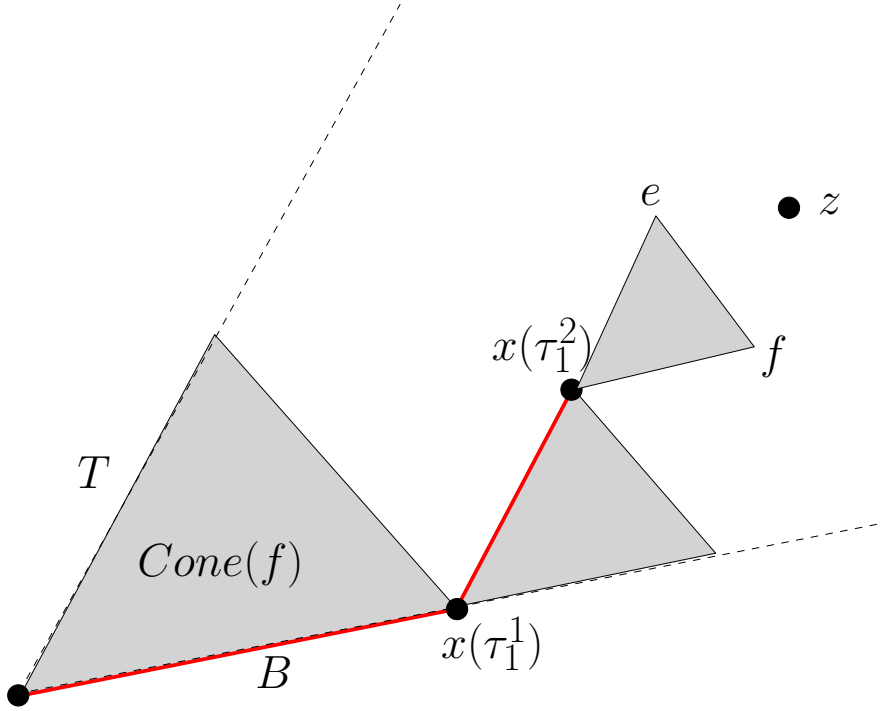
- P_1 plays mixed action $p \in \Delta\{T, B\}$ and P_2 plays $f(p)$,
- payoff $x(\tau_1^1)$ belongs to segment ab

Sketch of proof: stage τ_1^1



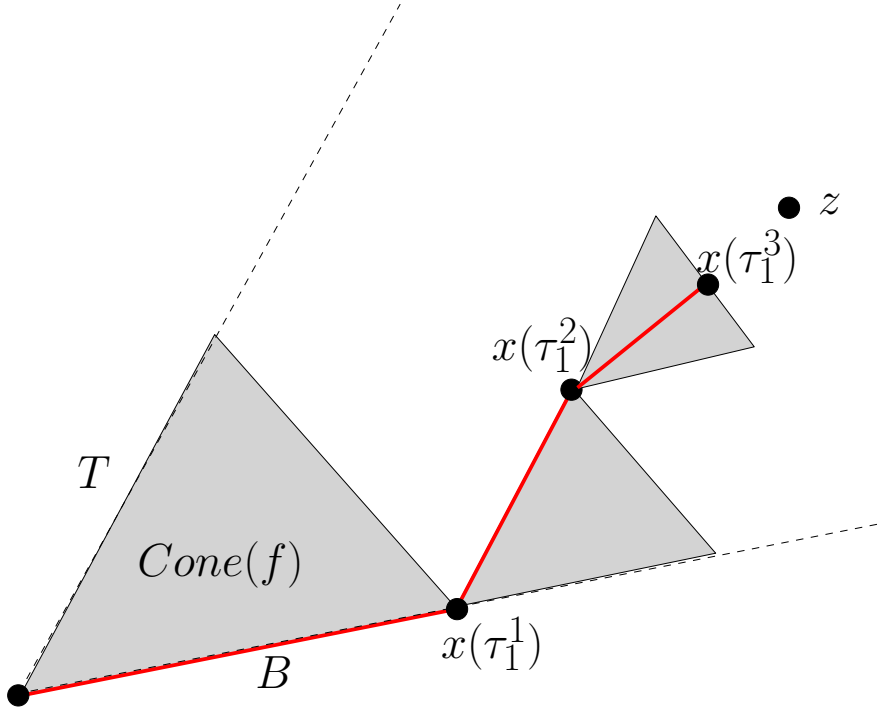
- suppose P_1 plays B in interval $0 \leq t \leq \tau_1^1$
- payoff $x(\tau_1^2)$ belongs to segment cd

Sketch of proof: stage τ_1^2



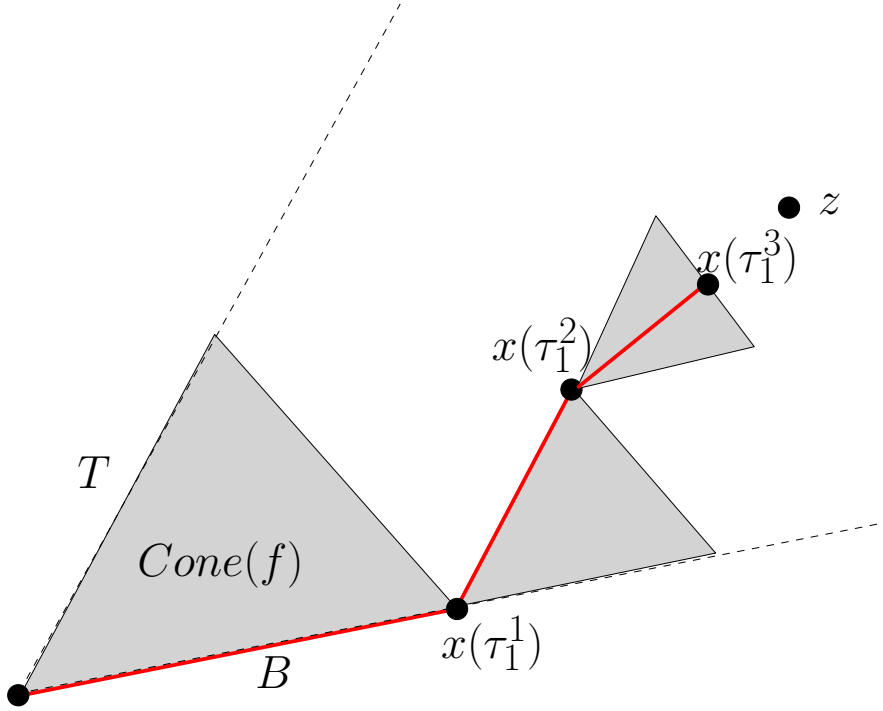
- suppose P_1 plays T in interval $\tau_1^1 \leq t \leq \tau_1^2$
- payoff $x(\tau_1^3)$ belongs to segment ef

Sketch of proof: stage τ_1^3



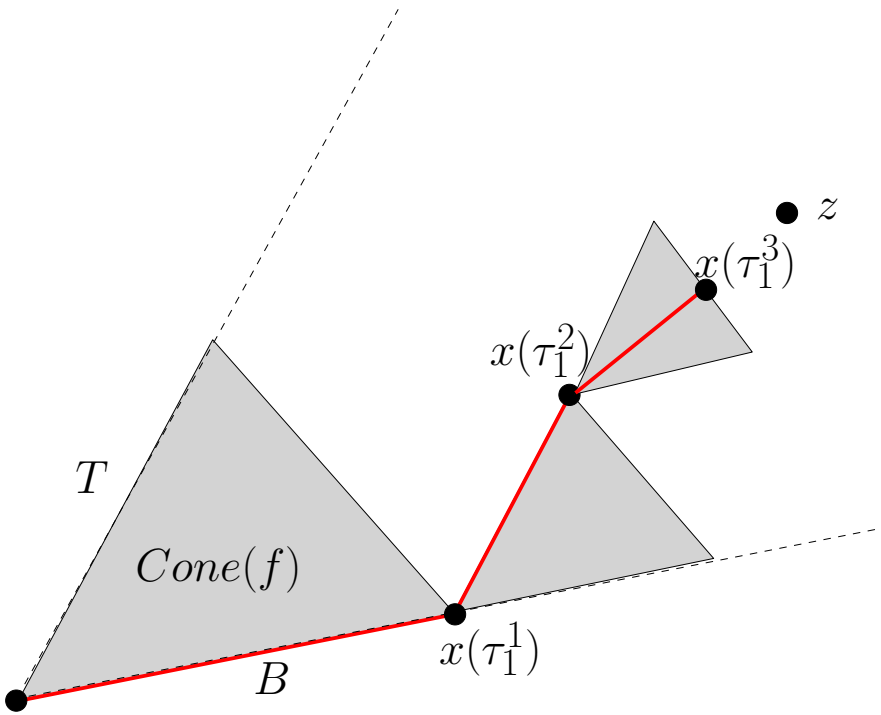
- suppose P_1 plays $\frac{1}{2}$ T and $\frac{1}{2}$ B in interval $\tau_1^2 \leq t \leq \tau_1^3$
- get to $x(\tau_1^3)$ very close to z

Sketch of proof: Necessity of 1.



In the game from τ_1^3 to ∞ total payoff has to be close to zero - true only if $\vec{0}$ is attainable

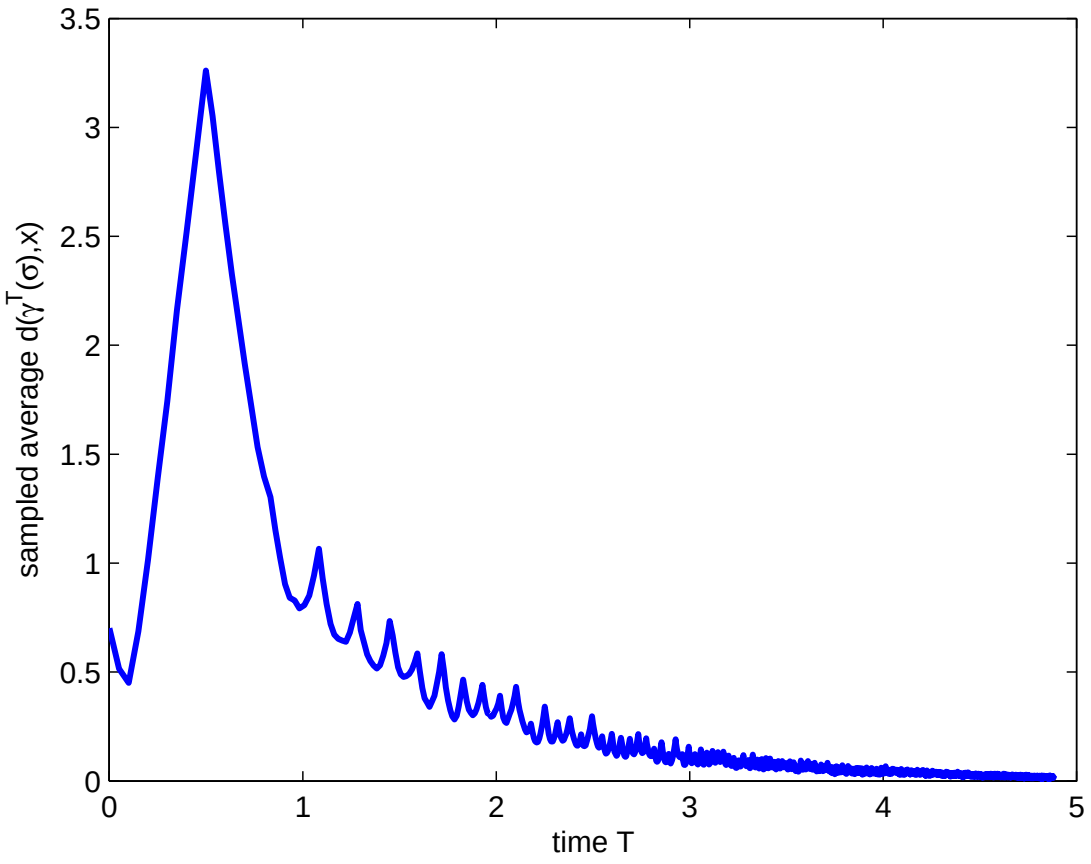
Sketch of proof: Necessity of 2.



all possible trajectories $\{x(\tau_1^k)\}_{k=0,\ldots,\infty}$ are within $Cone(f)$

Monte Carlo simulations

sampled average of $dist(z, x(t))$



Conclusions and further questions

- Games with vector payoffs:
  - approachability looked at average payoff
  - attainability looks at total payoff
- necessary and sufficient conditions for attainability
- Still to be done:
  - characterization of attainable sets
  - look at discounted payoff