Attainability in Repeated Games with Vector Payoffs

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University of Trento, 17 April 2013
Outline

Set-up
- What we know (approachability)
- What we do (attainability)

Motivations

Model and results
- Results
Repeated game with vector payoffs

- Two players play repeated game
- Vector payoffs are \(d\)-dimensional
- The total payoff up to stage \(t\) is \(x(t)\)
- Example for \(d = 2\) (in discrete-time)

\[
\begin{pmatrix}
(6, 7) & (1, 7) & (6, 2) & (1, 2) \\
(6, -4) & (1, -4) & (6, -9) & (1, -9) \\
(-3, -1) & (-8, -1) & (-3, -6) & (-8, -6) \\
(-3, 10) & (-8, 10) & (-3, 5) & (-8, 5)
\end{pmatrix}
\]
Repeated game with vector payoffs: stage 1

- Two players play repeated game
- vector payoffs are \( d \)-dimensional
- the total payoff up to stage \( t \) is \( x(t) \)
- example for \( d = 2 \) (in discrete-time)

\[
\begin{pmatrix}
(6,7) & (1, 7) & (6, 2) & (1, 2) \\
(6, -4) & (1, -4) & (6, -9) & (1, -9) \\
(-3, -1) & (-8, -1) & (-3, -6) & (-8, -6) \\
(-3, 10) & (-8, 10) & (-3, 5) & (-8, 5) \\
\end{pmatrix}
\]

- \( t = 1 \): (Top, Left) \( \rightarrow \) \( x(1) = (6, 7) \)
Repeated game with vector payoffs: stage 2

- Two players play repeated game
- Vector payoffs are \( d \)-dimensional
- The total payoff up to stage \( t \) is \( x(t) \)
- Example for \( d = 2 \) (in discrete-time)

\[
\begin{pmatrix}
(6, 7) & (1, 7) & (6, 2) & (1, 2) \\
(6, -4) & (1, -4) & (6, -9) & (1, -9) \\
(-3, -1) & (-8, -1) & (-3, -6) & (-8, -6) \\
(-3, 10) & (-8, 10) & (-3, 5) & (-8, 5)
\end{pmatrix}
\]

- \( t = 1 \): (Top,Left) \( \rightarrow \) \( x(1) = (6, 7) \)
- \( t = 2 \): (T,L)-(Bottom,Right) \( \rightarrow \) \( x(2) = (-2, 12) \)
Repeated game with vector payoffs: stage 3

- Two players play repeated game
- Vector payoffs are $d$-dimensional
- The total payoff up to stage $t$ is $x(t)$
- Example for $d = 2$ (in discrete-time)

$$
\begin{pmatrix}
(6,7) & (1,7) & (6,2) & (1,2) \\
(6,-4) & (1,-4) & (6,-9) & (1,-9) \\
(-3,-1) & (-8,-1) & (-3,-6) & (-8,-6) \\
(-3,10) & (-8,10) & (-3,5) & (-8,5)
\end{pmatrix}
$$

- $t = 1$: (Top,Left) → $x(1) = (6,7)$
- $t = 2$: (T,L)-(Bottom,Right) → $x(2) = (-2,12)$
- $t = 3$: (T,L)-(B,R)-(Top,Left) → $x(3) = (4,19)$
Approachability [Blackwell, ’56]

A set of payoff vectors $A$ is **approachable** by $P_1$ if she has a strategy such that the **average payoff** up to stage $t$, $\bar{x}(t) := \frac{x(t)}{t}$, converges to $A$, regardless of the strategy of $P_2$.

Example [Solan, Maschler, and Zamir, 2010]

<table>
<thead>
<tr>
<th></th>
<th>$L$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$(0,0)$</td>
<td>$(0,0)$</td>
</tr>
<tr>
<td>$B$</td>
<td>$(1,1)$</td>
<td>$(1,0)$</td>
</tr>
</tbody>
</table>

$C_1$ approachable when $P_1$ plays $T$ in every stage

$C_2$ approachable when $P_1$ plays $B$ in every stage

$C_3$ approachable when $P_1$ plays

\[
\begin{cases} 
B & \text{if } \bar{x}_1(t-1) + \bar{x}_2(t-1) < 1 \\
T & \text{otherwise}
\end{cases}
\]
Geometric insight

\[ A \text{ approachable if for any } \bar{x}(t) \in H^-, \exists p \text{ s.t. } R_1(p) \subset H^+. \]

- \( y(t) \) is projection of \( \bar{x}(t) \) onto \( A \)
- take supporting hyperplane (dashed line) for \( A \) in \( y(t) \)

\[ H = \{ z \in \mathbb{R}^d | \langle z - y(t), \bar{x}(t) - y(t) \rangle = 0 \} \]

- \( R_1(p) \) set of payoff vectors when \( P_1 \) plays mixed strategy \( p \).
just a few references

- Extension to infinite dimensional spaces
  Lehrer, “Approachability in infinite dimensional spaces,” IJGT, 31(2) 2002

- Connections to differential games
  Soulaimani, Quincampoix, Sorin “Repeated Games and Qualitative Differential Games...” SICON 48, 2009

- continuous-time approachability
The notion of attainability

A set of payoff vectors $A$ is **attainable** by $P_1$ if she has a strategy such that the **total payoff** up to stage $t$, $x(t)$, “converges” to $A$, regardless of the strategy of $P_2$.

Background and historical notes
Motivation: Control Theory

- Uncontrolled flows/demand \( w(t) \in W, \forall t \),
- Controlled flows/supply \( u(t) \in U, \forall t \).
- Buffer (excess supply) dynamics:
  \[
  \dot{x}(t) = Bu(t) - w(t), \quad x(0) = \zeta
  \]
- \( B \) incidence matrix.

We need to bound total excess supply $x(t) = \int_0^t \dot{x}(\tau)d\tau + \zeta$

reinterpret as continuous time repeated game where

- $P_1$ plays $u(t)$ and $P_2$ plays $w$
- $\dot{x}(t)$ is instantaneous payoff
- $x(t)$ is total payoff up to time $t$
back to first example

\[ f_1 \rightarrow w_1 \]
\[ f_2 \rightarrow w_2 \]
\[ f_3 \rightarrow w_2 \]

\[ u(t) \in \left\{ \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \end{bmatrix} \right\} \]

\[ w(t) \in \left\{ \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} \]

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
u_1(t) \\
u_2(t) \\
u_3(t)
\end{bmatrix} -
\begin{bmatrix}
w_1(t) \\
w_2(t)
\end{bmatrix}
\]

\[
\begin{pmatrix}
(6, 7) & (1, 7) & (6, 2) & (1, 2) \\
(6, -4) & (1, -4) & (6, -9) & (1, -9) \\
(-3, -1) & (-8, -1) & (-3, -6) & (-8, -6) \\
(-3, 10) & (-8, 10) & (-3, 5) & (-8, 5)
\end{pmatrix}
\]
simulation
The Model

- A repeated game \((A_1, A_2, g)\)
- \(A_i\) action space of \(P_i\)
- \(g : A_1 \times A_2 \rightarrow [-1, 1]^d\) is \(d\)-dimensional payoff
- \((a^t_i)_{t \in \mathbb{R}_+}\) is \textit{non-anticipating behavior strategy} for player \(i\)
  - \((a^t_i)_{t \in \mathbb{R}_+}\) takes values in \(\Delta(A_i)\)
  - \(\exists\) increasing sequence of times \(\tau^1_i < \tau^2_i < \tau^3_i < \ldots\) s.t. \(a^t_i\) is measurable w.r.t. the information \(\tau^k_i\), \(\tau^k_i \leq t < \tau^{k+1}_i\).
Attainability: definition

- $g_t =$ payoff at time $t$ given the mixed actions of the players
- $x(t) = \int_{\tau=0}^{t} g_{\tau} \text{(mixed action pairs at time } \tau) \, d\tau$

Def.: A set $A$ in $\mathbb{R}^d$ is **attainable** by $P_1$ if there is $T > 0$ such that for every $\epsilon > 0$ there is a strategy $\sigma_1$ of $P_1$ s.t.

$$\text{dist}(x(t)[\sigma_1, \sigma_2], A) \leq \epsilon, \quad \forall t \geq T, \forall \sigma_2.$$
Main results in a nutshell

**Thm. 1:** The following conditions are equivalent.

- **\( B1 \)** Vector \( \vec{0} \in \mathbb{R}^d \) is attainable by \( P_1 \);
- **\( B2 \)** \( v_\lambda \geq 0 \) for every \( \lambda \in \mathbb{R}^d \).

**Thm. 2:** Vector \( z \in \mathbb{R}^d (\neq \vec{0}) \) is attainable by \( P_1 \) \iff

- **\( B1 \)** The vector \( \vec{0} \in \mathbb{R}^d \) is attainable by \( P_1 \)
- and either one between **\( B3 \)** and **\( B4 \)** holds (yet to be introduced).

**Thm. 3:** The following statements are equivalent:

- **\( C1 \)** \( v_\lambda > 0 \) for every \( \lambda \in \mathbb{R}^d \);
- **\( C2 \)** Every vector \( z \in \mathbb{R}^d \) is attainable by player 1.
Main results in a nutshell

**Thm. 1:** The following conditions are equivalent.

- **B1** Vector $\vec{0} \in \mathbb{R}^d$ is attainable by $P_1$; $v_\lambda \geq 0$
- **B2** $v_\lambda \geq 0$ for every $\lambda \in \mathbb{R}^d$.

**Thm. 2:** Vector $z \in \mathbb{R}^d (\neq \vec{0})$ is attainable by $P_1$ \iff

- **B1** The vector $\vec{0} \in \mathbb{R}^d$ is attainable by $P_1$
- and either one between **B3** and **B4** holds (yet to be introduced)

**Thm. 3:** The following statements are equivalent:

- **C1** $v_\lambda > 0$ for every $\lambda \in \mathbb{R}^d$; $v_\lambda > 0$
- **C2** Every vector $z \in \mathbb{R}$ is attainable by player 1.
Value of projected game

\[ v_\lambda > 0 \text{ for every } \lambda \in \mathbb{R}^d. \]

\[
\begin{pmatrix}
(\# , \#) & (\# , \#) \\
(\# , \#) & (\# , \#)
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\langle \lambda, (\# , \#) \rangle & \langle \lambda, (\# , \#) \rangle \\
\langle \lambda, (\# , \#) \rangle & \langle \lambda, (\# , \#) \rangle
\end{pmatrix}
\]

\[ v_\lambda > 0 \text{ is value of projected game} \]
Theorem 2

Vector $z \in \mathbb{R}^d (\neq \vec{0})$ is attainable by P1 $\iff$

$\textbf{B1}$ The vector $\vec{0} \in \mathbb{R}^d$ is attainable by $P1$

$\textbf{B3}$ for every function $f : \Delta(A_1) \rightarrow \Delta(A_2)$, vector $z$ is in

$$\text{Cone}(f) := \{ y \in \mathbb{R}^d | y = \sum_{p \in A_1} \alpha_p g(p, f(p)) : \alpha_p \geq 0 \forall p \}$$
Sketch of proof

- $P_1$ plays mixed action $p \in \Delta\{T, B\}$ and $P_2$ plays $f(p)$,
- payoff $x(\tau_1^1)$ belongs to segment $ab$
Sketch of proof: stage $\tau_1^1$

- suppose $P_1$ plays B in interval $0 \leq t \leq \tau_1^1$
- payoff $x(\tau_1^2)$ belongs to segment cd
Sketch of proof: stage $\tau_1^2$

- Suppose $P_1$ plays $T$ in interval $\tau_1^1 \leq t \leq \tau_1^2$.
- Payoff $x(\tau_1^3)$ belongs to segment $ef$.
Sketch of proof: stage $\tau_1^3$

- suppose $P_1$ plays $\frac{1}{2}$ T and $\frac{1}{2}$ B in interval $\tau_1^2 \leq t \leq \tau_1^3$
- get to $x(\tau_1^3)$ very close to $z$
Sketch of proof: Necessity of 1.

In the game from $\tau_1^3$ to $\infty$ total payoff has to be close to zero - true only if $\vec{0}$ is attainable.
Sketch of proof: Necessity of 2.

all possible trajectories $\{x(\tau_1^k)\}_{k=0,...,\infty}$ are within $\text{Cone}(f)$
Monte Carlo simulations
sampled average of $\text{dist}(z, x(t))$
Conclusions and further questions

- Games with vector payoffs:
  - approachability looked at average payoff
  - attainability looks at total payoff
- necessary and sufficient conditions for attainability
- Still to be done:
  - characterization of attainable sets
  - look at discounted payoff