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AUTOMATIC CONTROL VIA PREISACH HYSTERESIS OF A FILTRATION MODEL WITH CAPILLARITY

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Abstract. We study an automatic control problem for the filtration through an earthy dam, given by controlling the boundary conditions, i.e. the level of the reservoirs. We also study an optimal control problem. The control is realized by an automatic law involving a Preisach operator of hysteresis. The problem is formulated in a weak sense in the framework of Sobolev spaces. Moreover, we study the case when the Preisach hysteresis law collapses to a single thermostatic one.

1 Introduction.

In this paper we study an automatic control problem for a filtration process of a fluid (typically water) through a porous medium (the so-called dam problem). Let Ω be the region occupied by the porous medium, and let u and s be the functions defined on $\Omega \times [0,T]$ representing the pressure of the fluid and the saturation of the porous medium. Our goal is to optimally drive the quantity $f(t) := \int_\Omega s(x,t)\varphi(x)dx$, where φ is a nonnegative functions of $L^1(\Omega)$. The quantity f is a weighted average of the saturation inside Ω, and may then be viewed as an information about the amount of water inside the medium. For instance, if φ is the characteristic function of a given region inside the dam, then f is exactly the amount of water at time t inside that region. To realize the goal, we suppose that we are able to control the boundary datum for the pressure u, which, in physically setting, corresponds (via the hydrostatic pressure) to the levels of the reservoirs surrounding the dam. Moreover, we are interesting in a automatic (feedback) control, in the sense that the evolution of the boundary value for pressure depends itself on the evolution of f. Note that, by its very definition, f depends on the saturation s which, together with the pressure u, depends on the boundary value. The naive idea is the following: if f is too large then the boundary value decreases, and if f is too small then the value increases. To realize such a feedback law, we suppose that the boundary value depends via an hysteresis relationship on f, and in doing that we also use the physical meaning of the boundary datum for pressure, i.e. the hydrostatic pressure of the reservoirs. The hysteresis relationship we are going to use is the one given by the so-called Preisach operator. We recall here that by hysteresis relationship we mean a relation between a time-dependent input and a time-dependent output which presents a particular kind of memory, the so-called rate-independent one.

In the paper [4] (which reports results of [3]), the present author has already studied a similar problem, where the hysteresis relationship is given by the so-called delayed relay (or thermostat). The delayed relay selects only two values, $+1$ and -1 (or "on/off") subject to the evolution of f. Hence, in that case, evolution of the boundary value presents elements of discontinuity in time. The Preisach model is instead a superposition of infinitely many weighted thermostats, which, under general assumption, guarantees the continuity in time. Hence, in this case, the automatic evolution of the boundary datum is more proportionally sensible to even small variations in the evolution of f.

The problem of controlling fluid flow inside porous media is of course of great importance for applications. For some example of mathematical treatments of such problems, see for instance Friedman-Yong [12], Friedman-Yaniro-Yong [11], and, more recently, Kelanemer [16], Barbu-Marinoschi [5].

The introduction of hysteresis effects in the feedback laws for automatic control processes often seems quite natural and also mandatory in order to prevent undesirable effects of the solutions of the problem, such as highly oscillatory behavior. For several examples of mathematical treatments of hysteresis automatic control problems see also Friedman-Hoffmann [10], Brokate-Friedman [6], Colli-Grasselli-Sprekels [9], Grasselli [14], Campbell-Macki [7], Gatti [13], and Cavaterra-Colombo [8].

As in [4], also in this paper, we use the weak formulation of the filtration problem given (and studied) by Alt-Luckhaus-Visintin [2] (see also Alt [1]). Such a weak formulation consists of a variational inequality in some suitable Sobolev spaces, an initial condition for the saturation s and a boundary condition for the pressure u. Of course, some constitutive relations must be added. In particular, the model takes account of a possible capillarity effect for the filtration process. This is done by the use of a constitutive relation for pressure versus saturation which allows strictly positive values of saturation even for negative values of pressure, and moreover by a suitable boundary condition that forces the positive part only of the pressure to be equal to the datum. In [2] the proof of the existence of a weak solution is given (see also [4]).

In Section 3 we address the existence of the solution of the automatic control problem (i.e. the variational inequality with a boundary condition for the pressure which, by the automatic hysteresis law, depends on f). The main tools are approximation with a delayed argument, compactness, continuity, and a suitable approximation of the set of test functions for the variational inequality.

In Otto [17], uniqueness and stability results for the weak formulation of the filtration process are proved. However, the stability result is given only with respect to the initial datum for the saturation, i.e. when the boundary datum for pressure does not change. The main reason for that seems to be the boundary condition (which physically takes account of capillarity effects) which leads to a non good dependence of the set of test functions with respect to convergence of data. It is then clear that the stability result of Otto is not suitable for proving uniqueness for our automatic control problem, where we just automatically act on the boundary. Indeed, uniqueness remains open. However, using also the uniqueness result of Otto, we prove in this paper a comparison result for the solutions of the filtration problem (without automatic control). This result is also interesting by itself, and, together with a order preserving property of the Preisach operator, leads to the remark that a maximal solution of the automatic control problem, as well as a minimal solution, can not exist. That is, for every possible set of more than one solutions, the corresponding functions f, as well as the corresponding reservoirs evolutions h, must transversally intersect infinitely many times. Hence, one can conclude that, at least for small times, the behaviors of solutions are "essentially the same". This argument is described in Section 4.

In Section 5 we address the asymptotic problem when the Preisach operator "converge" to a single relay, i.e. when the measures entering in the definitions of the Preisach operators converge to a Dirac measure. This also may in some sense justify the use of the Preisach operator instead of the single relay, in order to have an approximation with more regularity. However, the good limit hysteresis relation is in this case a generalization (a sort of convexification) of the delayed relay, i.e. a switching rule which also allows output's values in the whole interval $[-1,1]$. That is what in Visintin [19] is called the complete delayed relay.

In Section 6, we study an optimal control problem. The idea is to force f to stay as close as possible to a desired behavior. To do that, we probably have to frequently change the monotonicity of the level of reservoir, which is not so desirable. Hence we introduce a cost functional which takes account of the "distance" of f from the desired behavior and the total variation of the levels. What is natural to suppose be at our disposal is the choice of the hysteresis relationship in the automatic control law. In particular, this is exploited by choosing the measure which enters in the definition of the Preisach operator (i.e. the weight of all delayed relays). We can prove that, among a rather general set of measures, there exists a minimizing for the cost functional. The proof makes also use of the geometric interpretation of the evolution in the Preisach plane. An optimal control problem for a partial differential equation with Preisach hysteresis feedback law, where the aim is to optimally choose the hysteresis, is also studied in Friedman-Hoffmann [10] (see also Brokate-Friedman [6]). Up to our knowledge, this is one of the few times that such a kind of optimization problem is mathematically addressed. Looking to applications and to engineering literature, where, for automatically controlling mechanical systems, an artificial hysteresis is sometimes added (see for instance Tao-Kokotović [18], Hespanha-Liberzon-Morse [15]), it seems that more effort should be done in that direction.

Finally, for the mathematical model of hysteresis and their analysis, especially in connection with partial differential equations, we refer the reader to the book by Visintin [19] (see also the references therein).

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