# Robust optimality of linear saturated control in uncertain linear network flows

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***Abstract*— We propose a novel approach that, given a linear saturated feedback control policy, asks for the objective function that makes robust optimal such a policy. The approach is specialized to a linear network flow system with unknown but bounded demand and politopic bounds on controlled flows. All results are derived via the Hamilton-Jacobi-Isaacs and viscosity theory.**

**Keywords: Optimal control, Robust optimization, Inventory control, Viscosity solutions.**

## I. Introduction

Consider the problem of driving a continuous time state $z(t) \in \mathbb{R}^m$ within a target set $\mathcal{T} = \{\xi \in \mathbb{R}^m : |\xi| \leq \epsilon\}$ in a finite time $T \geq 0$ with $\epsilon \geq 0$ a-priori chosen and keeping the state within $\mathcal{T}$ from time $T$ on. Such a problem is shortly referred to as the $\epsilon$-*stabilizability* problem of $z(t)$. Define $u(t) \in \mathbb{R}^m$ the controlled flow vector, $w(t) \in \mathbb{R}^n$ an *Unknown But Bounded (UBB)* exogenous input (disturbance/demand) with $n < m$, and let $D \in \mathbb{R}^{n\times m}$ a given matrix, $\mathcal{U} = \{\mu \in \mathbb{R}^m : u^- \leq \mu \leq u^+\}$ and $\mathcal{W} = \{\eta \in \mathbb{R}^n : w^- \leq \eta \leq w^+\}$ be two hyper-boxes with assigned $u^+$, $u^-$, $w^+$ and $w^-$. Also, let $\sigma$ be a binary state such that $\sigma(t) = 0$ if $z(t) \notin \mathcal{T}$ and $\sigma(t) = 1$ if $z(t) \in \mathcal{T}$. The robust counterpart of the problem takes on the form

$$\min_{u\in U}\max_{w\in W} J(\zeta, u(.), w(.)) = \int_0^\infty e^{-\lambda(\sigma)t} g^\sigma(z(t), u(t))dt \quad (1)$$

$$\dot{z}(t) = u(t) - Dw(t),\ z(0) = \zeta \quad \text{for all } t \geq 0 \quad (2)$$

$$z(t) \in \mathcal{T} \quad \text{for all } t \geq T, \quad (3)$$

where we denote by $U = \{u : [0, +\infty[\to \mathcal{U}\}$ and by $W = \{w : [0, +\infty[\to \mathcal{W}\}$ the sets of measurable controls and demands respectively. From a game theoretic standpoint we will consider two players, player 1 playing $u$ and player 2 playing $w$. The state $z(t)$ has initial value $\zeta$ and integrates the discrepancy between the controlled flow $u(t)$ and $Dw(t)$ as described in (2). Controls $u(t)$ and demand $w(t)$ are bounded within hyperboxes by their definitions. Condition (3) guarantees the reachability of the target set from time $T$ on. Among all controls satisfying the above conditions (call it admissible controls or solution), we wish to find the one that minimizes the objective function (1) under the worst demand. The objective function is defined on an infinite horizon with discount factor $e^{-\lambda(\sigma)t}$ depending on $\sigma$. The reason for such a dependency on $\sigma$ will be clearer later on. The integrand in (1) is a function of $z$ and $u$ and its structure depends on $\sigma$ as follows

$$g^\sigma(z(t), u(t)) = \begin{cases} \tilde{g}(z(t), u(t)) & \text{if } \sigma = 0 \ (z(t) \notin \mathcal{T}) \\ \hat{g}(z(t), u(t)) & \text{if } \sigma = 1 \ (z(t) \in \mathcal{T}) \end{cases} \quad (4)$$

where $\tilde{g}(.)$ and $\hat{g}(.)$ have to be designed as explained below.

In a previous work [2], it has been shown that under certain conditions on the matrix $D$ (recalled below), the following (linear) saturated control policy drives the state $z$ within $\mathcal{T}$:

$$u(t) = sat_{[u^-,u^+]}(-kz(t)) := \quad (5)$$
$$:= \left(sat_{[u^-,u^+]}(-kz_1(t)), ..., sat_{[u^-,u^+]}(-kz_n(t))\right) \in \mathbb{R}^n,$$

with $k > 0$ and where

$$sat_{[\alpha,\beta]}(\xi_i) = \begin{cases} \beta, & \text{if } \xi_i > \beta, \\ \xi_i, & \text{if } \alpha \leq \xi_i \leq \beta, \\ \alpha, & \text{if } \xi_i < \alpha. \end{cases}$$

Then, we deduce that the saturated control policy returns an admissible solution for problem (1)-(3). In the light of this consideration, we focus on the following problem.

*Problem 1:* We wish to design the integrand $g^\sigma(.)$ of the objective function (1) in (4) such that the saturated control turns optimal for the min-max problem (1)-(3).

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