Pro-p Groups with Few Normal Subgroups

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1. On p-Groups of Finite Coclass

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A p-group G has coclass r if G is of class c and $|G| = p^{c+r}$.

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In 1980 Charles Leedham-Green and Mike Newman came with the five coclass conjectures in decreasing order of difficulty:

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Theorem (Donkin & Leedham-Green, Shalev & Zelmanov)
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The key point (for us) is that pro-p groups of finite coclass are p-adic analytic.

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A point worth noticing: finite rank implies PSG is easy. The other direction is harder.

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4. Pro-p groups of finite coclass are not closed under direct sum.

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Examples:

1. Let $\mathbb{Z}_p$ be the p-adic integers.

$$G_n = SL_d^n(\mathbb{Z}_p) = \ker(SL_d(\mathbb{Z}_p) \to SL_d(\mathbb{Z}_p/(p^n)).$$

$G = G_1$ is a pro-p group, $G_n = \gamma_n(G)$ and

$$|G_n/G_{n+1}| = p^{d^2-1}.$$

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$$J = \left\{t + a_1t^2 + a_2t^3 + \cdots \mid a_i \in \mathbb{F}_p\right\},$$

where the product is by composition.

$$|\gamma_n(J)/\gamma_{n+1}(J)| = \begin{cases} p & n \not\equiv 1 \bmod p-1 \\ p^2 & n \equiv 1 \bmod p-1. \end{cases}$$

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Goal: Find a good definition to avoid all the more difficult examples.

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A Pro-p group G has Polynomial Normal Subgroup Growth (PNSG) if there exists c such that $a_n^{\lhd}(G) \leq n^c$ for all n, where $a_n^{\lhd}(G)$ is the number of normal subgroups of index n.

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There is a soluble counter example, what about just infinite?

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A group G is called r-sandwich if there is r such that for all normal subgroup N of G there exists i such that $\gamma_i(G) \geq N \geq \gamma_{i+r}(G)$.

Theorem 1: Let G be a non-nilpotent pro-p group. Then G has finite obliquity if and only if it is sandwich. Moreover, in such a case, G is just infinite of finite width and has CNSG.

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Theorem 2: Let G be a non-nilpotent pro-p group with CNSG. Then G has a maximal finite normal subgroup K and G/K is just infinite. Moreover, G has finite width.

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Problem 2: Suppose G is hereditarily just infinite pro-p group with CNSG. Is it sandwich?

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1. $\tau(M)$ is an open subgroup of G;
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$$[G : H] > [G : \tau^{-1}(H)].$$

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$$[G : H] > [G : \tau^{-1}(H)].$$

We say that a period is uniform if there is a constant c such that for all H as above,

$$[G : H] = p^c[G : \tau^{-1}(H)].$$

Proposition: If G admits a period it admits a uniform period.

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Theorem 3: Suppose G is a non-abelian just infinite pro-p group which admits a period. Then G is sandwich, in particular it has CNSG. Moreover, there is d such for all big enough n, $a^{\lhd}_{p^n}(G) = a^{\lhd}_{p^{n+d}}(G)$.

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Moreover, every subgroup of finite index of G has all of the above properties too.

In addition, Branch groups and all the other known examples of hereditarily just infinite pro-p groups are all not CNSG.

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2. If G has finite obliquity or CNSG or a period, then every subgroup of finite index of G has finite obliquity or CNSG or a period respectively.

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2. If G has finite obliquity or CNSG or a period, then every subgroup of finite index of G has finite obliquity or CNSG or a period respectively.

3. If G has few normal subgroups , then there exists a constant c such that for all n, $a_n(G) \leq n^{c \log n}$.

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We define the period on J_k by

$$\tau_m(t(1 + f(t))) = t(1 + t^{p^m} f(t)).$$

Lemma: Let $\phi = a(t) \in J$ and $\psi = t + s(t) \in J_k$. Then

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Corollary: For $k \geq p^m$, the map τ_m induces a J-isomorphism from J_k/J_{k+p^m} onto J_{k+p^m}/J_{k+2p^m}.

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The fact that τ_m is a period follows from the sandwich property on the previous slide.