TRENTO, A.A. 2019/20
MATHEMATICS FOR DATA SCIENCE/BIOSTATISTICS
EXERCISE SHEET # 5

Important! In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

Important! Reduce all of the complete matrices in thwe following exercises to RREF forms.

Exercise 5.1. State whether the system in the unknowns x_1, x_2 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 3 \end{cases}$$

Exercise 5.2. State whether the system in the unknowns x_1, x_2 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 = 2 \\ 2x_1 + 2x_2 = 4 \end{cases}$$

Exercise 5.3. State whether the system in the unknowns x_1, x_2, x_3 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 = 2 \\ x_1 + 2x_2 = 3 \end{cases}$$

Exercise 5.4. State whether the system in the unknowns x_1, x_2, x_3 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases}$$

Exercise 5.5. State whether the system in the unknowns x_1, x_2, x_3 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \\ x_1 - x_2 - 3x_3 = 0 \end{cases}$$

Exercise 5.6. State whether the system in the unknowns x_1, x_2, x_3 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 2 \\ x_1 - x_2 - x_3 = 3 \end{cases}$$

Exercise 5.7. State whether the system in the unknowns x_1, x_2, x_3 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 2 \\ x_1 - x_2 - 3x_3 = 3 \end{cases}$$

Exercise 5.8. State whether the system in the unknowns x_1, x_2, x_3, x_4 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 - x_3 - 3x_4 = 1 \\ 3x_1 - 3x_2 - x_3 - 5x_4 = 0. \end{cases}$$

Exercise 5.9. State whether the system in the unknowns x_1, x_2, x_3, x_4 has solutions, and in case find them all.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 - x_3 - 3x_4 = 1 \\ 3x_1 - 3x_2 - x_3 - 5x_4 = 3. \end{cases}$$

Exercise 5.10.

(1) Define
   (a) systems of generators,
   (b) linear dependence,
   (c) linear independence,
   (d) bases.

(2) Show that the following are equivalent.
   (a) $v_1, \dots, v_m$ are linearly dependent, and
   (b) (at least) one of the v_i can be written as a linear combination of the others.

   Show that as a consequence, if $v_1, \dots, v_m$ are a system of generators of a space V, and they are linearly dependent, say v_m is a linear combination of $v_1, \dots, v_{m-1}$, then $v_1, \dots, v_{m-1}$ is a system of generators for V, that is, you can safely drop v_m.

(3) Show that the following are equivalent.
   (a) $v_1, \dots, v_m$ are a basis of the space V, and
   (b) Every element of V can be written *uniquely* as a linear combination of $v_1, \dots, v_m$.

Exercise 5.11. Show that if one of $v_1, \dots, v_m$ is zero, then $v_1, \dots, v_m$ are linearly dependent.

Exercise 5.12. For each of the following sysyem of vectors, state whether they are a basis of $\mathbf{R}^3$. When they are not, state explicitly which condition fails.

(1)

$$\begin{cases} v_1 = (1,0,0) \\ v_2 = (0,1,0) \\ v_3 = (0,0,1) \end{cases}$$

(2)

$$\begin{cases} v_1 = (1,0,0) \\ v_2 = (0,1,0) \end{cases}$$

(3)

$$\begin{cases} v_1 = (1,-1,0) \\ v_2 = (0,1,-1) \\ v_3 = (-1,0,1) \end{cases}$$

(4)

$$\begin{cases} v_1 = (1,-1,0) \\ v_2 = (0,1,-1) \\ v_3 = (1,0,1) \end{cases}$$