TRENTO, A.A. 2019/20
MATHEMATICS FOR DATA SCIENCE/BIOSTATISTICS
EXERCISE SHEET # 10

Important! In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

Notice! For technical reasons, sometimes in the following exercises we write something like "$e_1 + (-1)e_2$", which just means $e_1 - e_2$.

Exercise 10.1. Let $\mathcal{M}_{2\times 2}$ be the vector space of 2×2 real matrices.
We recall that $\dim(V) = 4$.

- Prove that the matrices
$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$
are linearly independent.
- Extend A, B to a basis of $\mathcal{M}_{2\times 2}$.

Exercise 10.2. Let $\mathcal{M}_{2\times 3}$ be the vector space of 2×3 real matrices.
We recall that $\dim(V) = 6$.

- Prove that the matrices
$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
are linearly independent.
- Extend A, B, C to a basis of $\mathcal{M}_{2\times 3}$.

Exercise 10.3. Let $\mathcal{M}_{2\times 2}$ be the vector space of 2×2 real matrices.
Let $V = span\{A, B, C, D\}$, where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Extract from A, B, C, D a basis of V.
Is $V = \mathcal{M}_{2\times 2}$?

Exercise 10.4. Let V be a vector space of dimension 2, and let e_1, e_2 be a base of V.
Consider the linear function $f : V \to V$ given by

$$\begin{cases} f(e_1) = e_1 + e_2 \\ f(e_2) = (-1)e_1 + 2e_2 \end{cases}$$

Write the matrix A of f with respect to the base e_1, e_2.

Consider the vectors

$$\begin{cases} g_1 = e_1 + (-1)e_2, \\ g_2 = 2e_1 + e_2. \end{cases}$$

(1) Show that g_1, g_2 are a base of V,
(2) write e_1, e_2 as linear combinations of g_1, g_2, and
(3) write the matrix B of f with respect to the base g_1, g_2.

Exercise 10.5. Let V be a vector space of dimension 2, and let e_1, e_2 be a base of V.

Consider the linear function $f : V \to V$ given by

$$\begin{cases} f(e_1) = 5e_1 + 2e_2 \\ f(e_2) = (-12)e_1 + (-5)e_2 \end{cases}$$

Write the matrix A of f with respect to the base e_1, e_2.

Consider the vectors

$$\begin{cases} g_1 = 3e_1 + e_2, \\ g_2 = 2e_1 + e_2. \end{cases}$$

(1) Show that g_1, g_2 are a base of V,
(2) write e_1, e_2 as linear combinations of g_1, g_2, and
(3) write the matrix B of f with respect to the base g_1, g_2.

Exercise 10.6. Let V be a vector space of dimension 2, and let e_1, e_2 be a base of V.

Consider the linear function $f : V \to V$ given by

$$\begin{cases} f(e_1) = (-4)e_1 + 10e_2 \\ f(e_2) = (-3)e_1 + 7e_2 \end{cases}$$

Write the matrix A of f with respect to the base e_1, e_2.

Consider the vectors

$$\begin{cases} g_1 = 3e_1 + (-5)e_2, \\ g_2 = (-1)e_1 + 2e_2. \end{cases}$$

(1) Show that g_1, g_2 are a base of V,
(2) write e_1, e_2 as linear combinations of g_1, g_2, and
(3) write the matrix B of f with respect to the base g_1, g_2.

Exercise 10.7. Use the rules of Laplace and Sarrus to compute some of the determinants of sheet #8.

Exercise 10.8. Let V be a vector space of dimension 3, and let e_1, e_2, e_3 be a base of V.

Consider the linear function $f : V \to V$ given by

$$\begin{array}{rcccccc} f(e_1) & = & e_1 & + & e_2 & & \\ f(e_2) & = & & & e_2 & + & e_3 \\ f(e_3) & = & e_1 & & & - & e_3 \end{array}$$

Write the matrix A of f with respect to the base e_1, e_2, e_3.

Consider the vectors

$$\begin{array}{rcl} g_1 &=& e_1, \\ g_2 &=& e_1 - e_2 - e_3, \\ g_3 &=& e_3. \end{array}$$

(1) Show that g_1, g_2, g_3 are a base of V,
(2) write e_1, e_2, e_3 as linear combinations of g_1, g_2, g_3, and
(3) write the matrix B of f with respect to the base g_1, g_2, g_3.