

TRENTO, A.A. 2019/20
MATHEMATICS FOR DATA SCIENCE/BIOSTATISTICS
EXERCISE SHEET # 11

Important! In solving the exercises

- explain what you are doing,
- explain why you are doing what you are doing, and
- spell out all intermediate steps.

Notice! For technical reasons, sometimes in the following exercises we write something like “ $e_1 + (-1)e_2$ ”, which just means $e_1 - e_2$.

Exercise 11.1. Define eigenvalues and eigenvectors.

Show that if v is an eigenvector for the linear function f with respect to the eigenvalue λ , and $a \neq 0$ is a number, then av is also an eigenvector for f with respect to the eigenvalue λ .

Exercise 11.2. Let V be a vector space of dimension 2, and let e_1, e_2 be a base.

Consider the linear function $f : V \rightarrow V$ given by

$$\begin{cases} f(e_1) = 5e_1 + 2e_2 \\ f(e_2) = (-12)e_1 + (-5)e_2 \end{cases}$$

Write the matrix A of f with respect to the base e_1, e_2 .

Compute the characteristic polynomial (in the variable λ)

$$\det(A - \lambda I),$$

where I is the 2×2 identity matrix. Find the eigenvalues λ_1, λ_2 if A as the solutions to the equation

$$\det(A - \lambda I) = 0.$$

For each eigenvalue λ_i , find an eigenvector g_i . Show that g_1, g_2 is a base of V , and write the matrix B of f with respect to the base g_1, g_2 .

Exercise 11.3. Let V be a vector space of dimension 2, and let e_1, e_2 be a base.

Consider the linear function $f : V \rightarrow V$ given by

$$\begin{cases} f(e_1) = (-4)e_1 + 10e_2 \\ f(e_2) = (-3)e_1 + 7e_2 \end{cases}$$

Write the matrix A of f with respect to the base e_1, e_2 .

Compute the characteristic polynomial (in the variable λ)

$$\det(A - \lambda I),$$

where I is the 2×2 identity matrix. Find the eigenvalues λ_1, λ_2 if A as the solutions to the equation

$$\det(A - \lambda I) = 0.$$

For each eigenvalue λ_i , find an eigenvector g_i . Show that g_1, g_2 is a base of V , and write the matrix B of f with respect to the base g_1, g_2 .

Exercise 11.4. Let V be a vector space of dimension n , and let $f : V \rightarrow V$ be a linear function. Suppose V has a base e_1, \dots, e_n consisting of eigenvectors of f , that is, there are numbers $\lambda_1, \dots, \lambda_n$ such that

$$f(e_i) = \lambda_i e_i$$

for $i = 1, \dots, n$.

Show that the matrix of f with respect to the base e_1, \dots, e_n is

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

Exercise 11.5. Compute the characteristic polynomial of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and show that A does not have real eigenvalues.

Exercise 11.6. Compute the characteristic polynomial of

$$A = \begin{bmatrix} -9 & -4 \\ 25 & 11 \end{bmatrix}.$$

- (1) Show that the roots of the characteristic polynomial are 1, 1.
- (2) Show that the eigenvectors with respect to the eigenvalue 1 form a subspace of dimension 1 (minus the zero vector).
- (3) If g_1 is one of these eigenvectors, choose arbitrarily a vector g_2 such that g_1, g_2 is a base, and write the matrix of A with respect to this base.

Exercise 11.7. For each of the following matrices A ,

- (1) compute the characteristic polynomial,
- (2) compute the eigenvalues,
- (3) state whether A is diagonalisable or not.
 - (a) If it is not diagonalisable, please stop.
 - (b) If it is diagonalisable, find a base of eigenvectors, and a matrix S such that $S^{-1}AS$ is diagonal.

$$\begin{bmatrix} 5 & 3 \\ -4 & -2 \end{bmatrix}, \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 5/2 & -1/2 & -1 \\ 1/2 & 3/2 & -1 \\ -1/2 & -1/2 & 2 \end{bmatrix}, \begin{bmatrix} 3/2 & 0 & -1/2 \\ 1/2 & 1 & -1/2 \\ -1/2 & 0 & 3/2 \end{bmatrix}, \\ \begin{bmatrix} 2 & -1/2 & -1/2 \\ 1 & 1/2 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix}, \begin{bmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 4 & 2 \\ -1 & -1 & -1 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 4 & -6 \\ -4 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}.$$

If at some point while working out the exercise you are able to say that the matrix is diagonalisable (what do I know, perhaps the eigenvalues are distinct), then spell it out!

Exercise 11.8. Define diagonalisable matrices.

Exercise 11.9. Define the algebraic and geometric multiplicity of an eigenvalue.