Metric nilpotent Lie algebras of dimension 5

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16-17.06.2017, GTG, University of Trento

Preliminaries

Definition

Let $\mathfrak{g}$ be a Lie algebra and G be the corresponding connected and simply connected Lie group. A metric Lie algebra $(\mathfrak{g}, \langle ., . \rangle)$ is a Lie algebra $\mathfrak{g}$ together with a Euclidean inner product $\langle ., . \rangle$ on $\mathfrak{g}$. This inner product $\langle ., . \rangle$ on $\mathfrak{g}$ induces a left invariant Riemannian metric on the Lie group G in a natural way.
Let $(\mathfrak{n}, \langle ., . \rangle)$ be a nilpotent metric Lie algebra. The corresponding nilpotent Lie group N endowed with the left-invariant metric arising from $\langle ., . \rangle$ is a Riemannian nilmanifold.

Denote $\mathcal{OA}(\mathfrak{n})$ the group of orthogonal automorphisms of $(\mathfrak{n}, \langle ., . \rangle)$, which preserve the inner product.
E. Wilson in *Isometry groups on homogeneous nilmanifolds*, Geom. Dedicata **12** (1982), 337-346, has been proved that the group $\mathcal{I}(N)$ of isometries of $(N, \langle ., . \rangle)$ (distance preserving bijections) is the semi-direct product $\mathcal{OA}(\mathfrak{n}) \ltimes N$ of the group $\mathcal{OA}(\mathfrak{n})$ and the group N itself, which is normal in the group of isometries.

E. Wilson described a classification procedure for the isometry equivalence classes of Riemannian nilmanifolds. This is applied by J. Lauret in *Homogeneous Nilmanifolds of Dimensions* 3 *and* 4, Geometriae Dedicata 68, (1997), 145-155, for the determination of the 3- and 4-dimensional Riemannian nilmanifolds up to isometry and their isometry groups. Sz. Homolya and O. Kowalski have classified in *Simply connected two-step homogeneous nilmanifolds of dimension* 5, Note Math. **26** (2006), 69-77, all 5-dimensional 2-step nilpotent Riemannian nilmanifolds and their isometry groups.

Together with P.T. Nagy we want to determine explicitly all 5-dimensional non 2-step Riemannian nilmanifolds and the groups of their isometries.

A subalgebra $\mathfrak{h}$ of a metric Lie algebra $\mathfrak{g}$ is totally geodesic if for all $Y, Z \in \mathfrak{h}$, $X \in \mathfrak{h}^\perp$ one has

$$\langle [X, Y], Z \rangle + \langle [X, Z], Y \rangle = 0.$$

This definition is chosen so that the corresponding Lie subgroup H of $\mathfrak{h}$ is a totally geodesic submanifold relative to the left invariant Riemannian metric defined by the inner product on the simply connected Lie group G of $\mathfrak{g}$. This notion is motivated by the fact that the left cosets of H define a totally geodesic foliation on G.

P.T. Nagy and Sz. Homolya in *Geodesic vectors and subalgebras in two-step nilpotent metric Lie algebras*, Adv. Geometry **15** (2015) have been proved that for 2-step nilpotent metric Lie algebras the linear structure of flat totally geodesic subalgebras does not depend on the choice of the inner product only on the isomorphism class of the nilpotent Lie algebra $\mathfrak{n}$.

Moreover, in G. Cairns, A. Hinić Galić, Yu. Nikolayevsky, *Totally geodesic subalgebras of filiform nilpotent Lie algebras*, J. Lie Theory, 23 (2013) the authors determine the maximal dimension of totally geodesic subalgebras of filiform nilpotent metric Lie algebras and show that this bound is attained.

Together with P.T. Nagy we wish to determine the standard filiform metric Lie algebras.

Canonical basis of non two-step nilpotent Lie algebras of dimension 5

We consider the non two-step nilpotent Lie algebras of dimension 5 which are not direct products of Lie algebras of lower dimension. According to W. A. de Graaf: *Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2*, J. Algebra **309** (2007), 640 – 653, the list of these Lie algebras is given up to isomorphism by the non-vanishing commutators with respect to a distinguished basis $\{E_1, E_2, \dots\}$:

$$\begin{aligned}
\mathfrak{l}_{5,5}: &\quad [E_1, E_2] = E_4, \; [E_1, E_4] = E_5, \; [E_2, E_3] = E_5;\\
\mathfrak{l}_{5,6}: &\quad [E_1, E_2] = E_3, \; [E_1, E_3] = E_4, \; [E_1, E_4] = E_5,\\
&\quad [E_2, E_3] = E_5;\\
\mathfrak{l}_{5,7}: &\quad [E_1, E_2] = E_3, \; [E_1, E_3] = E_4, \; [E_1, E_4] = E_5;\\
\mathfrak{l}_{5,9}: &\quad [E_1, E_2] = E_3, \; [E_1, E_3] = E_4, \; [E_2, E_3] = E_5.
\end{aligned}$$

This basis we call the *canonical basis* of the corresponding Lie algebra.

Heuristic procedure for the classification of metric Lie algebras

We use the following heuristic procedure for the classification of metric Lie algebras up to isometric isomorphisms:

1. Select a basis $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$ of the Lie algebra $\mathfrak{g}$, such that the commutation relations have a simple form.
2. For an inner product $\langle ., . \rangle$ on $\mathfrak{g}$ let $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ be the orthonormal basis of the form $F_i = \sum_{k=i}^{n} a_{ik} E_k$ with $a_{ik} \in \mathbb{R}$ and $a_{ii} > 0$ obtained from $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$ by the Gram-Schmidt process. These bases parametrize the inner products on $\mathfrak{g}$.
3. Compute the Lie bracket expressions with respect to the basis $\mathcal{F}$ and find their general shapes depending on real parameters. Find for each Lie bracket operation given by these real parameters all possible inner products. The class of these inner products determines the class of isometric isomorphic metric Lie algebras.

1. Choose for any class of isometric isomorphic metric Lie algebras a representing inner product expressed by its orthonormal basis $\mathcal{F}$. These inner products together with the Lie bracket operation give a classification of metric Lie algebras.

We illustrate this classification method on the Lie algebra $\mathfrak{l}_{5,5}$:
$[E_1, E_2] = E_4$, $[E_1, E_4] = E_5$, $[E_2, E_3] = E_5$.
We find the following series $\mathfrak{i}_5 < \mathfrak{i}_4 < \cdots < \mathfrak{i}_1 < \mathfrak{l}_{5,5} = \mathfrak{i}_0$ of ideals of $\mathfrak{l}_{5,5}$ with $\dim(\mathfrak{i}_k/\mathfrak{i}_{k+1}) = 1$: the center $Z(\mathfrak{l}_{5,5})$ of $\mathfrak{l}_{5,5}$ is $\mathbb{R}\,E_5$, the commutator subalgebra $\mathfrak{l}'_{5,5}$ is $\mathbb{R}\,E_4 + \mathbb{R}\,E_5$. The preimage $\pi^{-1}(Z(\mathfrak{l}_{5,5}/Z(\mathfrak{l}_{5,5})))$ of the center $Z(\mathfrak{l}_{5,5}/Z(\mathfrak{l}_{5,5}))$ of the factor algebra $\mathfrak{l}_{5,5}/Z(\mathfrak{l}_{5,5})$ in $\mathfrak{l}_{5,5}$ is $\mathbb{R}\,E_3 + \mathbb{R}\,E_4 + \mathbb{R}\,E_5$ and the centralizer $C_{\mathfrak{l}_{5,5}}(\mathfrak{l}'_{5,5})$ of $\mathfrak{l}'_{5,5}$ is $\mathbb{R}\,E_2 + \mathbb{R}\,E_3 + \mathbb{R}\,E_4 + \mathbb{R}\,E_5$. The Gram-Schmidt process applied to the ordered canonical basis $(E_5, E_4, E_3, E_2, E_1)$ yields an orthonormal basis $\{F_1, F_2, F_3, F_4, F_5\}$ of $\mathfrak{l}_{5,5}$, where the vector F_i is a positive multiple of E_i modulo the subspace $\langle E_j; j > i\rangle$ generated by $\{E_j; j > i\}$ and orthogonal to $\langle E_j; j > i\rangle$.

We have $F_5 \in Z(\mathfrak{l}_{5,5})$, $F_4 \in \mathfrak{l}'_{5,5}$, $F_3 \in \pi^{-1}(Z(\mathfrak{l}_{5,5}/Z(\mathfrak{l}_{5,5})))$ and $F_2 \in C_{\mathfrak{l}_{5,5}}(\mathfrak{l}'_{5,5})$. Hence $\mathfrak{l}_{5,5}$ has the orthogonal direct sum decomposition $\mathfrak{l}_{5,5} = \mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_5$ into one-dimensional subspaces $\mathbb{R}F_1, \ldots, \mathbb{R}F_5$ which is determined uniquely by the algebraic and metric structure of $(\mathfrak{l}_{5,5}, \langle ., . \rangle)$.
Since $F_i = \sum_{k=i}^{5} a_{ik} E_k$ with $a_{ii} > 0$ we have
(1)
$[F_1, F_2] = aF_4 + bF_5,\ [F_1, F_3] = cF_5,\ [F_1, F_4] = dF_5,\ [F_2, F_3] = fF_5,$

$a, d, f > 0,\ \ a, b, c, d, f \in \mathbb{R}$, where

$$a = \frac{a_{11}a_{22}}{a_{44}}, \quad b = \frac{a_{44}\left(a_{11}a_{24} + a_{12}a_{23} - a_{13}a_{22}\right) - a_{11}a_{22}a_{45}}{a_{44}a_{55}},$$

$$c = \frac{a_{11}a_{34} + a_{12}a_{33}}{a_{55}}, \quad d = \frac{a_{11}a_{44}}{a_{55}}, \quad f = \frac{a_{22}a_{33}}{a_{55}}.$$

Definition

Let $\{G_1, G_2, G_3, G_4, G_5\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^5$ and a, b, c, d, f real numbers with $a, d, f \neq 0$. Let $\mathfrak{n}_{5,5}(a, b, c, d, f)$ denote the metric Lie algebra defined on $\mathbb{E}^5$ by the non-vanishing commutators

$$[G_1, G_2] = aG_4 + bG_5, [G_1, G_3] = cG_5, [G_1, G_4] = dG_5, [G_2, G_3] = fG_5.$$

The map $E_1 \mapsto E_1$, $E_2 \mapsto adE_2 + bE_4$, $E_3 \mapsto \frac{f}{ad}E_3 + cE_4$, $E_4 \mapsto dE_4$, $E_5 \mapsto E_5$ is an isomorphism $\mathfrak{l}_{5,5} \to \mathfrak{n}_{5,5}(a, b, c, d, f)$.

Changing the orthonormal basis: $\tilde{F}_1 = -F_1$, $\tilde{F}_2 = -F_2$, $\tilde{F}_3 = F_3$, $\tilde{F}_4 = F_4$, $\tilde{F}_5 = -F_5$ we obtain

$$[\tilde{F}_1, \tilde{F}_2] = a\tilde{F}_4 - b\tilde{F}_5, \ [\tilde{F}_1, \tilde{F}_3] = c\tilde{F}_5, \ [\tilde{F}_1, \tilde{F}_4] = d\tilde{F}_5, \ [\tilde{F}_2, \tilde{F}_3] = f\tilde{F}_5.$$

Similarly, with the change of the basis: $\tilde{F}_1 = F_1$, $\tilde{F}_2 = -F_2$, $\tilde{F}_3 = F_3$, $\tilde{F}_4 = -F_4$, $\tilde{F}_5 = -F_5$ one has

$$[\tilde{F}_1, \tilde{F}_2] = a\tilde{F}_4 + b\tilde{F}_5, [\tilde{F}_1, \tilde{F}_3] = -c\tilde{F}_5, [\tilde{F}_1, \tilde{F}_4] = d\tilde{F}_5, [\tilde{F}_2, \tilde{F}_3] = f\tilde{F}_5.$$

Hence there is an orthonormal basis such that in the commutators (1) the coefficients b and c are non-negative.

Theorem

Let $\langle ., . \rangle$ be an inner product on the 5-dimensional three-step nilpotent Lie algebra $\mathfrak{l}_{5,5}$. There is a unique metric Lie algebra $\mathfrak{n}_{5,5}(a, b, c, d, f)$ with $a, d, f > 0$, $b, c \geq 0$, which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{5,5}, \langle ., . \rangle)$.

It turns out that up to one exceptional class all higher-step nilpotent metric Lie algebras of dimension 5 have such orthogonal direct sum decomposition:

Definition

Let $(\mathfrak{g}, \langle ., . \rangle)$ be a metric Lie algebra of dimension n. An orthogonal direct sum decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_n$ on one-dimensional subspaces $V_1, \ldots, V_n$ will be called a framing of $(\mathfrak{g}, \langle ., . \rangle)$, if it is determined uniquely by the algebraic and metric structure of $(\mathfrak{g}, \langle ., . \rangle)$. The metric Lie algebra $(\mathfrak{g}, \langle ., . \rangle)$ is called framed, if it has a framing.

Let $(\mathfrak{g}, \langle ., . \rangle)$ and $(\mathfrak{g}^*, \langle ., . \rangle^*)$ be framed metric Lie algebras of dimension n with framings $\mathfrak{g} = \mathbb{R}\, e_1 \oplus \cdots \oplus \mathbb{R}\, e_n$ and $\mathfrak{g}^* = \mathbb{R}\, e_1^* \oplus \cdots \oplus \mathbb{R}\, e_n^*$, where $(e_1, \ldots, e_n)$, respectively $(e_1^*, \ldots, e_n^*)$ are orthonormal bases.
If $\Phi : \mathfrak{g} \to \mathfrak{g}^*$ is an isometric isomorphism, it maps $\mathbb{R}\, e_i \to \mathbb{R}\, e_i^*$, $i = 1, \ldots, n$, i.e. $\Phi(e_i) = \varepsilon_i e_i^*$ with $\varepsilon_i = \pm 1$. Hence

$$\Phi[e_i, e_j] = \Sigma_{k=1}^n c_{i,j}^k \varepsilon_k e_k^* = [\Phi(e_i), \Phi(e_j)]^* = \varepsilon_i \varepsilon_j \Sigma_{k=1}^n {c^*}_{i,j}^k e_k^*,$$

consequently $|c_{i,j}^k| = |{c^*}_{i,j}^k|$.

Lemma

Assume that the commutators $[.,.]$ in $\mathfrak{g}$ and $[.,.]^*$ in $\mathfrak{g}^*$ are given in the form $[e_i, e_j] = \Sigma_{k=1}^n c_{i,j}^k e_k$, $[e_i^*, e_j^*]^* = \Sigma_{k=1}^n {c^*}_{i,j}^k e_k^*$, $i, j, k = 1, \ldots, n$. Then one has $c_{i,j}^k = \pm {c^*}_{i,j}^k$ for all $i, j, k = 1, \ldots, n$. Particularly, if $c_{i,j}^k \geq 0$ and ${c^*}_{i,j}^k \geq 0$ then $c_{i,j}^k = {c^*}_{i,j}^k$.

Theorem

The connected component of the isometry group $\mathcal{I}(N)$ of a simply connected Riemannian nilmanifold $(N, \langle ., . \rangle)$ corresponding to the framed metric Lie algebra $(\mathfrak{n}, \langle ., . \rangle)$ is isomorphic to the Lie group N.

Proof.

Lemma 1 yields that any orthogonal automorphism $\Phi : \mathfrak{n} \to \mathfrak{n}$ is given by $\Phi(e_i) = \varepsilon_i e_i$ with $\varepsilon_i = \pm 1$. Hence the group $\mathcal{OA}(\mathfrak{n})$ of orthogonal automorphisms of $(\mathfrak{n}, \langle ., . \rangle)$ is a subgroup of $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where the number of factors $\leq \dim \mathfrak{n}$. Since $\mathcal{I}(N) \cong N \rtimes \mathcal{OA}(\mathfrak{n})$ the assertion follows. □

Hence the Lie algebra $\mathfrak{n}_{5,5}(a, b, c, d, f)$, $a > 0$, $b \geq 0$, $c \geq 0$, $d > 0$, $f > 0$ is isometrically isomorphic to $\mathfrak{n}_{5,5}(a^*, b^*, c^*, d^*, f^*)$, $a^* > 0$, $b^* \geq 0$, $c^* \geq 0$, $d^* > 0$, $f^* > 0$ precisely if $a = a^*$, $b = b^*$, $c = c^*$, $d = d^*$, $f = f^*$.

$\mathfrak{n}_{5,5}(a,b,c,d,f)$: $[G_1,G_2]=aG_4+bG_5$, $[G_1,G_3]=cG_5$, $[G_1,G_4]=dG_5$, $[G_2,G_3]=fG_5$.
If $T:\mathfrak{n}_{5,5}(a,b,c,d,f)\to\mathfrak{n}_{5,5}(a,b,c,d,f)$ is an orthogonal automorphism of $\mathfrak{n}_{5,5}(a,b,c,d,f)$ then $TG_i=\varepsilon_i G_i$ and $[TG_i,TG_j]=[\varepsilon_i G_i,\varepsilon_j G_j]=T[G_i,G_j]$ for $i,j=1,\dots,5$, where $\varepsilon_i,\varepsilon_j=\pm 1$.
Let $b=c=0$. From the Lie brackets $[\varepsilon_1 G_1,\varepsilon_2 G_2]=a\varepsilon_4 G_4$, $[\varepsilon_1 G_1,\varepsilon_4 G_4]=d\varepsilon_5 G_5$, $[\varepsilon_2 G_2,\varepsilon_3 G_3]=f\varepsilon_5 G_5$ we obtain $\varepsilon_4=\varepsilon_1\varepsilon_2$, $\varepsilon_5=\varepsilon_1\varepsilon_4=\varepsilon_2\varepsilon_3$. Hence $\varepsilon_2=\varepsilon_5=\varepsilon_2\varepsilon_3$, and $\varepsilon_3=1$, $\varepsilon_1\varepsilon_4=\varepsilon_2=\varepsilon_5$. It follows that the group of orthogonal automorphisms of $\mathfrak{n}_{(a,0,0,d,f)}$ is isomorphic to the group $\mathbb{Z}_2\times\mathbb{Z}_2$.
If $b=0$, $c>0$, then we have in addition $[\varepsilon_1 G_1,\varepsilon_3 G_3]=c\varepsilon_5 G_5$, which yields that $\varepsilon_1\varepsilon_3=\varepsilon_5$. Hence we get $\varepsilon_1=\varepsilon_2=\varepsilon_5$, $\varepsilon_3=1=\varepsilon_4$. The group of orthogonal automorphisms of $\mathfrak{n}_{(a,0,c,d,f)}$ is isomorphic to the group $\mathbb{Z}_2$.

Theorem

Let $\langle .,. \rangle$ be an inner product on the 5-dimensional three-step nilpotent Lie algebra $\mathfrak{l}_{5,5}$.

(1) *There is a unique metric Lie algebra $\mathfrak{n}_{5,5}(a,b,c,d,f)$ with $a,d,f > 0$, $b,c \geq 0$, which is isometrically isomorphic to the metric Lie algebra $(\mathfrak{l}_{5,5}, \langle .,. \rangle)$.*

(2) *The group of orthogonal automorphisms of $\mathfrak{n}_{5,5}(a,b,c,d,f)$ is isomorphic to the matrix group:*

1. *for $b = c = 0$:*

(2)

$$\left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_1\varepsilon_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_4 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1\varepsilon_4 \end{pmatrix}, \quad \varepsilon_1,\ \varepsilon_4 = \pm 1 \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

1. for $b = 0,\ c > 0$:

$$(3)\qquad \left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 \end{pmatrix}, \qquad \varepsilon_1 = \pm 1 \right\} \cong \mathbb{Z}_2,$$

2. for $b > 0,\ c = 0$:

$$(4)\qquad \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2 \end{pmatrix}, \qquad \varepsilon_2 = \pm 1 \right\} \cong \mathbb{Z}_2,$$

3. if $b > 0,\ c > 0$, then it is trivial

with respect to the basis $\{G_1, G_2, G_3, G_4, G_5\}$.

The Lie algebra $\mathfrak{l}_{5,9}$ having two-dimensional center

Definition

Let $\{G_1, G_2, G_3, G_4, G_5\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^5$. Let $\mathfrak{n}_{5,9}(l, m, n, p, q)$, $l, m, n, p, q \in \mathbb{R}$ with $l, p, q \neq 0$, denote the metric Lie algebra defined on $\mathbb{E}^5$ by the non-vanishing commutators

$$(5)\quad [G_1, G_2] = lG_3 + mG_4 + nG_5,\ [G_1, G_3] = pG_4,\ [G_2, G_3] = qG_5.$$

If $n = 0$, $p = q$ we denote $\tilde{\mathfrak{n}}_{5,9}(l, m, p) = \mathfrak{n}_{5,9}(l, m, 0, p, p)$. The map $G_1 \mapsto G_1 + \frac{n}{q} G_3$, $G_2 \mapsto G_2 - \frac{m}{p} G_3$, $G_3 \mapsto lG_3$, $G_4 \mapsto plG_4$, $G_5 \mapsto qlG_5$ is an isomorphism $\mathfrak{n}_{5,9}(l, m, n, p, q)$ to $\mathfrak{l}_{5,9}$.

Theorem

The metric Lie algebra $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ is isometrically isomorphic to a unique $\mathfrak{n}_{5,9}(l, m, n, p, q)$ with $l, m, n, p, q \in \mathbb{R}$ such that $l > 0$, $q > p > 0$ and $m, n \geq 0$, or to a unique $\tilde{\mathfrak{n}}_{5,9}(l, m, p)$ with $l, m, p \in \mathbb{R}$ such that $l, p > 0$ and $m \geq 0$.

The Lie algebra $\mathfrak{l}_{5,9}$ is defined by: $[E_1, E_2] = E_3$, $[E_1, E_3] = E_4$, $[E_2, E_3] = E_5$.
The center $Z(\mathfrak{l}_{5,9})$ of $\mathfrak{l}_{5,9}$ is $\mathbb{R}\, E_4 + \mathbb{R}\, E_5$ and the commutator subalgebra $\mathfrak{l}'_{5,9}$ is $\mathbb{R}\, E_3 + \mathbb{R}\, E_4 + \mathbb{R}\, E_5$. Let be given an inner product $\langle .,. \rangle$ on $\mathfrak{l}_{5,9}$ and apply the Gram-Schmidt process to the ordered canonical basis $(E_5, E_4, E_3, E_2, E_1)$ of $\mathfrak{l}_{5,9}$. We obtain an orthonormal basis $E_i^* = \sum_{k=i}^{5} a_{ik} E_k$, $i = 5, \dots, 1$ of $\mathfrak{l}_{5,9}$ with $a_{ii} > 0$ such that

$$[E_1^*, E_2^*] = lE_3^* + mE_4^* + nE_5^*,\ [E_1^*, E_3^*] = pE_4^* + rE_5^*,\ [E_2^*, E_3^*] = qE_5^*,$$

where $l = \frac{a_{11} a_{22}}{a_{33}} > 0$, $p = \frac{a_{11} a_{33}}{a_{44}} > 0$, $q = \frac{a_{22} a_{33}}{a_{55}} > 0$.

To distinguish a 1-dimensional subspace of the center $Z(\mathfrak{l}_{5,9}) = \langle E_4^*, E_5^* \rangle$ and a 1-dimensional subspace of the $(\mathfrak{l}'_{5,9})^\perp = \langle E_1^*, E_2^* \rangle$ we consider the orthogonal unit vectors $F(t)$, $F(t+\frac{\pi}{2})$ in $(\mathfrak{l}'_{5,9})^\perp$:

$$F(t) = \cos tE_1^* + \sin tE_2^*, \quad F(t+\frac{\pi}{2}) = -\sin tE_1^* + \cos tE_2^*.$$

We have $[F(t), F(t+\frac{\pi}{2})] = [E_1^*, E_2^*]$ and $\Phi(t) = [F(t), E_3^*] = \frac{a_{11}a_{33}}{a_{44}} \cos tE_4^* +$

$$\frac{a_{33}}{a_{55}} \left[\frac{a_{12}a_{44} - a_{11}a_{45}}{a_{44}} \cos t + a_{22} \sin t \right] E_5^*,$$

and $\Phi(t+\frac{\pi}{2}) = [F(t+\frac{\pi}{2}), E_3^*] = -\frac{a_{11}a_{33}}{a_{44}} \sin tE_4^* +$

$$\frac{a_{33}}{a_{55}} \left[\frac{-a_{12}a_{44} + a_{11}a_{45}}{a_{44}} \sin t + a_{22} \cos t \right] E_5^*.$$

$\Phi(t_0)$ and $\Phi(t_0 + \frac{\pi}{2})$ are orthogonal if

$$\frac{1}{2}\left(a_{22}^2 a_{44}^2 - a_{11}^2 a_{55}^2 - (a_{12}a_{44} - a_{11}a_{45})^2\right)\sin 2t_0 +$$

$$a_{22}a_{44}\left(a_{12}a_{44} - a_{11}a_{45}\right)\cos 2t_0 = 0.$$

Lemma

The vectors $\Phi(t)$ and $\Phi(t + \frac{\pi}{2})$ are orthogonal for any $t \in \mathbb{R}$ if and only if

$$a_{12}a_{44} = a_{11}a_{45} \quad \text{and} \quad a_{11}a_{55} = a_{22}a_{44}. \tag{6}$$

Otherwise, there exists a unique $0 \leq t_0 < \frac{\pi}{2}$ such that $\Phi(t)$ and $\Phi(t + \frac{\pi}{2})$ are orthogonal if and only if $t = t_0 + k\frac{\pi}{2}$, $k \in \mathbb{Z}$.

In the second case we have either $\| \Phi(t_0) \| < \| \Phi(t_0 + \frac{\pi}{2}) \|$ and hence we define

$$F_1 = F(t_0),\ F_2 = F(t_0 + \frac{\pi}{2}),\ F_3 = E_3^*,\ F_4 = \frac{\Phi(t_0)}{\| \Phi(t_0) \|},$$

$$F_5 = \frac{\Phi(t_0 + \frac{\pi}{2})}{\| \Phi(t_0 + \frac{\pi}{2}) \|}, \tag{7}$$

or if $\| \Phi(t_0) \| = \| \Phi(t_0 + \pi) \| > \| \Phi(t_0 + \frac{\pi}{2}) \|$, then we define

$$F_1 = F(t_0 + \frac{\pi}{2}),\ F_2 = F(t_0 + \pi),\ F_3 = E_3^*,\ F_4 = \frac{\Phi(t_0 + \frac{\pi}{2})}{\| \Phi(t_0 + \frac{\pi}{2}) \|},$$

$$F_5 = \frac{\Phi(t_0 + \pi)}{\| \Phi(t_0 + \pi) \|}. \tag{8}$$

From this construction we obtain that

$$(9) \quad [F_1, F_2] = lF_3 + mF_4 + nF_5, \ [F_1, F_3] = pF_4, \ [F_2, F_3] = qF_5$$

with $l > 0$, $q > p > 0$. Using the basis change $\tilde{F}_1 = -F_1$, $\tilde{F}_2 = F_2$, $\tilde{F}_3 = -F_3$, $\tilde{F}_4 = F_4$, $\tilde{F}_5 = -F_5$ one has $[\tilde{F}_1, \tilde{F}_2] = l\tilde{F}_3 - m\tilde{F}_4 + n\tilde{F}_5$, $[\tilde{F}_1, \tilde{F}_3] = p\tilde{F}_4$, $[\tilde{F}_2, \tilde{F}_3] = q\tilde{F}_5$. With the basis change $\tilde{F}_1 = F_1$, $\tilde{F}_2 = -F_2$, $\tilde{F}_3 = -F_3$, $\tilde{F}_4 = -F_4$, $\tilde{F}_5 = F_5$ one has $[\tilde{F}_1, \tilde{F}_2] = l\tilde{F}_3 + m\tilde{F}_4 - n\tilde{F}_5$, $[\tilde{F}_1, \tilde{F}_3] = p\tilde{F}_4$, $[\tilde{F}_2, \tilde{F}_3] = q\tilde{F}_5$. Hence we can choose an orthonormal basis such that in the commutators (9) the coefficients satisfy $m, n \geq 0$. The one-dimensional subspaces $\mathbb{R}\, F_1$, $\mathbb{R}\, F_2$, $\mathbb{R}\, F_3$, $\mathbb{R}\, F_4$, $\mathbb{R}\, F_5$ form a framing of $(\mathfrak{l}_{5,9}, \langle ., . \rangle)$ since the subspace $\mathbb{R}F_5 \subset Z(\mathfrak{l}_{5,9})$ is generated by the vector in $\{\Phi(t_0), \Phi(t_0 + \frac{\pi}{2})\}$ having greater norm. The subspace $\mathbb{R}F_4 \subset Z(\mathfrak{l}_{5,9})$ is orthogonal to $\mathbb{R}F_5$. The subspace $\mathbb{R}\, F_3$ is contained in the commutator subalgebra and orthogonal to the center. The orthogonal one-dimensional subspaces $\mathbb{R}\, F_1$, $\mathbb{R}\, F_2$ are orthogonal to the commutator subalgebra and the subspace $\mathbb{R}[F_2, F_3]$ is contained in $\mathbb{R}F_5$.

If $\Phi(t)$ and $\Phi(t+\frac{\pi}{2})$ are orthogonal for any $t \in \mathbb{R}$, then $\| \Phi(t) \| = \| \Phi(t+\frac{\pi}{2}) \| = \frac{a_{11} a_{33}}{a_{44}} = \text{const}$. For E_4^*, E_5^* we obtain $E_4^* = \frac{a_{44}}{a_{11} a_{33}} \left(\cos t\Phi(t) - \sin t\Phi(t+\frac{\pi}{2})\right)$, $E_5^* = \frac{a_{44}}{a_{11} a_{33}} \left(\sin t\Phi(t) + \cos t\Phi(t+\frac{\pi}{2})\right)$. The Lie bracket $[F(t), F(t+\frac{\pi}{2})] = [E_1^*, E_2^*] = lE_3^* + mE_4^* + nE_5^*$, $l > 0$, can be written into the form $l\,E_3^* +$

$$+\frac{a_{44}}{a_{11} a_{33}} \left\{(m\cos t + n\sin t)\Phi(t) + (-m\sin t + n\cos t)\Phi(t+\frac{\pi}{2})\right\}.$$

If $m = n = 0$ we put $F_i = E_i^*$. For $(m, n) \neq (0, 0)$ there is unique $t_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ such that the solutions of the equation $-m\sin t + n\cos t = 0$ are $t_0 + k\,\frac{\pi}{2}$, $k \in \mathbb{Z}$. Then we define $F_1 = F(t_0)$, $F_2 = F(t_0 + \frac{\pi}{2})$,

$$F_3 = E_3^*,\ F_4 = \frac{a_{44}}{a_{11} a_{33}}\Phi(t_0),\ F_5 = \frac{a_{44}}{a_{11} a_{33}}\Phi(t_0 + \frac{\pi}{2}).$$

We obtain that the non-vanishing brackets $[F_i, F_j]$, $i, j = 1, \dots, 5$, have the shape

$$(10) \qquad [F_1, F_2] = lF_3 + mF_4,\ [F_1, F_3] = pF_4,\ [F_2, F_3] = pF_5,$$

with some $m \in \mathbb{R}$, $l, p > 0$. Using the isometric isomorphism $\tilde{F_1} = -F_1$, $\tilde{F_2} = F_2$, $\tilde{F_3} = -F_3$, $\tilde{F_4} = F_4$, $\tilde{F_5} = -F_5$ we obtain $[\tilde{F_1}, \tilde{F_2}] = l\tilde{F_3} - m\tilde{F_4}$, $[\tilde{F_1}, \tilde{F_3}] = p\tilde{F_4}$, $[\tilde{F_2}, \tilde{F_3}] = p\tilde{F_5}$. Hence we can assume $m \geq 0$.
If $m \neq 0$, then the subspace $\mathbb{R}\, F_5 \subset Z(\tilde{\mathfrak{n}}_{5,9}(l, m, p)$ is orthogonal to the Lie bracket of any two vectors contained in $\mathbb{R}\, F_1 \oplus \mathbb{R}\, F_2$. The subspace $\mathbb{R}F_4 \subset Z(\tilde{\mathfrak{n}}_{5,9}(l, m, p))$ is orthogonal to $\mathbb{R}F_5$. The subspace $\mathbb{R}\, F_3$ is contained in the commutator subalgebra and orthogonal to the center. The orthogonal one-dimensional subspaces $\mathbb{R}\, F_1$, $\mathbb{R}\, F_2$ are orthogonal to the commutator subalgebra and the subspace $\mathbb{R}[F_2, F_3]$ is contained in $\mathbb{R}F_5$. Hence the subspaces $\mathbb{R}\, F_1$, $\mathbb{R}\, F_2$, $\mathbb{R}\, F_3$, $\mathbb{R}\, F_4$, $\mathbb{R}\, F_5$ form a framing of the metric Lie algebra $(\tilde{\mathfrak{n}}_{5,9}(l, m, p), \langle ., . \rangle)$.

If $m = 0$ then we have

$$(11) \qquad [F_1, F_2] = lF_3,\ [F_1, F_3] = pF_4,\ [F_2, F_3] = pF_5.$$

From this we can see that $Z(\tilde{\mathfrak{n}}_{5,9}(l, 0, p))$ is orthogonal to the Lie bracket of any two vectors contained in $\mathbb{R} F_1 \oplus \mathbb{R} F_2$. Hence the metric Lie algebra $(\tilde{\mathfrak{n}}_{5,9}(l, 0, p), \langle ., . \rangle)$ is not framed. In this case for any isometric isomorphism $\Phi : \tilde{\mathfrak{n}}_{5,9}(l, 0, p) \to \tilde{\mathfrak{n}}_{5,9}(l^*, m^*, p^*)$ one has $\Phi(F_1) = \cos tF_1 \pm \sin tF_2$, $\Phi(F_2) = \mp \sin tF_1 + \cos tF_2$,
$\Phi(F_3) = \varepsilon F_3$, $\Phi(F_4) = \cos tF_4 \pm \sin tF_5$,
$\Phi(F_5) = \mp \sin tF_4 + \cos tF_5$ from which easily follows $m^* = 0$,
$l = l^*$, $p = p^*$.

Theorem

Let $\langle .,. \rangle$ be an inner product on the 5-dimensional three-step nilpotent Lie algebra $\mathfrak{l}_{5,9}$.

(1) *The metric Lie algebra $(\mathfrak{l}_{5,9}, \langle .,. \rangle)$ is isometrically isomorphic to a unique $\mathfrak{n}_{5,9}(l,m,n,p,q)$ with $l,m,n,p,q \in \mathbb{R}$ such that $l > 0$, $q > p > 0$ and $m,n \geq 0$, or to a unique $\tilde{\mathfrak{n}}_{5,9}(l,m,p)$ with $l,m,p \in \mathbb{R}$ such that $l,p > 0$ and $m \geq 0$.*

(2) *The groups of orthogonal automorphisms are the following matrix groups with respect to the basis $\{G_1, G_2, G_3, G_4, G_5\}$:*

(A) *for $\mathfrak{n}_{5,9}(l,m,n,p,q)$*

(i) *if $m = n = 0$:*

$$\left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 \end{pmatrix}, \quad \varepsilon_1,\ \varepsilon_2 = \pm 1 \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \tag{12}$$

(ii) if $m = 0,\ n > 0$:

$$(13)\qquad \left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 \end{pmatrix}, \qquad \varepsilon_1 = \pm 1 \right\} \cong \mathbb{Z}_2,$$

(iii) if $m > 0,\ n = 0$:

$$(14)\qquad \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \varepsilon_2 = \pm 1 \right\} \cong \mathbb{Z}_2,$$

(iv) if $m, n > 0$, then it is trivial;

For the Lie algebra $\tilde{\mathfrak{n}}_{5,9}(l, m, p)$: (i) if $m = 0$:

(15)
$$\left\{ \begin{pmatrix} \cos t & \varepsilon_2 \sin t & 0 & 0 & 0 \\ -\sin t & \varepsilon_2 \cos t & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \cos t & \sin t \\ 0 & 0 & 0 & -\varepsilon_2 \sin t & \cos t \end{pmatrix}, \varepsilon_2 = \pm 1, t \in [0, 2\pi) \right\},$$

(ii) if $m > 0$, then it is the group

(16)
$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \varepsilon_2 = \pm 1 \right\} \cong \mathbb{Z}_2.$$

Theorem

Every filiform metric Lie algebra is a framed metric Lie algebra.

Proof.

Every filiform nilpotent Lie algebra $\mathfrak{g}$ has a basis $\{E_1, \cdots, E_n\}$:

(17) $$[E_1, E_i] = E_{i+1}, \text{ for all } i \geq 2,$$

(defining relations of the standard filiform Lie algebras)

(18) $$[E_i, E_j] \in \mathfrak{g}^{i+j} = \text{Span}(E_{i+j}, \cdots, E_n), \text{ for all } i, j : i+j \neq n+1$$

and there exists $\alpha \in \mathbb{R}$ with

(19) $$[E_i, E_{n-i+1}] = (-1)^i \alpha E_n, \text{ for all } 2 \leq i \leq n-1.$$

If n is odd, then $\alpha = 0$. One set $E_i = 0$ for $i > n$ (cf. Theorem 4.1 in G. Cairns, A. Hinić Galić, Yu. Nikolayevsky, *Totally geodesic subalgebras of nilpotent Lie algebras*, J. Lie Theory, 23 (2013), 1023 – 1049)

Proof.

The lower central series $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}, \dots, \mathcal{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})] = \mathbb{R}\, E_{i+2} + \cdots + \mathbb{R}\, E_n, \dots, \mathcal{C}^{n-2}(\mathfrak{g})$ of $\mathfrak{g}$ forms a series of invariant ideals with $\dim(\mathcal{C}^i(\mathfrak{g})/\mathcal{C}^{i+1}(\mathfrak{g})) = 1$. The Gram-Schmidt process applied to the ordered basis $(E_n, E_{n-1}, \dots, E_2, E_1)$ yields an orthonormal basis $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ such that the commutator subalgebra $\mathfrak{g}' = \mathcal{C}^1(\mathfrak{g})$ has a framing $\mathfrak{g}' = \mathbb{R}\, F_3 \oplus \cdots \oplus \mathbb{R}\, F_n$. The orthogonal complementer of the commutator subalgebra has dimension 2.

If $\mathfrak{g}$ is a standard filiform Lie algebra, then the centralizer $C_{\mathfrak{g}}(\mathfrak{g}')$ of the commutator algebra $\mathfrak{g}'$ is the ideal $\mathbb{R}\, E_2 + \cdots + \mathbb{R}\, E_n$. Hence the decomposition $\mathfrak{g} = \mathbb{R}\, F_1 \oplus \cdots \oplus \mathbb{R}\, F_n$ is a framing of the metric standard filiform algebra $(\mathfrak{g}, \langle ., . \rangle)$. □

Proof.

From the commutator relations (17-19) we have $[E_2, \mathcal{C}^1(\mathfrak{g})] \subset \mathcal{C}^3(\mathfrak{g}) = \langle E_5, \cdots, E_n \rangle$. The factor Lie algebra $\mathfrak{g}/\mathcal{C}^3(\mathfrak{g}) = \langle \bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4 \rangle$ is determined by the non-vanishing Lie brackets $[\bar{E}_1, \bar{E}_2] = \bar{E}_3$, $[\bar{E}_1, \bar{E}_3] = \bar{E}_4$. Hence it is isomorphic to the 4-dimensional standard filiform Lie algebra $\mathfrak{s}_4$. The centre $Z(\mathfrak{g}/\mathcal{C}^3(\mathfrak{g}))$ of the factor Lie algebra $\mathfrak{g}/\mathcal{C}^3(\mathfrak{g})$ is $\mathbb{R}\,\bar{E}_4$, the commutator subalgebra $(\mathfrak{g}/\mathcal{C}^3(\mathfrak{g}))'$ of $\mathfrak{g}/\mathcal{C}^3(\mathfrak{g})$ is $\mathbb{R}\,\bar{E}_3 + \mathbb{R}\,\bar{E}_4$ and the centralizer $C_{\mathfrak{g}/\mathcal{C}^3(\mathfrak{g})}(\mathfrak{g}/\mathcal{C}^3(\mathfrak{g}))'$ of the commutator subalgebra $(\mathfrak{g}/\mathcal{C}^3(\mathfrak{g}))'$ of the factor algebra $\mathfrak{g}/\mathcal{C}^3(\mathfrak{g})$ is $\mathbb{R}\,\bar{E}_2 + \mathbb{R}\,\bar{E}_3 + \mathbb{R}\,\bar{E}_4$. Therefore the preimage of the centralizer $C_{\mathfrak{g}/\mathcal{C}^3(\mathfrak{g})}(\mathfrak{g}/\mathcal{C}^3(\mathfrak{g}))'$ in $\mathfrak{g}$ is $\mathbb{R}\,E_2 + \mathbb{R}\,E_3 + \mathbb{R}\,E_4 + \mathbb{R}\,E_5 + \cdots + \mathbb{R}\,E_n$. Applying the Gram-Schmidt process to the ordered basis $(E_n, \cdots, E_1)$ we obtain an orthonormal basis $\{F_1, \cdots, F_n\}$ such that the decomposition $\mathbb{R}\,F_1 \oplus \cdots \oplus \mathbb{R}\,F_n$ is a framing of the metric filiform algebra $(\mathfrak{g}, \langle ., . \rangle)$. □

$\mathfrak{s}_n : [E_1, E_2] = E_3, \ldots, [E_1, E_i] = E_{i+1}, \ldots, [E_1, E_{n-1}] = E_n.$
$[G_1, G_2] = c_{2,2} G_3 + c_{2,3} G_4 + \cdots + c_{2,n-1} G_n$
$[G_1, G_i] = c_{i,i} G_{i+1} + \cdots + c_{i,n-1} G_n$
$[G_1, G_{n-1}] = c_{n-1,n-1} G_n.$

Definition

Let $\{G_1, \ldots, G_n\}$ be an orthonormal basis in the n-dimensional Euclidean vector space $\mathbb{E}^n$ and let $\mathcal{C} = \{c_{j,k} \in \mathbb{R};\ 2 \leq k \leq j \leq n-1\}$ be a lower triangular $n-2 \times n-2$ matrix with positive diagonal elements. We denote by $\mathfrak{n}_{\mathcal{C}}$ the Lie algebra and by $[.,.]_{\mathcal{C}}$ its Lie bracket defined on $\mathbb{E}^n$ by the non-vanishing commutators

$$(20) \quad [G_1, G_i]_{\mathcal{C}} = -[G_i, G_1]_{\mathcal{C}} = \sum_{t=i}^{n-1} c_{t,i}\, G_{t+1}, \quad i = 2, \ldots, n-1.$$

The Lie algebra $\mathfrak{n}_{\mathcal{C}}$ equipped with the inner product $\langle .,. \rangle_{\mathcal{C}}$ induced by the Euclidean inner product of $\mathbb{E}^n$ is a metric Lie algebra $(\mathfrak{n}_{\mathcal{C}}, \langle .,. \rangle_{\mathcal{C}})$.

Using an isometric isomorphism we achive that the sign of the coefficients $c_{k,i}$ such that $k - i$ is odd simultaneously changed. If the set $\mathcal{P} = \{(k,i) : \; c_{k,i} \neq 0 \text{ and } k - i \text{ is odd}\}$ is not empty, then there is a suitable orthonormal basis satisfying $c_{k_0,i_0} > 0$ for the minimal element (k_0, i_0) of $\mathcal{P}$ with respect to the anti-lexicographic ordering of pairs.

Theorem

Let $\langle ., . \rangle$ be an inner product on the n-dimensional standard filiform nilpotent Lie algebra $\mathfrak{s}_n$.
There is a unique metric Lie algebra $(\mathfrak{n}_{\mathcal{C}}, \langle ., . \rangle_{\mathcal{C}})$ satisfying

1. *$(\mathfrak{n}_{\mathcal{C}}, \langle ., . \rangle_{\mathcal{C}})$ is isometrically isomorphic to $(\mathfrak{s}_n, \langle ., . \rangle)$,*
2. *if the set $\mathcal{P} = \{(k,i) : \; c_{k,i} \neq 0 \text{ and } k - i \text{ is odd}\}$ is not empty then $c_{k_0,i_0} > 0$ for the minimal element (k_0, i_0) of $\mathcal{P}$ with respect to the anti-lexicographic ordering of pairs.*

Theorem

The group of orthogonal automorphisms of $\mathfrak{n}_{\mathcal{C}}$ *is the matrix group*

1. *if* $\{(i,k):\ c_{k,i} \neq 0$ *and* $k-i$ *is odd*$\} = \emptyset$*:*

   (21)
$$\left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon_1^2\varepsilon_2 & 0 & \dots & 0 \\ 0 & 0 & \varepsilon_1^3\varepsilon_2 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1^n\varepsilon_2 \end{pmatrix}, \varepsilon_1,\ \varepsilon_2 = \pm 1 \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

2. *if* $\{(i,k):\ c_{k,i} \neq 0$ *and* $k-i$ *is odd*$\} \neq \emptyset$*:*

$$\left\{ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{pmatrix}, \varepsilon_2 = \pm 1 \right\} \cong \mathbb{Z}_2, \qquad (22)$$

with respect to the basis $\{G_1, \dots, G_n\}$.

$\mathfrak{l}_{5,6}$: $[E_1, E_2] = E_3$, $[E_1, E_3] = E_4$, $[E_1, E_4] = E_5$, $[E_2, E_3] = E_5$

Definition

Let $\{G_1, G_2, G_3, G_4, G_5\}$ be an orthonormal basis in the Euclidean vector space $\mathbb{E}^5$ and a, b, c, d, f, g, h real numbers with $a, d, g, h \neq 0$. The metric Lie algebra defined on $\mathbb{E}^5$ by the non-vanishing commutators $[G_1, G_2] = aG_3 + bG_4 + cG_5$, $[G_1, G_3] = dG_4 + fG_5$, $[G_1, G_4] = gG_5$, $[G_2, G_3] = hG_5$ will be denoted by $\mathfrak{n}(a, b, c, d, f, g, h)$.

The map $E_1 \mapsto E_1$, $E_2 \mapsto w(adg\, E_2 + bg\, E_3 + (c - \frac{bf}{d})E_4)$,

$$E_3 \mapsto wdg\, E_3,\ E_4 \mapsto w(g\, E_4 - \frac{f}{d}E_5),\ E_5 \mapsto wE_5,$$

is an isomorphism $\mathfrak{l}_{5,6} \to \mathfrak{n}(a, b, c, d, f, g, h)$, where $w = \frac{h}{ad^2g^2}$.

Theorem

Let $\langle .,. \rangle$ be an inner product on the 5-dimensional filiform but not standard filiform nilpotent Lie algebra $\mathfrak{l}_{5,6}$.

(1) *The metric Lie algebra $(\mathfrak{l}_{5,6}, \langle .,. \rangle)$ is isometrically isomorphic to a unique metric Lie algebra $\mathfrak{n}(a,b,c,d,f,g,h)$ with $a,b,c,d,f,g,h \in \mathbb{R}$ such that either $a,b,d,g,h > 0$, or $a,d,g,h > 0$, $b = 0$, $f \geq 0$.*

(2) *The groups of orthogonal automorphisms of $\mathfrak{n}(a,b,c,d,f,g,h)$ are the following matrix groups with respect to the basis $\{G_1, G_2, G_3, G_4, G_5\}$:*

(i) *for $b = f = 0$:*

$$\left\{ \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 \end{pmatrix}, \quad \varepsilon_1 = \pm 1 \right\} \cong \mathbb{Z}_2, \tag{23}$$

(ii) *for $b^2 + f^2 \neq 0$, it is trivial.*