On a Question of Yu. N. Mukhin

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Where to go?

- (1970) Y. N. Mukhin described the strongly TQH *LCA-groups*.
  (1986) He dealt with the modular compact groups.
  (1984) Posed Problem 9.32 in the Kourovka note book:

  Classify the locally compact groups G with product XY of any closed subgroups X and Y a closed subgroup of G.

  Strongly topological quasi-Hamiltonian Groups

  (1988) He classified all locally compact groups satisfying $\overline{XY} = \overline{YX}$.

  Topologically quasi-Hamiltonian groups, TQH

The Compact p-Case (Hofmann & Russo 2015)

A compact p-group G is *near-abelian* if it contains an abelian closed subgroup A such that each of its closed subgroups is normal in G and G/A is monothetic. For every odd prime p the following statements are equivalent:

- G is near-abelian;
- G is TQH;
- G is strongly-TQH;
- G is strict inverse limit of finite near-abelian groups;
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For $p = 2$ the dihedral groups D_8 must not be involved in G.

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What is the strategy?

- define (locally) compact near-abelian groups and describe their structure. Learn years later about Mukhin's Problem 9.32.
- Useful facts? .
  1. It is known that modular and TQH-groups can be nonabelian only if they are totally disconnected (Kümmich, Mukhin).
  2. Mukhin solved the question for groups with a discrete subgroup isomorphic to $\mathbb{Z}$.

  For us it was enough to consider only PERIODIC groups.

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  - the integers $\mathbb{Z}$;
  - discrete cyclic groups $\mathbb{Z}(n)$ having order n;
  - p-adic integers $\mathbb{Z}_p$;
  - Tori $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and their cartesian products $\mathbb{T}^k$ for $k \in \mathbb{N}$ or $k = \mathbb{N}$;

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- A locally compact group G is INDUCTIVELY MONOTHETIC (IMG) if every finite subset is contained in a monothetic subgroup.

  1. If discrete – then it is a subgroup of $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$;
  2. If infinite compact – then it is either connected of dimension 1 or procyclic (the inverse limit finite cyclic groups);
  3. If periodic – then it is a local product of its p-primary groups (J. BRACONNIER, 1948).

- The 2-dimensional torus $\mathbb{T} \times \mathbb{T}$ is monothetic, but not IMG, as it contains a subgroup isomorphic to $\mathbb{Z}(p) \times \mathbb{Z}(p)$ which is not monothetic.

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- The locally compact group G is NEAR-ABELIAN, if it contains a closed abelian subgroup A with inductively monothetic quotient G/A, and every closed subgroup of A is normal in G.
- 2 simple examples:
  - The p-adic integers $H := \mathbb{Z}_p$ act on Prüfer's group $A := \mathbb{Z}(p^\infty)$ when considered as a $\mathbb{Z}_p$-module; this gives rise to a near-abelian group extension of A by H.
  - The group $\mathbb{Z}(2)$ acts on the reals $\mathbb{R}$ by inversion. The semidirect product $\mathbb{R} \rtimes \mathbb{Z}(2)$ is near-abelian.

For solving question 9.32 the *A-nontrivial* near-abelian groups matter: $G/C_G(A)$ does neither act trivially nor by inversion on A.

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Theorem *Let G be an A-nontrivial near-abelian group. Then it admits a direct decomposition $G = G_1 \times G_2$ and all of the following holds:*

1. *G_1, G_2 are closed subgroup of G and G_1 is abelian; and*
2. *There is a closed inductively monothetic subgroup H of G_2 with $G_2 = (A \cap G_2)H$; and*
3. *G_1 and G_2 are* coprime.

The class of near-abelian groups is closed under passing to subgroups and quotients and contains strict inverse limits with compact kernels.

Theorem: *If in a locally compact group G every topologically finitely generated subgroup is near-abelian so is G.*

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What is the Plan? Cont'd.

- Strongly-TQH $\Longrightarrow$ TQH.

  Hence we need to know the structure of periodic TQH-groups.

- Every *periodic* TQH-group is the local product of p-groups. Each of these p-groups is of the form (see Hofmann & Russo 2015)

$$G = A\overline{\langle b \rangle}$$

  with A abelian and b a topological generator of a procyclic p-group (isomorphic either to $\mathbb{Z}_p$ or $\mathbb{Z}(p^n)$). There is $s \geq 1$ ($s \geq 2$ if $p = 2$) and for all $a \in A$

$$a^b = a^{1+p^s}.$$

- TQH $\Rightarrow$ near-abelian. $\quad$ $G = AH$, H IMG

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Mukhin's Example

- Let I be an index set and $B := \prod_{i \in I} B_i$ cartesian product of cyclic groups B_i of prime order p_i.
- Set $C := \bigoplus_{i \in I} C_i$ for groups C_i of prime order p_i.
- It turns out that the local product

$$B \times C = \prod_{i \in I}^{\mathrm{loc}} (B_i \times C_i, C_i)$$

  - is abelian, hence near-abelian and TQH.
  - is NOT strongly-TQH.

As Mukhin classified the abelian strongly-TQH groups, and TQH-groups are near-abelian, it suffices to assume A strongly-TQH.

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- Let I be an index set and $B := \prod_{i \in I} B_i$ cartesian product of cyclic groups B_i of prime order p_i.
- Set $C := \bigoplus_{i \in I} C_i$ for groups C_i of prime order p_i.
- It turns out that the local product

$$B \times C = \prod_{i \in I}^{\text{loc}} (B_i \times C_i, C_i)$$

  - is abelian, hence near-abelian and TQH.
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Solving question 9.32, i.e., Mukhins Problem

Theorem (W. Herfort, K.H. Hofmann, F.G. Russo, 2016)
Let G be periodic TQH with strongly-TQH basis A:

(A): G strongly-TQH $\Leftrightarrow$ $G/A \cap H$ strongly-TQH.

(B): *If, in addition, $A \cap H = \{1\}$, then G is strongly-TQH, if and only if*

$$G = G_1 \times G_2 \times H_0 \times A_0 \times D, \quad \text{with coprime factors, where}$$

- G_1 *is the direct product of finitely many p_i-groups;*
- G_2 *is compact:*
- H_0 *is IMG;*
- A_0 *is abelian; and*
- D *is discrete and quasi-Hamiltonian.*

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A Study of Mainly Periodic Locally Compact groups.

- Chabauty Topology.
- Sylow- and Hall-Theory for Topologically Locally Finite Groups (Schur-Zassenhaus Splitting Theorem).
- Scalar Automorphisms.
- Master Graph Depicting Scalar actions.
- Inductively Monothetic Groups and their Classification.
- Some Divisibility Questions in LCA groups.
- Near-Abelian Groups (Hall- and Sylow Theory, Existence of the Scaling Group H, etc.).
- Applications: Topologically modular, TQH- and strongly-TQH Groups.

Grazie per l'attenzione!