On the representability of actions for topological groups

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Joint work with Francis Borceux and Maria Manuel Clementino

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Conversely, from an action we obtain a split extension via the semidirect product construction.

Actions and automorphisms

An action of B on X is a map $B \times X \to X$ such that

1. $b \cdot (x_1 x_2) = (b \cdot x_1)(b \cdot x_2)$;
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which sends every group B to the set $\mathrm{SplExt}(B, X)$, is representable, i.e. it is of the form $\mathrm{Hom}_{\mathit{Gp}}(-, \mathrm{Aut}(X))$, where Aut(X) is the representing object.

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Theorem (Schreier-Mac Lane)

$$Ext(B, X, \varphi) \cong H^2(B, Z(X)),$$

whenever $Ext(B, X, \varphi)$ is not empty.

Other examples

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Actions of rings are equivalent to split extensions (actions are just left and right multiplication), but actions of rings are not representable, in general.

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Actions are homomorphisms $B \to \mathrm{CAut}(X)$, where $\mathrm{CAut}(X)$ is the group of continuous automomorphisms of X (with continuous inverse), but the problem is to endow $\mathrm{CAut}(X)$ with a suitable topology.

Cartesian closed categories

Definition

A category $\mathcal{C}$ with finite products is Cartesian closed if, for every $Y \in \mathcal{C}$, the functor

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Theorem (Borceux, Janelidze, Kelly)

In the category $Gp(\mathcal{C})$ of internal groups in a Cartesian closed category, actions are representable.

Internal groups

An internal group in $\mathcal{C}$ is an object G in $\mathcal{C}$ with a multiplication $m\colon G \times G \to G$, a unit $e\colon 1 \to G$ and an inverse $i\colon G \to G$ that are morphisms in $\mathcal{C}$.

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Sketch of the proof: if $X, Y \in Gp(\mathcal{C})$, the object $\mathrm{Hom}(X, Y)$ is the equalizer

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where $u(f)(x_1, x_2) = f(x_1 x_2)$ and $v(f)(x_1, x_2) = f(x_1)f(x_2)$. Then $\mathrm{Aut}(X)$ is the internal group of invertible elements in $\mathrm{Hom}(X, X)$.

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Definition

A filter on a set X is a non-empty subset $\mathfrak{x}$ of $P(X)$ such that

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$$\text{Reflexivity: } \forall\, x \in X \quad \dot{x} \to x,$$

where $\dot{x} = \{\, A \subseteq X \mid x \in A \,\}$.

$$\text{Transitivity: if } \mathfrak{X} \to \mathfrak{x} \text{ and } \mathfrak{x} \to x, \text{ then } \mu(\mathfrak{X}) \to x,$$

where $A \in \mu(\mathfrak{X})$ iff $\{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\} \in \mathfrak{X}$.

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Given a topological space X, $X^X \in PsTop$ is actually a topological space if and only if X is quasi-locally compact.

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Theorem

If X is a quasi-locally compact group, then actions on X are represented by $CAut(X)$.

The condition is sufficient but not necessary: $(\mathbb{Q}, +)$, with the euclidean topology, is not quasi-locally compact, but the actions on it are represented by $CAut(\mathbb{Q}, +) = (\mathbb{Q}^*, \cdot)$, with the euclidean topology.