Vanishing class sizes and p-nilpotency in finite groups

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Some general notation

In this talk, every group is assumed to be a finite group.

Given a group G, we denote by $\mathrm{Irr}(G)$ the set of *irreducible complex characters* of G, and we set

$$\mathrm{cd}(G) = \{\chi(1) \,:\, \chi \in \mathrm{Irr}(G)\}.$$

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Arithmetical structure of $\mathrm{cd}(G)$ and group structure of G

There is a deep interplay between the "arithmetical structure" of $\mathrm{cd}(G)$ and the group structure of G. One celebrated instance:

Theorem (Ito-Michler)

Let G be a group and p a prime. If every element in $\mathrm{cd}(G)$ is not divisible by p, then G has an (abelian) normal Sylow p-subgroup.

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Some other sets of positive integers associated with G

Other significant sets of positive integers associated with a group G:

- $\mathrm{o}(G) = \{o(g) \,:\, g \in G\}$.
- $\mathrm{cs}(G) = \{|g^G| \,:\, g \in G\}$.

Now, denoting by $\mathrm{Van}(G)$ the set of the *vanishing elements* of G (i.e., the elements on which some irreducible character of G takes value 0), we set

$$\mathrm{vo}(G) = \{o(g) \,:\, g \in \mathrm{Van}(G)\},$$

and

$$\mathrm{vcs}(G) = \{|g^G| \,:\, g \in \mathrm{Van}(G)\}.$$

We will deal with problems of “Ito-Michler type” concerning the above sets of integers.

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Zeros of characters: the starting point

The analysis concerning zeros of characters starts from the following classical result by W. Burnside.

Theorem

Let G be a group, and χ an irreducible character of G such that $\chi(1) > 1$. Then there exists $g \in G$ such that $\chi(g) = 0$.

Zeros of characters: the starting point

This result has been improved in several directions. For instance:

Theorem (Malle, Navarro, Olsson; 2000)

Let $\chi \in \mathrm{Irr}(G)$, $\chi(1) > 1$. Then there exists a prime number p and a p-element $g \in G$ such that $\chi(g) = 0$.

Recall that, if $\chi \in \mathrm{Irr}(G)$ vanishes on a p-element of G, then $\chi(1)$ is divisible by p. From this fact we immediately get:

Corollary

Let $\chi \in \mathrm{Irr}(G)$, $\chi(1) > 1$. If $\chi(1)$ is a π-number, then there exists a π-element $g \in G$ such that $\chi(g) = 0$.

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Vanishing elements

Let $\mathcal{R}$ be a row in the character table of a group G. Burnside's Theorem says:

$\mathcal{R}$ contains zeros $\iff$ $\mathcal{R}$ corresponds to a nonlinear character.

Vanishing elements

Let now $\mathcal{C}$ be a column in the character table of G. Following the standard "duality" between characters and conjugacy classes, it would be tempting to conjecture:

$\mathcal{C}$ contains zeros $\iff$ $\mathcal{C}$ corresponds to a noncentral conjugacy class.

Part "$\Longrightarrow$" of the previous statement is true but, although "$\Longleftarrow$" holds for nilpotent groups (Isaacs, Navarro, Wolf; 1999), it does not hold in general (consider Sym(3)). What is true is:

Theorem (Isaacs, Navarro, Wolf; 1999)

Let G be a solvable group, and $g \in G$ an element of odd order. If g is a nonvanishing element of G, then $g \in \mathbf{F}(G)$.

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A7

	1a	2a	3a	3b	4a	5a	6a	7a	7b
X.1	1	1	1	1	1	1	1	1	1
X.2	6	2	3	.	.	1	-1	-1	-1
X.3	10	-2	1	1	.	.	1	A	/A
X.4	10	-2	1	1	.	.	1	/A	A
X.5	14	2	2	-1	.	-1	2	.	.
X.6	14	2	-1	2	.	-1	-1	.	.
X.7	15	-1	3	.	-1	.	-1	1	1
X.8	21	1	-3	.	-1	1	1	.	.
X.9	35	-1	-1	-1	1	.	-1	.	.

Vanishing elements

The above theorem is false if we drop solvability. On the other hand, we have the following.

Theorem (Dolfi, Navarro, P., Sanus, Tiep; 2010)

Let G be a group, and $g \in G$ an element whose order is coprime to 6. If g is a nonvanishing element of G, then $g \in \mathbf{F}(G)$.

In this case the assumption on the order can not be removed.

Vanishing elements: a brief digression on Brauer characters

One may consider similar problems in the context of Brauer characters.
A contribution in this direction:

Theorem (Dolfi, P., Sanus; 2017)

Let $p > 3$ be a prime number, let G be a solvable group, and let $g \in G$ be such that $p \nmid o(g)$. If no irreducible p-Brauer character of G vanishes on g, then $g \in \mathbf{O}_{pp'pp'}(G)$ (i.e., g lies in a normal subgroup of G whose p-length and p'-length are both at most 2), with possible exceptions if $p \in \{5, 7\}$ and $o(g)$ is divisible by 2 or 3.

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The following elementary observations turn out to be critical in order to detect vanishing elements.

Proposition

Let $N \trianglelefteq G$, and $\theta \in \mathrm{Irr}(N)$. Then $G \setminus \bigcup_{g \in G} I_G(\theta^g) \subseteq \mathrm{Van}(G)$.

Proposition

Let $N \trianglelefteq G$, and p a prime. If there exists $\theta \in \mathrm{Irr}(N)$ such that $p \nmid \dfrac{|N|}{\theta(1)}$, then every $g \in N$ with $p \mid o(g)$ lies in $\mathrm{Van}(G)$.

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If we want to detect vanishing elements, the following theorem is a very useful one.

Theorem (Bianchi, Brough, Camina, P.; preprint 2017)

Let A be an abelian minimal normal subgroup of G. Let N/M be a chief factor of G such that $|N/M|$ is coprime with $|A|$ and $\mathbf{C}_N(A) = M$. Then

(a) $N \setminus M \subseteq \mathrm{Van}(G)$.

(b) *There exist $x \in N \setminus M$ and $\theta \in \mathrm{Irr}(A)$ such that $x \notin \bigcup_{g \in G} I_G(\theta^g)$.*

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Some "Ito-Michler type" theorems

Let G be a group, p a prime, and $P \in \mathrm{Syl}_p(G)$. Then:

p does not divide $\chi(1)$ for every $\chi \in \mathrm{Irr}(G)$

$\Downarrow$ (Ito-Michler)

$\mathrm{vo}(G)$ does not contain any p-power
(i.e., $\chi(x) \neq 0$ for every $\chi \in \mathrm{Irr}(G)$ and $x \in P$)

$\Downarrow$ (Dolfi, P., Sanus, Spiga; 2009)

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Now, set $\mathrm{Irr}(1_P^G) = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi_P, 1_P \rangle \neq 0\}$.

Then (Malle, Navarro; 2012):

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Some "Ito-Michler type" theorems

Next, we focus on conjugacy class sizes. It is an easy exercise to prove the following

Remark

Let G be a group and p a prime. Then p does not divide any number in $\mathrm{cs}(G)$ if and only if G has a central Sylow p-subgroup (i.e., G has a p-complement H that is a direct factor, and G/H is abelian).

What if the "Ito-Michler assumption" is required only for the sizes of the *vanishing* conjugacy classes? In this case, we get

Theorem (Dolfi, P., Sanus; 2010)

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Now, we aim to drop the p-solvability assumption. Let B_0 be the principal p-block of G, and define

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$$\mathrm{Van}(B_0) = \{x \in G \mid \chi(x) = 0 \text{ for some } \chi \in \mathrm{Irr}(B_0)\}.$$

Theorem (Dolfi, Malle, P., Sanus; preprint 2017)

Let G be a group and p a prime. Then p does not divide $|x^G|$ for every $x \in \mathrm{Van}(B_0)$ if and only if G has a normal p-complement H and G/H is abelian.

(Note that, if G has a p-complement H, then $\mathrm{Irr}(1_H^G) \subseteq \mathrm{Irr}(B_0)$.)