Vanishing class sizes and $p$-nilpotency in finite groups

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GTG - Trento, 16 June 2017
In this talk, every group is assumed to be a finite group.

Given a group $G$, we denote by $\text{Irr}(G)$ the set of irreducible complex characters of $G$, and we set

$$\text{cd}(G) = \{ \chi(1) : \chi \in \text{Irr}(G) \}.$$
Some general notation

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Arithmetical structure of $\text{cd}(G)$ and group structure of $G$

There is a deep interplay between the “arithmetical structure” of $\text{cd}(G)$ and the group structure of $G$. One celebrated instance:

**Theorem (Ito-Michler)**

Let $G$ be a group and $p$ a prime. If every element in $\text{cd}(G)$ is not divisible by $p$, then $G$ has an (abelian) normal Sylow $p$-subgroup.
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Some other sets of positive integers associated with $G$

Other significant sets of positive integers associated with a group $G$:

- $\text{o}(G) = \{o(g) : g \in G\}$.
- $\text{cs}(G) = \{|g^G| : g \in G\}$.

Now, denoting by Van($G$) the set of the vanishing elements of $G$ (i.e., the elements on which some irreducible character of $G$ takes value 0), we set

- $\text{vo}(G) = \{o(g) : g \in \text{Van}(G)\}$,
- $\text{vcs}(G) = \{|g^G| : g \in \text{Van}(G)\}$.

We will deal with problems of “Ito-Michler type” concerning the above sets of integers.
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Zeros of characters: the starting point

The analysis concerning zeros of characters starts from the following classical result by W. Burnside.

**Theorem**

Let $G$ be a group, and $\chi$ an irreducible character of $G$ such that $\chi(1) > 1$. Then there exists $g \in G$ such that $\chi(g) = 0$. 
Zeros of characters: the starting point

This result has been improved in several directions. For instance:

**Theorem (Malle, Navarro, Olsson; 2000)**

Let $\chi \in \text{Irr}(G)$, $\chi(1) > 1$. Then there exists a prime number $p$ and a $p$-element $g \in G$ such that $\chi(g) = 0$.

Recall that, if $\chi \in \text{Irr}(G)$ vanishes on a $p$-element of $G$, then $\chi(1)$ is divisible by $p$. From this fact we immediately get:

**Corollary**

Let $\chi \in \text{Irr}(G)$, $\chi(1) > 1$. If $\chi(1)$ is a $\pi$-number, then there exists a $\pi$-element $g \in G$ such that $\chi(g) = 0$. 
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Vanishing elements

Let $\mathcal{R}$ be a row in the character table of a group $G$. Burnside’s Theorem says:

$\mathcal{R}$ contains zeros $\iff$ $\mathcal{R}$ corresponds to a nonlinear character.
Vanishing elements

Let now $C$ be a column in the character table of $G$. Following the standard “duality” between characters and conjugacy classes, it would be tempting to conjecture:

\[ C \text{ contains zeros } \iff C \text{ corresponds to a noncentral conjugacy class.} \]

Part “⇒” of the previous statement is true but, although “⇐” holds for nilpotent groups (Isaacs, Navarro, Wolf; 1999), it does not hold in general (consider Sym(3)). What is true is:

**Theorem (Isaacs, Navarro, Wolf; 1999)**

Let $G$ be a solvable group, and $g \in G$ an element of odd order. If $g$ is a nonvanishing element of $G$, then $g \in F(G)$. 

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The above theorem is false if we drop solvability. On the other hand, we have the following.

**Theorem (Dolfi, Navarro, P., Sanus, Tiep; 2010)**

Let $G$ be a group, and $g \in G$ an element whose order is coprime to 6. If $g$ is a nonvanishing element of $G$, then $g \in F(G)$.

In this case the assumption on the order can not be removed.
Vanishing elements: a brief digression on Brauer characters

One may consider similar problems in the context of Brauer characters. A contribution in this direction:

**Theorem (Dolfi, P., Sanus; 2017)**

Let $p > 3$ be a prime number, let $G$ be a solvable group, and let $g \in G$ be such that $p \nmid o(g)$. If no irreducible $p$-Brauer character of $G$ vanishes on $g$, then $g \in O_{pp'pp'}(G)$ (i.e., $g$ lies in a normal subgroup of $G$ whose $p$-length and $p'$-length are both at most 2), with possible exceptions if $p \in \{5, 7\}$ and $o(g)$ is divisible by 2 or 3.
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The following elementary observations turn out to be critical in order to detect vanishing elements.

Proposition
Let $N \trianglelefteq G$, and $\theta \in \text{Irr}(N)$. Then $G \setminus \bigcup_{g \in G} I_G(\theta^g) \subseteq \text{Van}(G)$.

Proposition
Let $N \trianglelefteq G$, and $p$ a prime. If there exists $\theta \in \text{Irr}(N)$ such that $p \nmid \frac{|N|}{\theta(1)}$, then every $g \in N$ with $p \mid o(g)$ lies in $\text{Van}(G)$. 

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If we want to detect vanishing elements, the following theorem is a very useful one.

**Theorem (Bianchi, Brough, Camina, P.; preprint 2017)**

*Let $A$ be an abelian minimal normal subgroup of $G$. Let $N/M$ be a chief factor of $G$ such that $|N/M|$ is coprime with $|A|$ and $C_N(A) = M$. Then*

(a) $N \setminus M \subseteq \text{Van}(G)$.

(b) There exist $x \in N \setminus M$ and $\theta \in \text{Irr}(A)$ such that $x \notin \bigcup_{g \in G} I_G(\theta^g)$. 

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Some “Ito-Michler type” theorems

Let $G$ be a group, $p$ a prime, and $P \in \text{Syl}_p(G)$. Then:

$p$ does not divide $\chi(1)$ for every $\chi \in \text{Irr}(G)$

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vo($G$) does not contain any $p$-power
(i.e., $\chi(x) \neq 0$ for every $\chi \in \text{Irr}(G)$ and $x \in P$)

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Now, set $\text{Irr}(1^G_P) = \{ \chi \in \text{Irr}(G) \mid \langle \chi_P, 1_P \rangle \neq 0 \}$. Then (Malle, Navarro; 2012):

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Next, we focus on conjugacy class sizes. It is an easy exercise to prove the following

**Remark**

Let $G$ be a group and $p$ a prime. Then $p$ does not divide any number in $\text{cs}(G)$ if and only if $G$ has a central Sylow $p$-subgroup (i.e., $G$ has a $p$-complement $H$ that is a direct factor, and $G/H$ is abelian).

What if the “Ito-Michler assumption” is required only for the sizes of the vanishing conjugacy classes? In this case, we get

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Let $G$ be a group and $p$ a prime. Assume that there exists a $p$-complement $H$ of $G$.

In view of Malle and Navarro’s work, we set

$$\text{Van}(1^G_H) = \{ x \in G \mid \chi(x) = 0 \text{ for some } \chi \in \text{Irr}(G) \text{ with } \langle \chi_H, 1_H \rangle \neq 0 \}.$$ 

Theorem (Dolfi, Malle, P., Sanus; preprint 2017)

Let $p$ be a prime, $G$ a $p$-solvable group, and $H$ a $p$-complement of $G$.

Then $p$ does not divide $|x^G|$ for every $x \in \text{Van}(1^G_H)$ if and only if $H \trianglelefteq G$ and $G/H$ is abelian.
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Now, we aim to drop the $p$-solvability assumption. Let $B_0$ be the principal $p$-block of $G$, and define

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Let $G$ be a group and $p$ a prime. Then $p$ does not divide $|x^G|$ for every $x \in \text{Van}(B_0)$ if and only if $G$ has a normal $p$-complement $H$ and $G/H$ is abelian.

(Note that, if $G$ has a $p$-complement $H$, then $\text{Irr}(1^G_H) \subseteq \text{Irr}(B_0)$.)
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