Analytic Moufang loops and Malcev algebras.

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Groups and Topological Groups

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Loops and Groups

Given a loop $(L, \cdot, /, \backslash, 1)$.

left multiplication $L_a x = ax$
right multiplication $R_b y = yb$

By definition all mappings L_a, R_b are bijections.

left multiplication group $LMult(L)$ - group generated by $\{L_a\}_{a \in L}$

right multiplication group $RMult(L)$ - group generated by $\{R_b\}_{b \in L}$

multiplication group $Mult(L)$ - group generated by $\{L_a, R_b\}_{a,b \in L}$

inner mapping group $Int(L) = \{\phi \in Mult(L) \mid \phi(1) = 1\}$

left inner mapping group $LInt(L) = \{\phi \in LMult(L) \mid \phi(1) = 1\}$

right inner mapping group $RInt(L) = \{\phi \in RMult(L) \mid \phi(1) = 1\}$

As common a **normal subloop** is the kernel of loop homomorphism.

A subloop is normal if and only if it is invariant under inner mappings.

Some characteristic subloops

associator $(x, y, z) = x(yz)\backslash(xy)z$
(L, L, L) - subloop generated by all associators.

commutator $[x, y] = xy\backslash yx$
$[L, L]$ - subloop generated by all commutators.

left nucleus $N_l = \{u \in L \mid (u, x, y) = 1 \text{ for all } x, y \in L\}$
analogously the *middle nucleus* N_m and the *right nucleus* N_r
nucleus $N(L) = N_l \cap N_m \cap N_r$

center $C(L) = \{u \in N(L) \mid [u, x] = 1 \text{ for all } x \in L\}$

radical-defect $\Delta(L) = (L, L, L) \cap [L, L]$

Moufang loops, automorphic loops and left automorphic loops

$$(xy)(zx) = (x(yz))x \quad \text{(Moufang identity)}$$

In a Moufang loop M all nuclei coincide and $N(L)$ as well as $C(L)$ is normal.

A loop A is called *automorphic* if $Int(A) \subseteq Aut(A)$.

A loop A is called *left automorphic* if $LInt(A) \subseteq Aut(A)$.

Obviosly in automorphic loop all characteristic subloops are normal.

Automorphic loops form a variety, which is a subvariety of the variety of left automorphic loops.

Left automorphic and automorphic Moufang loops

Commutative Moufang loops (CML) are automorphic.

Left automorphic Moufang loops are right automorphic Moufang loops and vice verse.

It is known, that

- if M is left automorphic Moufang loop then $M/N(M)$ is a CML,
- if M is automorphic Moufang loop then $M/N(M)$ is a CML of exponent 3.

Free automorphic Moufang loops.

A.Grishkov, P.Plaumann and LS (2012) have shown the following theorem:

Theorem

Let F_n be a free group with a basis $\{x_1, \ldots, x_n\}$ and let C_n be a free CML with a basis $\{y_1, \ldots, y_n\}$.

Then the subloops A_n of the direct product $F_n \times C_n$ generated by the elements (x_i, y_i) where $i \in \{1, \ldots, n\}$ is a free automorphic Moufang loop of rank n.
In particular, $\Delta(A_n) = 1$.

Open Question Describe the structure of a free left automorphic Moufang loop.

Smooth left-automorphic Moufang Loops I

The tangent algebra of a smooth Moufang loop is a Malcev algebra, i.e. A is an algebra over $\mathbb{R}$ with the defining identities:

$$x^2 = 0, \;\; J(x, y, xz) = J(x, y, z)x.$$

The tangent algebra of a smooth automorphic Moufang loop is a Lie algebra, i.e. the following identities hold:

$$x^2 = 0, \;\; J(x, y, z) = 0$$

The tangent algebra of a smooth left-automorphic Moufang loop is a Malcev algebra with the following additional identity

$$J(x, y, zw) = 0. \tag{1}$$

It is clear that this algebra obeys the identities:

$$x^2 = 0, \;\; J(x, y, xz) = J(x, y, z)x = 0. \tag{2}$$

Some history

Malcev, A. I. (1955).*Analytics Loops.* Mat. Sbornik

Sagle, A.(1961) *Malcev algebras.* Trans. Amer. Math. Soc.

Kuz'min, E. N. (1971). *The connection between Mal'cev algebra and analytic Moufang loops.* Algebra and Logic

Kerdman, F. S.(1979) *Analytic Moufang loops in the large.* Dokl. Akad. Nauk SSSR

Kinyon M., Kunen K. ,Phillips J. D. (2002) *Every diassociative A-loop is Moufang.* Proc. Amer. Math. Soc.

Carrillo Catalan R., LS (2004) *On smooth power-alternative loops.* Comm. Algebra

Smooth left-automorphic Moufang Loops II

Let us call the Malcev algebras satisfying (1) Malcev algebras of first type and the Malcev algebras satisfying (2) Malcev algebras of second type.

Theorem

The varieties of Malcev algebras of first type and of second type are different.

Example

Here we give an example of a 23-dimensional algebras of second type which is not an algebra of first type.
Let F be a free anti commutative algebra generated by $X = \{x_1, x_2, x_3, x_4\}$, nilpotent of class 4, it means $F^4 = 0$. Let I be a subspace with with a basis of all X-words, $w = w(x_1, x_2, x_3, x_4)$, such that some letter x_i appears in w two or more times. It is clear that I is an ideal of F. Let's denote by $\mathbb{A}$ the factor algebra F/I. Then a basis of $\mathbb{A}$ has 22 elements:

Example I

$B = \cup_{i=1}^{3} B_i$ with

$$
\begin{aligned}
B_1 =& X, \\
B_2 =& \{[x_i, x_j] \mid 1 \leq i < j \leq 4\}, \\
B_3 =& \{[x_i, x_j, x_k] \mid 1 \leq i < j \leq 4, 1 \leq k \leq 4,\ k \neq i, j\}.
\end{aligned}
$$

The algebra A is a Malcev algebra.
Let's define an antisymmetric bilinear function $\psi : \mathrm{A} \times \mathrm{A} \longmapsto k$ given by the following values:

$$
\begin{array}{ll}
\psi([x_1, x_2], [x_3, x_4]) = 2, & \psi([x_1, x_3], [x_2, x_4]) = -2, \\
\psi([x_1, x_4], [x_2, x_3]) = 2, & \psi([x_2, x_3, x_1], x_4) = -3, \\
\psi([x_2, x_4, x_1], x_3) = 3, & \psi([x_2, x_4, x_3], x_1) = -1, \\
\psi([x_3, x_4, x_1], x_2) = -3, & \psi([x_3, x_4, x_2], x_1) = 1,
\end{array}
$$

and $\psi(v, w) = 0$ for all other values.

Example II

Consider a space $\tilde{\mathrm{A}} = \mathrm{A} \oplus kv$ and define a product on $\tilde{\mathrm{A}}$ as follows:

$$[(a, \alpha v), (b, \beta v)] = ([a, b], \psi(a, b)) \tag{3}$$

Direct computations show that $\tilde{\mathrm{A}}$ is a Malcev algebra of second type . On the other hand, if we set $x_i = (x_i, 0)$, then

$$\begin{aligned} [J(x_1, x_2, x_3), x_4] &= [x_1, x_2, x_3, x_4] + [x_2, x_3, x_1, x_4] - [x_1, x_3, x_2, x_4] \\ &= (0, -3v) \neq 0. \end{aligned}$$

Hence $\tilde{\mathrm{A}}$ is not a Malcev algebra of the first type.

Almost left-automorphic Moufang Loops

Question What kind of smooth loop corresponds to a Malcev algebra of second type?
Definition A Moufang loop L with the property that every three elements of L generate a left automorphic subloop will be called an *almost left automorphic* Moufang loop.
Due to Malcev -Kuzmin theory, we have

Theorem

A Malcev algebra of the second type is, in fact, a tangent algebra of a local almost left automorphic Moufang loop. The tangent algebra of every smooth almost left automorphic Moufang loop is a Malcev algebra of the second type.

Now we discuss the question of the existence of global left automorphic Moufang loop which corresponds to the given Malcev algebra of the first type and the existence of global almost left automorphic Moufang loop, which corresponds to the given Malcev algebra of the second type.

Global Moufang loops

Let $\mathcal{M}A_n$ be a variety of Malcev algebras with the identity $J(x_1x_2...x_n, y, z) = 0,\ n \in \mathbb{N}$.
In particular, a Malcev algebra $A \in \mathcal{M}A_2$ is a tangent algebra of some smooth left automorphic Moufang loop M. Let us call a variety of smooth Moufang loops with the identity $([...[x_1, x_2], x_3]..., x_k], y, z) = 1$ the variety of k-generalized left automorphic Moufang loops.
The tangent algebra of smooth k-generalized left automorphic Moufang loops is an algebra from the variety $\mathcal{M}A_k$. For any algebra from $\mathcal{M}A_k$ there exists a local smooth k-generalized left automorphic Moufang loop.

Theorem

A local smooth k- generalized left automorphic Moufang loop defines a global smooth k-generalized left automorphic Moufang loop.

One question on Malcev algebras

S.V. Pchelintsev posted the problem whether Malcev algebras of first type and of second type are special, in the sense that they can be embedded in a commutator algebra of some alternative algebra.

Theorem

Malcev algebras of first type are special.

(joint work in progress with A. Grishkov, I. Kashuba, M. Rasskazova)

The proof of this theorem can be based on results of Pchelintsev ("Speciality of metabelian Malcev Algebras", Mathematical Notes, 2003). In this paper he considered metabelian Malcev algebras and developed auxiliar identities which can be used to show that every Malcev algebra of nilpotency class 5 is special. This helps to prove our Theorem.

Let me note that we are working on another proof of the theorem.

Conjecture Malcev algebras of second type are special.

Last page

My co-authors on the correspondece between smooth left-automorphic Moufang loops and smooth almost left automorphic Moufang loops and their tangent Malcev algebras are :
A. Grishkov, R. Carrillo-Catalan and M. Rasskazova.

Thank you