Countably Recognizable Group Classes

Università degli Studi di Napoli Federico II

Marco Trombetti

Gruppen und topologische Gruppen

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Let G be an **infinite** group.

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Usually, if particular proper subgroups of G **share** a given structure, then the structure of G itself is **known**.

The set of **all proper** subgroups

The set of **all proper** subgroups
The set of all **large** proper subgroups

The set of **all proper** subgroups

The set of all **large** proper subgroups

F. de Giovanni – M.T. (2016)

If G is a soluble group of **cardinality** $\aleph_1$ in which all proper subgroup of cardinality $\aleph_1$ are abelian, then G is **abelian**.

The set of **all proper** subgroups
The set of all **large** proper subgroups
The set of all **small** proper subgroups

Definition

A group class $\mathfrak{X}$ is said to be **countably recognizable** when a group G is an $\mathfrak{X}$-group everytime all countable subgroups of G have the property $\mathfrak{X}$.

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Countably recognizable classes of groups were introduced and studied by **Reinhold Baer** in 1962, but already in 1950 it was proved respectively by **S.N. Černikov** and Baer that being hypercentral and hyperabelian are countably recognizable properties.

Definition

A group class $\mathfrak{X}$ is said to be **local** when G is an $\mathfrak{X}$-group whenever all its finite subsets are contained in a subgroups which is an $\mathfrak{X}$-group.

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Examples

The class of nilpotent groups of bounded class
The class of soluble groups of bounded length
...

Some History

Baer gave a lot of interesting examples of countably recognizable properties which are not local; for instance, it follows from Baer results, that if $\mathfrak{X}$ is a countably recognizable group class which is closed by subgroups and homomorphic images, then the class of groups admitting an ascending normal series with $\mathfrak{X}$-factors is still countably recognizable.

Some History

Later, many other countably recognizable group classes were discovered.

B.H. Neumann — the class of residually finite groups

R.E. Phillips — the class of groups whose subgroups have all maximal subgroups having finite index.

M.R. Dixon, **M.J. Evans** e **H. Smith**
the class of (finite rank)-by-nilpotent (-by-soluble) groups.

Some History

G. Higman proved that being free is not countably recognizable.

M.I. Kargapolov proved that having a non-trivial abelian subgroup which is ascendant in the group G is not countably recognizable.

Definition

A group G is said to be an F**C-group** if every element of G has only finitely many conjugates in G.

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Recall

Let G be a group. The elements of G having finitely many conjugates in G form a subgroup $FC(G)$ which is known as the F**C-center** of G.

Let G be a group. We define the **upper FC-central series** of G as the ascending series $\{FC_\alpha(G)\}_\alpha$ defined by setting $FC_0(G) = \{1\}$,

$$FC_{\alpha+1}(G)/FC_\alpha(G) = FC(G/FC_\alpha(G))$$

for each **ordinal** α, and

$$FC_\lambda(G) = \bigcup_{\alpha<\lambda} FC_\alpha(G)$$

for each **limit ordinal** λ.

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The group G is called **FC-hypercentral** if the upper FC-central series reachs G.

The group G is called **FC-nilpotent** if the upper FC-central series reachs G after finitely many steps.

A group G is called **nilpotent-by-finite** when G has a normal nilpotent subgroup N such that G/N is finite.

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nilpotent-by-finite $\implies$ FC-nilpotent $\implies$ FC-hypercentral

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nilpotent-by-finite $\implies$ FC-nilpotent $\implies$ FC-hypercentral

Theorem
F. de Giovanni – M.T.

The class of FC-nilpotent groups is **countably recognizable.**

Definition

Let $\mathfrak{X}$ be a class of groups. Then we define $\mathfrak{X}C$ to be the class of groups such that $G/C_G(x^G) \in \mathfrak{X}$ for each $g \in G$.

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Let $\mathfrak{X}$ be a countably recognizable group class which is also closed by subgroups. Then the class $\mathfrak{X}C$ is countably recognizable.

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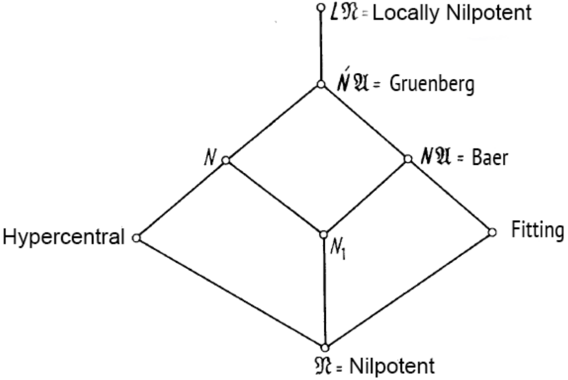
Theorem

F. de Giovanni – M.T.

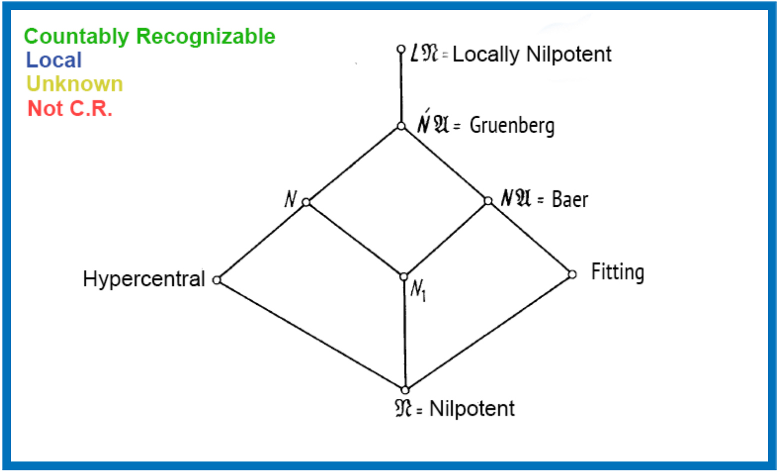
Let $\mathfrak{X}$ be a countably recognizable group class which is also closed by subgroups. Then the class $\mathfrak{X}C$ is countably recognizable.

Definition

Let FC^0 to be the class of finite groups. For each non-negative integer n we set FC^{n+1} to be the class of groups G such that $G/C_G(\langle x\rangle^G) \in FC^n$ for each $x \in G$.



$L\mathfrak{N}$ = Locally Nilpotent
$\acute{N}\mathfrak{A}$ = Gruenberg
N
$N\mathfrak{A}$ = Baer
Hypercentral
N_1
Fitting
$\mathfrak{N}$ = Nilpotent



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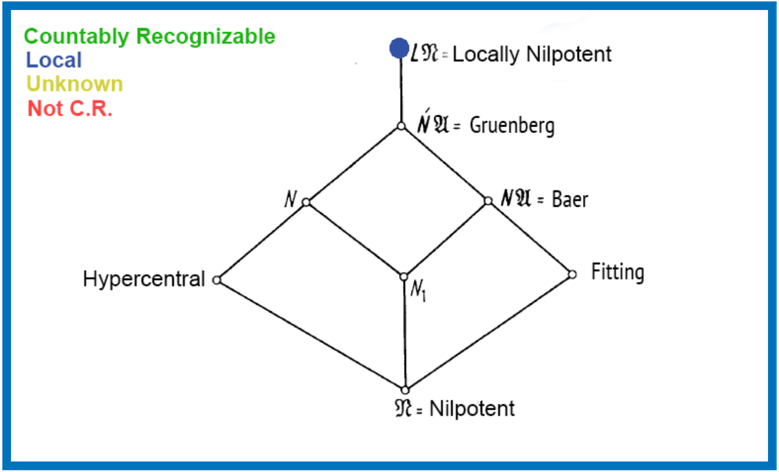

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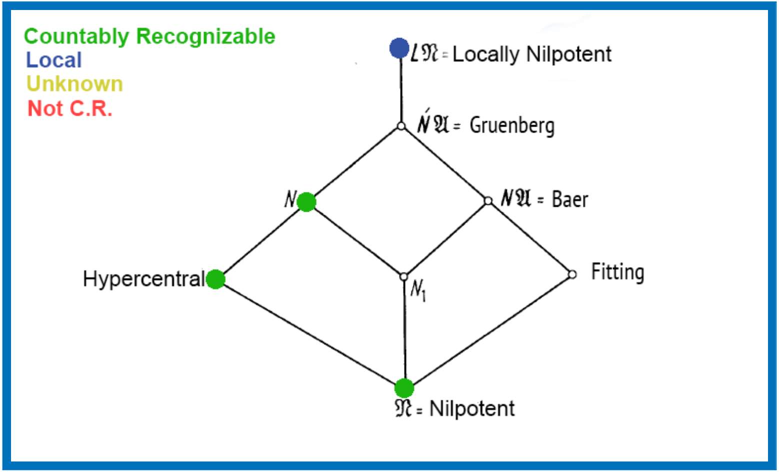

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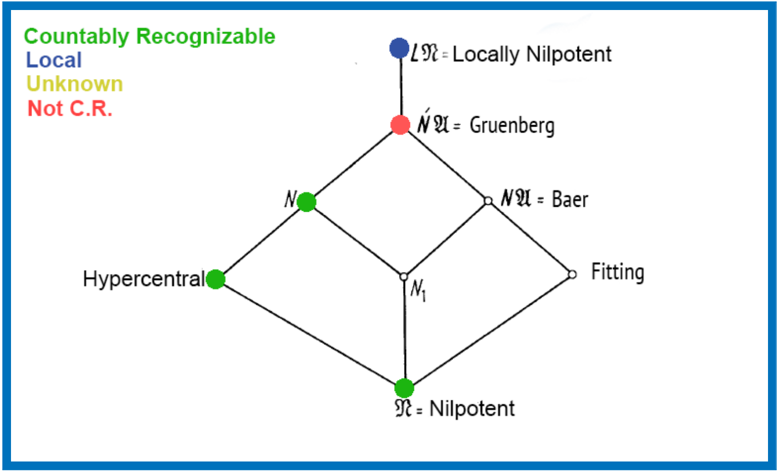

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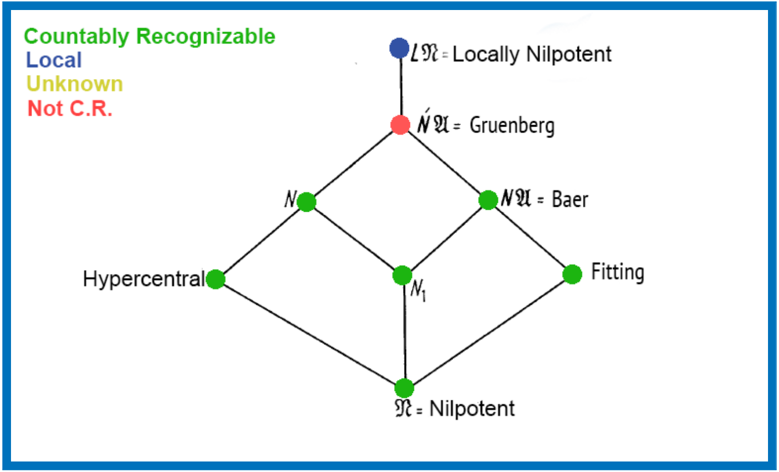

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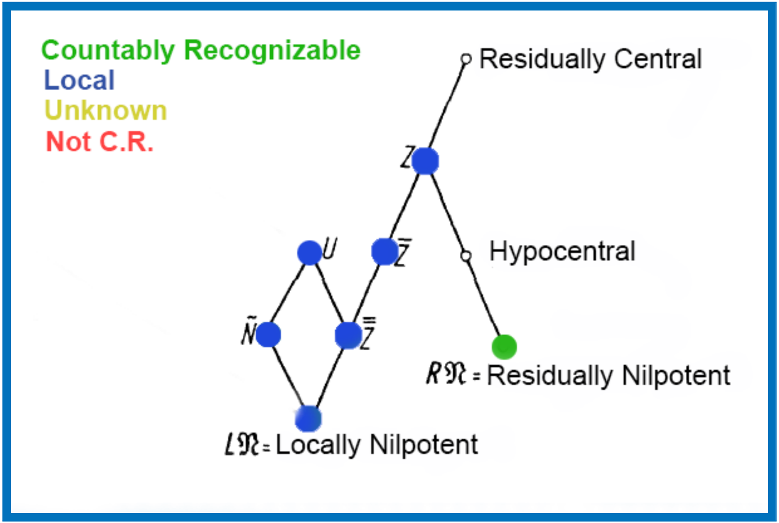
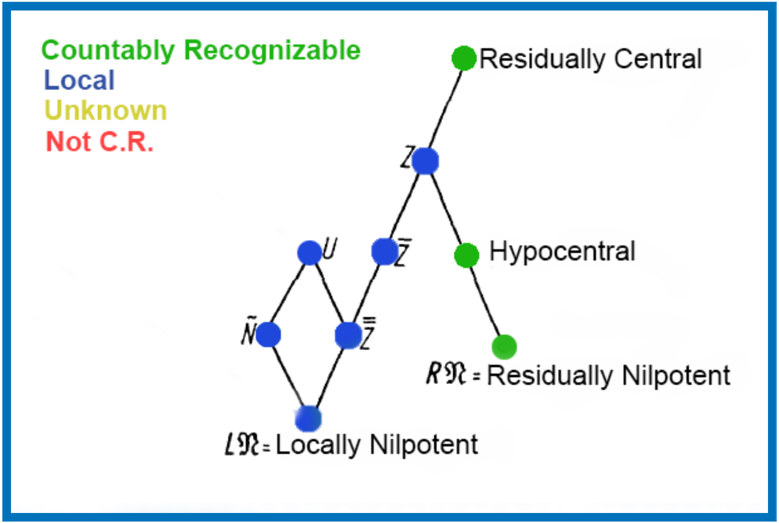

Countably Recognizable
Local
Unknown
Not C.R.
Residually Central
Z
U
$\bar{Z}$
Hypocentral
$\tilde{N}$
$\bar{\bar{Z}}$
$R\mathfrak{N}$ = Residually Nilpotent
$L\mathfrak{N}$ = Locally Nilpotent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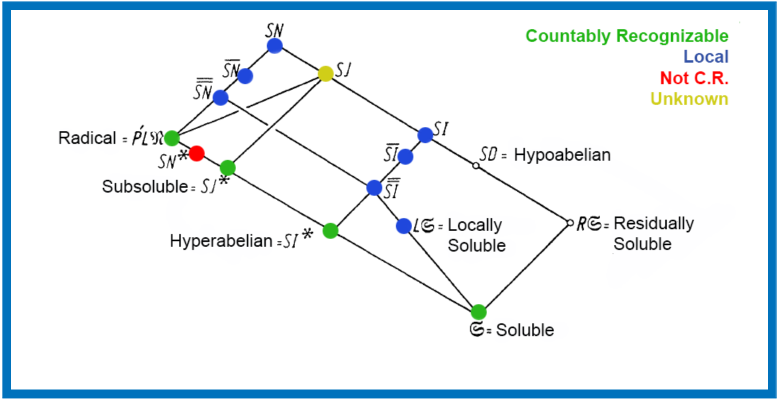
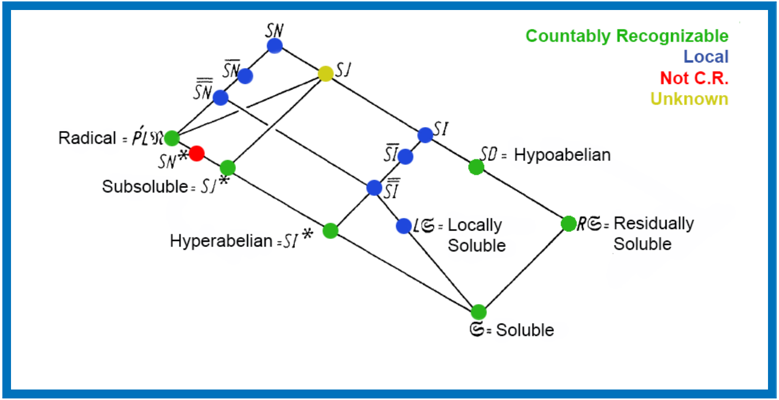


Countably Recognizable
Local
Not C.R.
Unknown
SN
$\overline{SN}$
$\overline{\overline{SN}}$
SJ
SI
$\overline{SI}$
$\overline{\overline{SI}}$
Radical = $\acute{P}L\mathfrak{N}$
SN^*
Subsoluble = SJ^*
Hyperabelian = SI^*
SD = Hypoabelian
$L\mathfrak{S}$ = Locally Soluble
$R\mathfrak{S}$ = Residually Soluble
$\mathfrak{S}$ = Soluble



Countably Recognizable
Local
Not C.R.
Unknown
SN
SJ
Radical = PLℜ
SN*
Subsoluble = SJ*
SI
SD = Hypoabelian
Hyperabelian = SI*
LS = Locally Soluble
RS = Residually Soluble
S = Soluble

Definition

A group G is called **minimax** if it admits a finite series whose factors satisfying either the minimal or the maximal condition on subgroups.

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The class of minimax **soluble** groups is countably recognizable

Theorem

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The class of **minimax groups** is countably recognizable

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To be closed (with resp. to the profinite topology) has **countable character**.

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Theorem
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The property of having all subgroups closed in the profinite topology is countably recognizable.

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