Exercise 8 : the Bogoliubov vacuum

$$H_{Bog} = E_0(N) + \sum_k \hbar\omega_k\, b_k^\dagger b_k$$

ground state $|g\rangle$: $b_k|g\rangle = 0$

* momentum distribution in ground state

$$\begin{pmatrix}\Lambda(r)\\ \Lambda^\dagger(r)\end{pmatrix} = \sum_k \begin{pmatrix}u_k(r)\\ v_k(r)\end{pmatrix} b_k + \begin{pmatrix}v_k^*(r)\\ u_k^*(r)\end{pmatrix} b_k^\dagger$$

$\langle n_k\rangle = \langle \delta\psi^\dagger(k)\,\delta\psi(k)\rangle =$ BEC does not contribute

$$= U_k^2 \langle b_k^\dagger b_k\rangle + V_k^2 \langle b_k b_k^\dagger\rangle = V_k^2 =$$

$$= \frac{1}{2}\left(\frac{E_k+\mu}{\omega_k} - 1\right)$$

* $k\to\infty$: $\langle n_k\rangle \to 0$ as $\frac{5}{2}\frac{\mu^2}{E_k^2}$
* $k\to 0$: $\langle n_k\rangle \to \infty$ as $\frac{\mu}{2\hbar c_s k} = \frac{m c_s}{2\hbar k}$

* Wiener-Khintchine theorem :

$$\int d^3R\, d^3\rho\, e^{ik\rho} \langle \psi^\dagger(R)\,\psi(R+\rho)\rangle =$$

$$= \int d^3R \int d^3\rho\, e^{ik\rho} \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3k''}{(2\pi)^3}\, e^{-ik'R}\, e^{ik''(R+\rho)} \cdot$$

$(2\pi)^3\delta(k'-k'')$

$$\cdot \langle \psi^\dagger(k')\,\psi(k'')\rangle =$$

$$= \int d^3 r \, e^{-ikr} \int \frac{d^3 k'}{(2\pi)^3} \, e^{ik'r} \, \langle \psi^+(k') \psi(k') \rangle =$$

$(2\pi)^3 \delta(k-k')$

$$= \langle \psi^+(k) \psi(k) \rangle$$

* Asymptotic behaviours of $\langle \psi^+(r) \psi(r') \rangle = \rho^{(1)}(r,r')$

- two-body collision wave function

$$\psi(r-r') \simeq 1 - a/|r-r'| \quad \text{for } |r-r'| \to 0.$$

$$\tilde{\psi}(k) \simeq a/k^2 \quad , \quad n(k) \simeq a^2/k^4 \sim \mu^2/E_k^2$$

$\Longrightarrow$ explains large k, small $|r-r'|$ behaviour

- long distance : $\rho^{(1)}(r-r') \simeq \int \frac{d^3 k}{(2\pi)^3} \frac{m c_s}{2\hbar k} = \frac{m c_s}{4\pi^2 \hbar |r-r'|^2} = \frac{1}{4\pi^2 \xi |r-r'|^2}$

$\to$ only length scale is ξ.

* sudden jump in a

- ground state $|g\rangle$ depends on a : can not follow sudden jump.

- excitations created in the system
- detectable as density fluctuations with

$$G^{(2)}(x,x') = G^{(2)}(|x-x'|)$$